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The orbit-preserving transformation groups associated with a measurable flow

By Masasi KOWADA

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§ 0. Introduction.

The purpose of this paper is to expose a general approach to the study of a measurable flow on a standard space with a probability measure.

For a given measurable flow $\mathfrak{T}$ we introduce the group $\mathcal{G}$ of bimeasurable transformations which transform the orbits of the flow $\mathfrak{T}$ onto another orbits; we call such a transformation in $\mathcal{G}$ an orbit-preserving transformation. Such group is related with many problems in the theory of flow. An orbit-preserving transformation yields a new flow, say a time changed flow of $\mathfrak{T}$ which is metrically isomorphic to the flow $\mathfrak{T}$. In this sense, such group $\mathcal{G}$ makes the flow $\mathfrak{T}$ invariant and gives us informations about the geometry of trajectories of $\mathfrak{T}$. Moreover the group $\mathcal{G}$ determines the subfamily of time change functions by which the time changed flows of $\mathfrak{T}$ are metrically isomorphic to $\mathfrak{T}$.

The notion of time change of flow was introduced by E. Hopf [**2**], and is extensively studied by G. Maruyama [**5**] and H. Totoki [**9**]. Our approach is different from them in the point of view of global analysis; for example, we ask in what class of flows the given flow $\mathfrak{T}$ is typical one.

Our method is worked by appealing to a cohomologous class of (one-) cocycles of the group $\mathcal{G}$; the notion of additive functionals of a flow which was introduced by G. Maruyama is just cocycle of the group $\mathcal{G}$ with respect to the additive group $\boldsymbol{R}$ of real numbers, and a time change function of flow is an inverse function in time variable of an additive functional, although our definition of time change functions is slightly different from his. These notions are defined in the sections 2 and 3.

The group $\mathcal{G}$ contains important subgroups. As one of them we are concerned with the subgroup $\mathcal{G}_s$ in the sections 3.3 and 3.4. The group $\mathcal{G}_s$ consists of all transformations which transform each orbit of $\mathfrak{T}$ onto itself. This group $\mathcal{G}_s$ is related with, for example, the time change of an analytic flow defined by a differential equation on the 2-dimensional torus which was treated by I. Arnold [10] and A. Kolmogorov [11]. In the section 3.3, we extend their results by our method.

The group $\mathcal{G}$ also contains a subgroup $\mathcal{A}$. A transformation $\sigma \in \mathcal{G}$ belongs to $\mathcal{A}$, if σ is an automorphism and the function $\tau_\sigma = \tau_\sigma(t, \omega)$ defined by

$$\sigma T_t \sigma^{-1} \omega = T_{\tau_\sigma(t,\omega)} \omega , \qquad \omega \in \Omega$$

satisfies admissible condition;

$$\tau_\sigma = \tau_\sigma(t, \omega) \text{ is differentiable at } t = 0 .$$

In terminology of Ya. G. Sinai [8], the flow $\mathcal{T}$ is a *transversal flow* of $\sigma \in \mathcal{A}$, which was successfully introduced by him to investigate an automorphism σ (or flow). On the while, our object of study is not σ but the very flow $\mathcal{T}$. Thus, in this sense, our approach is dual to that of Ya. G. Sinai.

The structures of groups $\mathcal{A}$ is related with the value of the entropy of the flow $\mathcal{T}$. In the section 4.1, we show that if the entropy of $\mathcal{T}$, $h(\mathcal{T})$ is positive finite, the group $\mathcal{A}$ consists of orbit-preserving automorphisms which commute with all T_t, $t \in (-\infty, \infty)$. In other section of 4 we study the ergodicity and spectrum of $\mathcal{T}$ appealing to the structure of the group $\mathfrak{A}$ and $\mathcal{A}$.

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§ 1. Preliminaries.

Some notions used in this work are a little different from usual ones in ergodic theory.

Throughout this paper $(\Omega, \mathfrak{B}, P)$ is a standard space, where P is a probability (Radon) measure on Ω.

Two spaces $(\Omega, \mathfrak{B}, P)$ and $(\Omega^0, \mathfrak{B}^0, P^0)$ are isomorphic if there exists a bimeasurable mapping θ from Ω onto Ω^0 such that $P^0(\theta E) = P(E)$, $E \in \mathfrak{B}$. An automorphism on Ω is an $1-1$ bimeasurable and measure-preserving transformation on Ω. We mean by a flow a 1-parameter group of automorphisms on Ω.

A flow $\{T_t\}$ is said to be measurable if the following condition is satisfied;

$$\{(t, \omega) ;\ T_t \omega \in B \in \mathfrak{B}\} \in \mathfrak{B}_R \times \mathfrak{B} ,$$

where $\mathfrak{B}_R$ is the topological Borel field in $\boldsymbol{R}$, the real line.

Throughout this paper, by a flow, we mean a measurable flow and we denote it by $\mathcal{T} = (\Omega, \mathfrak{B}, P, T_t)$.

Two flows $\mathcal{T} = (\Omega, \mathfrak{B}, P, T_t)$ and $\mathcal{T}^0 = (\Omega^0, \mathfrak{B}^0, P^0, T_t^0)$ are said to be (metrically) isomorphic, if there exists an isomorphism σ from Ω onto Ω^0 such that $\sigma^{-1} T_t^0 \sigma \omega = T_t \omega$, a.e. $\omega(dP)$ for all $t \in \boldsymbol{R}$, where the abbreviation a.e. $\omega(dP)$ means that an assertion foregoing to it hold for almost every ω with respect to the measure P. A flow $\mathfrak{T}=(\Omega, \mathfrak{B}, P, T_t)$ is said to be ergodic, if an invariant $\mathfrak{B}$-set E, i.e., $T_tE=E$ for all $t\in \boldsymbol{R}$, has measure 0 or 1.

Periodic point ω of the flow $\mathfrak{T}$ is an element of Ω such that $T_t\omega=\omega$ for some nonzero $t\in\boldsymbol{R}$. Throughout this paper, we assume that the set of all periodic points is a null set.

It is well known that an ergodic measurable flow $\mathfrak{T}$ can be represented by the special flow $\mathfrak{T}^*=(\Omega^*, \mathfrak{B}^*, P^*, T_t^*)$. The special flow $\mathfrak{T}^*$ can be constructed as follows. Let $(\Omega^0, \mathfrak{B}^0)$ be a standard space with measure m, θ be a positive real valued Borel function on Ω^0 and let T be an automorphism on (Ω^0, m). Let μ be the Lebesgue measure on $\boldsymbol{R}$. Let Ω^* be the set of all pairs of points (p, t) with $0\leqq t\leqq\theta(p)$ and $\mathfrak{B}^*$ be the restriction of $\mathfrak{B}^0\times\mathfrak{B}_{\boldsymbol{R}}$ to Ω^*. Put $P^*=m\times\mu/N$, where N is a normalizer, and define the 1-parameter group $\{T_t^*\}$ by

$$T_t^*(p, u)=\begin{cases}(T^np,\ t+u-\Sigma_0^{n-1}\theta(T^kp)) \\ \qquad\qquad \text{for } n>0 \text{ and } \Sigma_0^{n-1}\theta(T^kp)\leqq t+u<\Sigma_0^n\theta(T^kp) \\ (p,\ t+u) \quad \text{for } 0\leqq u+t<\theta(p) \\ (T^{-n}p,\ t+u+\Sigma_1^n\theta(T^{-k}p)) \\ \qquad\qquad \text{for } n<0 \text{ and } -\Sigma_0^n\theta(T^{-k}p)<t+u\leqq-\Sigma_0^{n-1}\theta(T^{-k}p)\,.\end{cases}$$

Then $\mathfrak{T}^*=(\Omega^*, \mathfrak{B}^*, P^*, T_t^*)$ is a measurable flow.

Put $F(p, u)=\theta(p)$ and $G(p, u)=u$, and then F and G are measurable functions on Ω^*.

§ 2. The orbit-preserving transformation group.

Suppose we are given a measurable flow $\mathfrak{T}=(\Omega, \mathfrak{B}, P, T_t)$ on a standard space $(\Omega, \mathfrak{B})$ with a probability measure P. Throughout this paper, we assume that the set of all periodic points of $\mathfrak{T}$ is a null set. Concerning this flow $\mathfrak{T}$, we wish to introduce groups of bimeasurable transformations on $(\Omega, \mathfrak{B})$, which make the flow $\mathfrak{T}$ invariant in the following sense.

DEFINITION 2.1. Let O_ω be the orbit of ω under the flow $\mathfrak{T}$,

$$O_\omega=\{T_t\omega\,;\ -\infty<t<\infty\}\,.$$

Let σ be a bimeasurable point transformation on Ω which transforms every orbit onto another orbit;

$$\sigma(O_\omega)=O_\eta \qquad \omega, \eta\in\Omega\,. \tag{2.1}$$

We say such mapping σ is *orbit-preserving transformation* and we denote by $\mathcal{G}$ the set of all orbit-preserving transformations on Ω. Clearly, $\mathcal{G}$ forms a group under the ordinary multiplication.

We identify σ_2 with $\sigma_1 \in \mathcal{G}$, if the measure of the set $\{\omega \in \Omega ; \sigma_1\omega \neq \sigma_2\omega\}$ is zero. This identification is an equivalence relation which is compatible with respect to the group operation.

If no confusion is likely to occur, we use the notation $\mathcal{G}$ for the quotient group of $\mathcal{G}$ with respect to the above equivalence relation.

The group $\mathcal{G}$ contains some subgroups which play an important role in the study of the flow $\mathcal{T}$. We shall give the definitions of them.

DEFINITION 2.2. A transformation $\sigma \in \mathcal{G}$ is said to be *strictly orbit-preserving*, if σ transforms every orbit onto itself; $\sigma(O_\omega) = O_\omega$. We denote by $\mathcal{G}_s$ the subgroup of all strictly orbit-preserving transformations on Ω.

By C we mean the set of all transformations commuting with every T_t. The set C also makes a subgroup of $\mathcal{G}$.

Let $\mathfrak{A}$ and $\mathfrak{A}_s$ be the intersection of $\mathcal{G}$ and $\mathcal{G}_s$ with the set of all automorphisms on Ω, respectively. They also form subgroups of $\mathcal{G}$. We use sometimes the notion $\mathcal{T}$ for the group $\{T_t\}$.

We can easily see that all the groups introduced above are metrical invariants of the flow. Suppose that a flow $\mathcal{T}^0 = (\Omega^0, \mathfrak{B}^0, P^0, T_t^0)$ is isomorphic to the flow $\mathcal{T}$ with respect to an isomorphism θ. Then the group $\mathcal{G}$ is isomorphic to the group $\mathcal{G}^0$ with respect to the isomorphism θ, where $\mathcal{G}^0$ is the orbit-preserving group associated with the flow $\mathcal{T}^0$. This situation is same to any other groups introduced above.

PROPOSITION 2.1. *The groups $\mathcal{G}$, $\mathcal{G}_s$, $\mathfrak{A}$, $\mathfrak{A}_s$, and C are metrical invariants of the flow $\mathcal{T}$.*

Note that the equivalence relation $\sim$ in $\mathcal{G}$ defined by C which will be introduced in the next section is also preserved by the isomorphism θ.

§ 3. The group $\mathcal{G}$ and the time change functions.

3.1. The time change functions of the flow $\mathcal{T}$.

The group $\mathcal{G}$ is closely related with the time change functions of the flow $\mathcal{T}$. The notion of time change function of a flow was introduced by G. Maruyama [5], from which ours is slightly different (refer also to H. Totoki [9]).

DEFINITION 3.1. A time change function $\tau = \tau(t, \omega)$ of the flow $\mathcal{T}$ is a real valued function defined on $\boldsymbol{R}$ which satisfies the followings;

(1) $\tau(t, \omega)$ is a finite valued and $1-1$ mapping from $\boldsymbol{R}$ onto itself for a. e. ω.

(2) $\tau(t+s, \omega) = \tau(t, \omega) + \tau(s, T_{\tau(t,\omega)}\omega)$ for a. e. ω.

(3) $\tau(0, \omega) = 0$ a. e. ω.

The set of all time change functions of the flow $\mathcal{T}$ is denoted by $\boldsymbol{F}$. When a time change function $\tau(t, \omega)$ is Borel measurable in (t, ω) we say τ is a *measurable* time change function.

Now we shall consider the relation between $\mathcal{G}$ and $\boldsymbol{F}$. Since any transformation $\sigma \in \mathcal{G}$ is orbit-preserving, there corresponds to σ a function $\tau_\sigma(t, \omega)$ by the relation

$$\sigma T_t \sigma^{-1}\omega = T_{\tau_\sigma(t,\omega)} \qquad \text{a. e. } \omega\,. \tag{3.1}$$

To prove the measurability of τ_σ, $\sigma \in \mathcal{G}$, we identify the flow $\mathcal{T}$ with the special flow $\mathcal{T}^* = (\Omega^*, \mathfrak{B}^*, P^*, T_t^*)$ of $\mathcal{T}$ and we regard the group $\mathcal{G}$ and the functions $\tau = \tau(t, \omega)$ as ones associated with the flow $\mathcal{T}^*$. Let Ω_0 be the basic space, θ be the ceiling function and T be the basic automorphism. Let $G(p, x) = x$ and $F(p, x) = \theta(p)$, then G and F are Borel measurable. Let $\sigma \in \mathcal{G}$. We define the mappings $(p, t) \to R[(p, t)\,;\, \sigma]$ and $(p, t) \to L[(p, t)\,;\, \sigma]$ from Ω^* into Ω_0 and from Ω^* into $\boldsymbol{R}$, respectively by

$$\sigma(p, t) = (R[(p, t)\,;\, \sigma],\ L[(p, t)\,;\, \sigma])\,.$$

Since σ is orbit-preserving, for each (p, t), there corresponds an integer k such that $R[(p, t)\,;\, \sigma] = T^k p$. We define

$$K[(p, t)\,;\, \sigma] = k\,.$$

LEMMA 3.1.1. *The functions $R[\cdot\,;\, \sigma]$, $L[\cdot\,;\, \sigma]$ and $K[\cdot\,;\, \sigma]$ are measurable for any $\sigma \in \mathcal{G}$.*

PROOF. For any measurable subset $M \subset \Omega_0$, the subsets

$$M^* = \{(p, t) \in \Omega^*\,;\ p \in M\}$$

and

$$M^*(a, b) = \{(p, t)\,;\ a \leqq G(p, t) < b\} \cap M^*$$

are measurable. Since the set $\{(p, t)\,;\, R[(p, t)\,;\, \sigma] \in M\} = \sigma^{-1}(M^*)$ is measurable, $R[\cdot\,;\, \sigma]$ is measurable. The measurability of $L[\cdot\,;\, \sigma]$ is deduced from the equation $L[(p, t)\,;\, \sigma] = G(\sigma(p, t))$. Let $\{\zeta_n\}$ be the sequence of measurable partitions of Ω_0 which satisfies the followings

1) $\zeta_n \uparrow \varepsilon$

2) for any different points $p, q \in \Omega_0$, there exists a partition ζ_n which separates p and q, namely there exists $M \in \zeta_n$ such that $p \in M$ and $q \overline{\in} M$.

We denote the set $\{(p, t)\,;\, K(p, t) = k\}$ by E_k. Suppose $(p, t) \in E_k$. Then for any n there exists $M_n \in \zeta_n$ such that $T^k p = R[(p, t)\,;\, \sigma] \in M_n$ and hence $(p, t) \in (T^{-k}M_n)^*$. Thus

$$E_k \subset \bigcap_n \bigcup_{M \in \zeta_n} (\{R[(p, t)\,;\, \sigma] \in M\} \cap (T^{-k}M)^*)\,.$$

Conversely, let $(p, t) \in \cap \cup (\{R[(p, t)\,;\, \sigma] \in M\} \cap (T^{-k}M)^*)$. Then there exists a set $M_n \in \zeta_n$ such that $R[(p, t)\,;\, \sigma] \in M_n$ and $(p, t) \in (T^{-k}M_n)^*$ for any n. Hence $R[(p, t)\,;\, \sigma] = T^k p$; if not, there exists a partition ζ_n which separates $R[(p, t)\,;\, \sigma]$ and $T^k p$. It follows

$$E_k=\bigcap_n \bigcup_{M\in\zeta_n}(\{R[(p,t);\sigma]\in M\}\cap(T^{-k}M)^*),$$

and therefore E_k is the measurable set.

THEOREM 3.1.1. 1) *$\tau_\sigma=\tau_\sigma(t,\omega)$ is the measurable time change function of $\mathfrak{T}$ for any $\sigma\in\mathfrak{G}$.*

2) $\tau_{\sigma_1\sigma_2}(t,\omega)=\tau_{\sigma_1}(\tau_{\sigma_2}(t,\sigma_1^{-1}\omega),\omega)$ *a. e.* $\omega(dP)$.

PROOF. Since the flow $\mathfrak{T}^*$ is measurable, the mappings $((p,t),s)\rightarrow R[(p,t);T_s^*]$ and $((p,t),s)\rightarrow L[(p,t);T_s^*]$ are measurable. Let

$$\Theta_k(p,t)=\begin{cases}\Sigma_0^{k-1}\theta(T^jp) & k\geqq 1\\ 0 & k=0\\ -\Sigma_{-1}^{-k}\theta(T^jp) & k\leqq -1\,.\end{cases}$$

Putting

$$g_k(p,t)=L[(p,t);\sigma^{-1}]+s-\Theta_k(R[(p,t);\sigma^{-1}],L[(p,t);\sigma^{-1}]),$$

we get

$$\begin{aligned}&\sigma T_s^*\sigma^{-1}(p,t)\\ &\quad=\sigma((T^kR[(p,t);\sigma^{-1}],L[(p,t);\sigma^{-1}]+s-\Theta_k(R[(p,t);\sigma^{-1}],L[(p,t);\sigma^{-1}]))\\ &\quad=(R[(T^kR[(p,t);\sigma^{-1}],g_k(p,t));\sigma],L[(T^kR[(p,t);\sigma^{-1}],g_k(p,t));\sigma]),\end{aligned}$$

when

$$\begin{aligned}&\Theta_k(R[(p,t);\sigma^{-1}],L[(p,t);\sigma^{-1}])\leqq L[(p,t);\sigma^{-1}]+s\\ &\qquad\qquad<\Theta_{k+1}(R[(p,t);\sigma^{-1}],L[(p,t);\sigma^{-1}]).\end{aligned}$$

On the while,

$$\begin{aligned}\sigma T_s^*\sigma^{-1}(p,t)&=T^*_{\tau_\sigma(s,(p,t))}(p,t)\\ &=(T^jp,t+\tau_\sigma(s,(p,t))-\Theta_j)\end{aligned}$$

when $\Theta_j\leqq t+\tau_\sigma(s,(p,t))<\Theta_{j+1}$. It follows

$$\left.\begin{aligned}&R[(T^kR[(p,t);\sigma^{-1}],g_k(p,t));\sigma]=T^jp\\ &L[(T^kR[(p,t);\sigma^{-1}],g_k(p,t));\sigma]=t+\tau_\sigma(s,(p,t))-\Theta_j\end{aligned}\right\}\qquad(3.1.2)$$

Put $K(p,t,s)=K[(p,t);T_s^*]$ and $E_k=\{((p,t),s);K(p,t,s)=k\}$. The similar considerations to the above lemma lead us to the equality

$$E_k=\bigcap_n\bigcup_{M\in\zeta_n}\{((p,t),s);R[(p,t);T_s^*]\in M,(p,t)\in(T^{-k}M)^*\},$$

and hence $K(p,t,s)$ is measurable. Put $J(p,t,s)=j$ if (p,t,s) satisfies (3.1.2). Then

$$\begin{aligned}&\{J(p,t,s)=j,K(p,t,s)=k\}=\{K(p,t,s)=k\}\\ &\qquad\cap\bigcap_n\bigcup_{M\in\zeta_n}\{(R[T^kR[(p,t);\sigma^{-1}],g_k(p,t,s)]\in M,(p,t)\in(T^{-j}M)^*\}\end{aligned}$$

and therefore the set $\{J(p, t, s)=j\} = \bigcup_k \{J(p, t, s)=j, K(p, t, s)=k\}$ is the measurable set. On the set $\{J(p, t, s)=j\}$, τ_σ has the form

$$\tau_\sigma(s, (p, t)) = L[T^k R[(p, t) ; \sigma^{-1}], g_k] + \Theta_j(T^k R[(p, t) ; \sigma^{-1}]) - t$$

and hence τ_σ is measurable. Another properties of τ_σ are easy to see and we omit them.

Thus we have the mapping: $\sigma \to \tau_\sigma$ from $\mathcal{G}$ into $\boldsymbol{F}$. Moreover the formula (3.1) gives us a time changed flow $\mathcal{S}_\sigma = (\Omega, \mathfrak{B}_\Omega, Q, S_t)$ of the flow $\mathcal{T}$, where $S_t\omega = T_{\tau_\sigma(t,\omega)}\omega$ and $Q(E) = P(\sigma^{-1}E)$, $E \in \mathfrak{B}_\Omega$.

LEMMA 3.1.2. *$\mathcal{S}_\sigma = (\Omega, \mathfrak{B}_\Omega, Q, S_t)$ is a measurable flow which is metrically isomorphic to the flow $\mathcal{T}$.*

Conversely let $\tau = \tau(t, \omega) \in \boldsymbol{F}$ be a measurable time change function and define an automorphism S_t by $S_t\omega = T_{\tau(t,\omega)}\omega$, $\omega \in \Omega$. Then we can easily see that $\{S_t\}$ is a group of bimeasurable transformations such that $S_{t+s} = S_t S_s$, $t, s \in \boldsymbol{R}$. Suppose that there exists a bimeasurable transformation σ on Ω such that

$$\sigma T_t \sigma^{-1}\omega = S_t\omega , \qquad \omega \in \Omega .$$

Then, with respect to the probability measure $Q(E) = P(\sigma^{-1}E)$, $E \in \mathfrak{B}_\Omega$, $\mathcal{S} = (\Omega, \mathfrak{B}_\Omega, Q, S_t)$ becomes a measurable flow which is isomorphic to the flow $\mathcal{T}$, with respect to σ. Clearly such transformation σ must be orbit-preserving and $\tau(t, \omega) = \tau_\sigma(t, \omega)$ a. e. $\omega(dP)$. We denote by $\boldsymbol{F}(\mathcal{G})$ the family of all $\tau_\sigma = \tau_\sigma(t, \omega)$, $\sigma \in \mathcal{G}$ which are classified mod 0.

Note that the mapping: $\sigma \to \tau_\sigma$ is not bijection. We say that two transformations σ_1 and σ_2 ($\in \mathcal{G}$) are *equivalent* if $\sigma_2^{-1}\sigma_1 \in C$ and denote $\sigma_1 \sim \sigma_2$. It is trivial to see that the relation $\sim$ is an equivalence relation. Denote by $\mathcal{G}/\sim$ the quotient space of $\mathcal{G}$ with respect to the above relation $\sim$, and by $[\sigma]$ the equivalence class with representative $\sigma \in \mathcal{G}$. From the relation $\sigma_1 \sim \sigma_2$, it follows $\tau_{\sigma_1} = \tau_{\sigma_2}$ a. e. ω, because of the followings;

$$\begin{aligned} T_{\tau_{\sigma_1}(t,\omega)}\omega &= \sigma_1 T_t \sigma_1^{-1}\omega = \sigma_2\sigma_2^{-1}\sigma_1 T_t \sigma_1^{-1}\omega \\ &= \sigma_2 T_t \sigma_2^{-1}\sigma_1\sigma_1^{-1}\omega = \sigma_2 T_t \sigma_2^{-1}\omega \\ &= T_{\tau_{\sigma_2}(t,\omega)}\omega . \end{aligned}$$

Thus we see that the mapping: $\mathcal{G}/\sim \ni [\sigma] \to \tau_\sigma \in \boldsymbol{F}(\mathcal{G})$ is a bijection. In the section 3.3, we shall give some results about the characterization of time change function in $\boldsymbol{F}(\mathcal{G}_s)$.

Summing up the above discussions, we obtain

PROPOSITION 3.1.1. *The element $[\sigma] \in \mathcal{G}/\sim$ induces the time changed flow $\mathcal{S}_{[\sigma]} = (\Omega, \mathfrak{B}, Q, S_t)$ which is isomorphic to the flow $\mathcal{T}$ via the time change function $\tau_\sigma \in \boldsymbol{F}(\mathcal{G})$. Moreover the mapping $\mathcal{G}/\sim \ni [\sigma] \to \tau_\sigma \in \boldsymbol{F}(\mathcal{G})$ is a bijection.*

There exists a flow $\mathfrak{T}$ such that $\boldsymbol{F}$ contains a time change function which does not come from the group $\mathcal{G}$. For example, let $\mathfrak{T}$ be a flow with discrete spectrum. Then there exists always a time changed flow $\mathcal{S}$ with continuous spectrum, so that $\mathcal{S}$ is not isomorphic to $\mathfrak{T}$ and the time change function does not belong to $\boldsymbol{F}(\mathcal{G})$.

3.2. The cohomology $H^1(\mathfrak{T}, \boldsymbol{R})$ and time change functions.

In this section, we discuss time change functions of the flow $\mathfrak{T}$ and cocycles of the group $\{T_t\}$. At first, we shall give definitions of cocycles and cohomology of a general dynamical system.

Let (X, B, μ) be a measure space and $\mathfrak{G}$ be a transformation group on X such that $\mu(gE)=0$ if $\mu(E)=0$, $E \in B$, $g \in \mathfrak{G}$. We say that a mapping φ from $\mathfrak{G} \times X$ into a group Σ is a *cocycle* of the dynamical system $(X, B, \mu, \mathfrak{G})$ with respect to the group Σ, if φ satisfies the equation

$$\varphi(g_2 g_1, \omega) = \varphi(g_1, \omega)\varphi(g_2, g_1\omega) \qquad \text{a.e. } \omega \text{ and for any } g_1, g_2 \in \mathfrak{G}\,. \tag{3.2.1}$$

We denote by $\tilde{H}^1(\mathfrak{G}, \Sigma)$ the set of all cocycles of $\mathfrak{G}$. Two cocycles φ and $\psi \in \tilde{H}^1(\mathfrak{G}, \Sigma)$ are said to be *homologous* with respect to a *coboundary* $h=h(\omega)$ if there exists a function $h=h(\omega)$ on X with values in Σ such that

$$\varphi(g, \omega)h(\omega) = \psi(g, \omega)h(g\omega) \qquad \text{a.e. } \omega\,. \tag{3.2.2}$$

As can be easily seen, the homologous relation is an equivalence relation. The group of homologous classes of cocycles is called the *cohomology* of $\mathfrak{G}$ with respect to Σ and is denoted by $H^1(\mathfrak{G}, \Sigma)$.

Now we consider the dynamical system $\mathfrak{T}=(\Omega, \mathfrak{B}_\Omega, P, T_t)$ and its cohomology $H^1(\mathfrak{T}, \boldsymbol{R})$. To each time change function $\tau \in \boldsymbol{F}$, we can construct an additive functional in the following way. Let $\tau=\tau(t, \omega)$ be a time change function. We define $\varphi=\varphi(u, \omega)$ the inverse function in u of $\tau=\tau(t, \omega)$ by

$$\varphi(u, \omega)=t\,, \quad \text{if} \quad \tau(t, \omega)=u\,.$$

By Definition 3.1, φ is well defined on $\boldsymbol{R}\times\Omega$.

LEMMA 3.2.1. *The functional φ satisfies*

a) $\varphi=\varphi(u, \omega)$ *is a finite valued* $1-1$ *mapping from* $\boldsymbol{R}$ *onto itself for a.e.* $\omega(dP)$

b) $\varphi(u+v, \omega)=\varphi(u, \omega)+\varphi(v, T_u\omega)$ *for a.e.* $\omega(dP)$

c) $\varphi(0, \omega)=0$ *for a.e.* $\omega(dP)$.

PROOF. It is obvious to see a) and c). We shall show only b). Put $\tau(t, \omega)=u$, $\tau(s, T_{\tau(t,\omega)}\omega)=v$, and $\tau(t+s, \omega)=\gamma$. Then it follows

$$\varphi(u+v, \omega)=\varphi(\gamma, \omega)=t+s=\varphi(u, \omega)+\varphi(v, T_u\omega)\,.$$

We shall agree to say that the functional φ defined above is an *additive functional* corresponding to a time change function τ (cf. G. Maruyama [**5**]).

By the above lemma, an additive functional of the flow $\mathfrak{T}$ is just a cocycle in $H^1(\mathfrak{T}, \boldsymbol{R})$ corresponding to a time change function, $\tau \in \boldsymbol{F}$. The additive functional corresponding to a time change function $\tau_\sigma \in \boldsymbol{F}(\mathfrak{G})$ enjoys special properties; one of which is the following lemma and it will be used in the section 3.3.

LEMMA 3.2.2. *Let $\tau_\sigma \in \boldsymbol{F}(\mathfrak{G})$ and let $\varphi_{\sigma^{-1}} = \varphi_{\sigma^{-1}}(u, \omega)$ be the additive functional corresponding to $\tau_{\sigma^{-1}} = \tau_{\sigma^{-1}}(u, \omega)$. Then*

$$\tau_\sigma(t, \sigma\omega) = \varphi_{\sigma^{-1}}(t, \omega) \qquad a.e.\ \omega\,,$$

so that τ_σ is measurable if and only if $\varphi_{\sigma^{-1}}$ is measurable.

PROOF. Let $\tau_{\sigma^{-1}}(t, \omega) = u$ and then $\varphi_{\sigma^{-1}}(u, \omega) = t$. It follows, $\sigma^{-1}T_t\sigma\omega = T_{\tau_{\sigma^{-1}}(t,\omega)}\omega = T_u\omega$. This implies,

$$T_{\varphi_{\sigma^{-1}}(u,\omega)}\sigma\omega = \sigma T_u\omega = \sigma T_u\sigma^{-1}\sigma\omega = T_{\tau_\sigma(u,\sigma\omega)}\sigma\omega\,.$$

This gives us the conclusion.

We shall give a sufficient condition for two cocycles φ and ψ being mutually homologous.

THEOREM 3.2.1. *Suppose that two measurable cocycles φ and ψ of the flow $\mathfrak{T}$ satisfy the condition*

$$-\infty < \lim_{S\to\infty} \frac{1}{S}\int_0^S [\varphi(t, \omega) - \psi(t, \omega)]dt < \infty \qquad a.e.\ \omega\,.$$

Then φ and ψ are homologous.

PROOF. Put

$$\tilde{h}(\omega) = \lim_{S\to\infty} \frac{1}{S}\int_0^S [\varphi(t, \omega) - \psi(t, \omega)]dt$$

and

$$B = \{\omega\,;\ \tilde{h}(\omega) < \infty\}\,.$$

Then it follows $T_tB \subset B$ and for every $\omega \in B$

$$\begin{aligned}
\tilde{h}(T_t\omega) &= \lim_{S\to\infty} \frac{1}{S}\int_0^S [\varphi(s, T_t\omega) - \psi(s, T_t\omega)]ds \\
&= \lim_{S\to\infty} \frac{1}{S}\int_0^S [\varphi(t+s, \omega) - \varphi(t, \omega) - \psi(t+s, \omega) + \psi(t, \omega)]ds \\
&= \tilde{h}(\omega) - \varphi(t, \omega) + \psi(t, \omega)\,.
\end{aligned}$$

Let $h = h(\omega)$ be a measurable extension of $\tilde{h} = \tilde{h}(\omega)$. Then

$$\varphi(t, \omega) + h(T_t\omega) = \psi(t, \omega) + h(\omega) \qquad \text{a.e. } \omega\,.$$

The following is related to the converse problem of the previous theorem.

THEOREM 3.2.2. *Suppose that a measurable cocycle* $\varphi(t, \omega)$ *is homologous to* $\psi(t, \omega)$:

$$\varphi(t, \omega)+h(T_t\omega)=\psi(t, \omega)+h(\omega) \qquad a.e.\ \omega .$$

If $h=h(\omega)$ *is integrable, we get*

$$-\infty<\lim_{S\to\infty}\frac{1}{S}\int_0^S[\varphi(s, \omega)-\psi(s, \omega)]ds<\infty \qquad a.e.\ \omega .$$

PROOF. We have

$$\frac{1}{S}\int_0^S[\varphi(\omega)-\psi(\omega)]ds=\frac{1}{S}\int_0^S[h(T_s\omega)-h(\omega)]ds$$
$$=\frac{1}{S}\int_0^S h(T_s\omega)ds-h(\omega) .$$

By the ergodic theorem, we get

$$\lim_{S\to\infty}\frac{1}{S}\int_0^S[\varphi-\psi]ds=\bar{h}(\omega)-h(\omega)$$

with some integrable $\mathcal{T}$-invariant function $\bar{h}(\omega)$. This completes the proof.

3.3. A characterization of time change functions $\tau_\sigma \in F(\mathcal{G}_s)$.

In this section, we shall give the geometrical interpretations of the previous discussions in 3.2.

We shall agree to say that a real valued measurable function f on Ω is *admissible* if

$$F_\omega(t)=f(T_t\omega)+t-f(\omega)$$

is the one-one onto map on $\boldsymbol{R}$ for a.e. ω.

THEOREM 3.3.1. *Let* f *be a real valued function on* Ω. *Then the transformation* σ *defined by*

$$\sigma\omega=T_{f(\omega)}\omega , \qquad \omega\in\Omega$$

is a strictly orbit-preserving transformation, namely $\sigma\in\mathcal{G}_s$, *if and only if* f *is admissible with respect to the flow* $\mathcal{T}$.

PROOF. We shall show that σ is the onto mapping. Let $\xi\in\Omega$ be an arbitrary element. Then, for some element $\omega\in O_\xi$ and some $s\in\boldsymbol{R}$, the element ξ has a form $\xi=T_s\omega$. We can find the time $t\in\boldsymbol{R}$ such that

$$s=f(T_t\omega)+t .$$

It follows

$$\xi=T_{f(T_t\omega)+t}\omega=T_{f(T_t\omega)}T_t\omega=\sigma T_t\omega .$$

To show that σ is $1-1$, suppose $\sigma\omega=\sigma\xi$. This implies $T_{f(\omega)}\omega=T_{f(\xi)}\xi$. Hence, by putting $t=f(\omega)-f(\xi)$, we obtain $\xi=T_t\omega$. It follows

$$T_{f(\omega)}\omega=\sigma\omega=\sigma\xi=\sigma T_t\omega=T_{f(T_t\omega)+t}\omega .$$

From this, it follows $f(T_t\omega)-f(\omega)+t=0$. Since f is admissible, we get $t=0$, namely $\omega=\xi$.

To prove the measurability of σ, we identify the flow $\mathfrak{T}$ with the special representation $\mathfrak{T}^*=(\Omega^*, \mathfrak{B}^*, P^*, T_t^*)$ of $\mathfrak{T}$ and we regard the group $\mathcal{G}_s$ and the admissible function f as ones associated with $\mathfrak{T}^*$. Let Ω_0 be the basic space, θ be the ceiling function and T be the basic automorphism. Let $G(p, x)=x$ and $F(p, x)=\theta(p)$. Then $G(p, x)$ and $F(p, x)$ are Borel measurable. Put

$$M^*=\{(p, t)\in\Omega^*;\ p\in M\}, \qquad M \text{ is measurable set in } \Omega_0,$$

$$M^*(a, b)=\{(p, t);\ a\leqq G(p, t)<b\}\cap M^*$$

and

$$\Theta_k(p, t)=\begin{cases} \Sigma_0^{k-1}\theta(T^jp) & k\geqq 1\\ -\Sigma_{-1}^{-k}\theta(T^jp) & k\leqq -1\\ 0 & k=0 .\end{cases}$$

We get

$$\begin{aligned}&\sigma^{-1}(M^*(a, b))\\ &=\bigcup_k\{(p, t);\ (T^kp, t+f(p, t)-\Theta_k(p))\in M^*(a, b),\ \Theta_k(p)\leqq t+f(p, t)<\Theta_{k+1}(p)\}\\ &=\bigcup_k[(T^{-k}M)^*\cap\{(p, t);\ a\leqq G(p, t)+f(p, t)-\Theta_k(p, t)<b\}\\ &\qquad\qquad\cap\{(p, t);\ \Theta_k(p, t)\leqq G(p, t)+f(p, t)\leqq\Theta_{k+1}(p, t)\}] .\end{aligned}$$

Since the right term is the measurable set and the family $\{M^*(a, b);\ M\in\mathfrak{B}_{\Omega_0}, (a, b)\subset \boldsymbol{R}\}$ generates $\mathfrak{B}^*$, σ is measurable. Since σ is a one-to-one onto measurable transformation on the standard space, σ^{-1} has also the same properties.

Conversely we suppose $\sigma\in G_s$. Recall the notations $R[(p, t); \sigma]$, $L[(p, t); \sigma]$ and $K[(p, t); \sigma]$ defined in the Lemma 3.1.1. As shown in the Lemma 3.1.1, the set $E_k=\{(p, t);\ K(p, t)=k\}$ is measurable and $\Omega^*=\bigcup_k E_k$. Since the function f has the form

$$f(p, t)=L[(p, t);\ \sigma]+\Theta_k(p, t)+G(p, t), \qquad (p, t)\in E_k ,$$

f is measurable. This completes the proof.

Let $\sigma\in G_s$. Since $\sigma\omega$ is in O_ω, we can find a time $t\in\boldsymbol{R}$ such that $\sigma\omega=T_t\omega$. Denote t by $f_\sigma(\omega)$. Moreover, to each $\sigma\in G_s$, as was shown in 3.1, there corresponds a time change function $\tau_\sigma\in\boldsymbol{F}(\mathcal{G}_s)$.

THEOREM 3.3.2. *A time change function $\tau_\sigma\in\boldsymbol{F}(\mathcal{G}_s)$ has the form;*

$$\tau_\sigma(t, \sigma\omega)=f_\sigma(T_t\omega)-f_\sigma(\omega)+t , \qquad a. e.\ \omega ,$$

where f_σ is admissible.

We denote by φ_0 the ordinary time, namely, $\varphi_0=\varphi_0(t, \omega)=t$ for any $\omega \in \Omega$. Note that φ_0 is also a cocycle in $H^1(\mathfrak{T}, \boldsymbol{R})$. We, now, can give a condition for a time change function $\tau=\tau(t, \omega)$ of the flow $\mathfrak{T}$ to be induced from a transformation $\sigma \in \mathcal{G}_s$.

THEOREM 3.3.3. *Let $\tau \in \boldsymbol{F}$ and φ be the additive functional corresponding to τ. Then $\tau=\tau(t, \omega)$ is induced from a transformation $\sigma \in \mathcal{G}_s$, if $\varphi=\varphi(u, \omega)$ is homologous to the ordinary time $\varphi_0=\varphi_0(t, \omega)=t$ with respect to an admissible coboundary function f, namely*

$$\varphi(t, \omega)=f(T_t\omega)-f(\omega)+t \qquad a.\,e.\ \omega\,.$$

PROOF. Suppose $\varphi(t, \omega)=f(T_t\omega)-f(\omega)+t$, where f is admissible. Then f yields a transformation σ in G_s and a time change function τ_σ. By Theorem 3.3.2 and Lemma 3.2.2, we get

$$\varphi(t, \omega)=\tau_\sigma(t, \sigma\omega)=\varphi_{\sigma^{-1}}(t, \omega) \qquad \text{a. e. } \omega\,,$$

where $\varphi_{\sigma^{-1}}$ is the additive functional corresponding to the time change function $\tau_{\sigma^{-1}}$. Hence it follows $\tau(t, \omega)=\tau_{\sigma^{-1}}(t, \omega)$, that is, $\tau \in \boldsymbol{F}(\mathcal{G}_s)$.

Combining the above theorem with Theorem 3.2.2, we get a sufficient condition for $\tau=\tau(t, \omega)$ to be induced from $\sigma \in \mathcal{G}_s$.

COROLLARY 3.3.1. *If the additive functional φ of a time change function τ is measurable and satisfies*

$$-\infty<\lim_{S\to\infty}\frac{1}{S}\int_0^S[\varphi(t, \omega)-t]dt<\infty \qquad a.\,e.\ \omega\,,$$

then $\tau=\tau(t, \omega)$ is induced from a transformation $\sigma \in \mathcal{G}_s$ and moreover the time changed flow of $\mathfrak{T}$ by τ is metrically isomorphic to the flow $\mathfrak{T}$.

APPLICATION TO THE FLOW ON THE TORUS. As an application of the previous discussions, we shall consider the flow which was studied by A. Kolmogorov and I. Arnold, and we shall give an extension of their result.

Let us consider the ergodic flow $\mathfrak{T}=(M_2, \mathfrak{B}, dxdy, T_t)$ on the 2-dimensional torus $\boldsymbol{M}_2$ with the normalized Lebesgue measure $dxdy$, where $\mathfrak{B}$ is the topological Borel field and T_t is defined by

$$\frac{dx}{dt}=1\,, \qquad \frac{dy}{dt}=\gamma \quad (\gamma \text{ is an irrational number})\,.$$

Let $K(x, y)$ be a real valued periodic function on $\boldsymbol{R}^2$ with period 1 such that

$$0<K(x, y) \in C^{(k)} \qquad (k \geqq 3)$$

and

$$\int_0^1\int_0^1 K(x, y)dxdy=1\,.$$

Define the additive functional $\varphi=\varphi(t, x, y)$ by

$$\varphi(t, x, y)=\int_0^t K(x+s, y+\gamma s)ds\,.$$

Then we get the time changed flow $\mathcal{S}=(\boldsymbol{M}_2, \mathfrak{B}, Q, S_t)$, where S_t is defined by

$$\frac{dx}{dt}=\frac{1}{K(x, y)}\,, \qquad \frac{dy}{dt}=\frac{1}{\gamma K(x, y)}$$

and

$$dQ(x, y)=K(x, y)dxdy\,.$$

Now our problem reads as follows;

Are two flows $\mathcal{T}$ and $\mathcal{S}$ isomorphic?

Suppose that γ satisfies an arithmetic condition such that there exist positive numbers L and H $(H<k-2)$ for which

$$|m+n\gamma|>\frac{L}{(|m|+|n|)^H} \tag{3.3.1}$$

holds for any integers m and n.

Let

$$K(x, y)=\sum c_{m,n}e^{2\pi i(mx+ny)}$$

be the Fourier expansion of $K(x, y)$. Then we get

$$\varphi(t, x, y)=t+\sum_{(m,n)\neq(0,0)} c_{m,n}\frac{e^{2\pi i(m+n\gamma)t}-1}{2\pi i(m+n\gamma)}e^{2\pi i(mx+ny)}$$

and

$$\lim_{S\to\infty}\frac{1}{S}\int_0^S[\varphi-t]dt=\sum_{(m,n)\neq(0,0)}\frac{c_{m,n}}{2\pi i(m+n\gamma)}e^{2\pi i(mx+ny)}\,.$$

We shall show that the right terms of the above equations are absolutely convergent series.

We get the following estimation of the Fourier coefficients $c_{m,n}$:

$$2^{k-1}\pi^k(|m|+|n|)^k|c_{m,n}|\leqq \mathrm{Max}\left\{\mathrm{Max}\frac{\partial^k}{\partial x^k}K,\ \mathrm{Max}\frac{\partial^k}{\partial y^k}K\right\}.$$

Denoting the right term by M, we get

$$\frac{|c_{m,n}|}{2\pi|m+n\gamma|}\leqq M/2^k\pi^{k+1}L(|m|+|n|)^{k-H}\,.$$

Let $N(j)$ be a number of the lattice points (m, n) for which $|m|+|n|=j$. Then

$$N(j)\leqq 2^2(j+1)\leqq 2^3 j\,.$$

It follows

$$\begin{aligned}\sum_{(m,n)\neq(0,0)}\frac{|c_{m,n}|}{2\pi|m+n\gamma|}&\leqq\frac{M}{2^k\pi^{k+1}}\sum_{(m,n)\neq(0,0)}(|m|+|n|)^{k-H}\\ &\leqq\frac{M}{2^k\pi^{k+1}}\sum_{j=1}j^{H-k+1}\,.\end{aligned}$$

Since $0<H<k-2$, the right term converges.

By Corollary 3.3.1, we conclude that the two flows $\mathfrak{T}$ and $\mathcal{S}$ are isomorphic with respect to a strictly orbit-preserving transformation.

THEOREM 3.3.4. *Let $\mathfrak{T}$ be the flow on the 2-dimensional torus $\boldsymbol{M}_2$ defined by*

$$dx/dt=1\,, \qquad dy/dt=\gamma$$

with the Lebesgue measure dxdy, and let $\mathcal{S}$ be the time changed flow of $\mathfrak{T}$ defined by

$$dx/dt=1/K(x, y)\,, \qquad dy/dt=1/\gamma K(x, y)$$

with the measure $dQ(x, y)=K(x, y)dxdy$. Suppose $0<K\in C^{(k)}$ and $\iint K\,dxdy=1$, and let γ be an irrational number for which there exist positive numbers L and H $(H<k-2)$ such that

$$|m+n\gamma|>\frac{L}{(|m|+|n|)^H} \qquad \textit{for any integers } m \textit{ and } n\,.$$

Then the time changed flow $\mathcal{S}$ is isomorphic to the flow $\mathfrak{T}$.

3.4. The cocycle $f_{\mathfrak{T}}$ in $H^1(\mathcal{G}_s, \boldsymbol{R})$.

In this section, we are concerned with the cocycle $f_{\mathfrak{T}}$ of the group $\mathcal{G}_s$ with respect to $\boldsymbol{R}$, which is determined by the flow $\mathfrak{T}$. As was shown in 3.3, to each $\sigma\in\mathcal{G}_s$, there corresponds a function $f_\sigma=f_\sigma(\omega)$ such that $\sigma\omega=T_{f_\sigma(\omega)}\omega$. The family $\{f_\sigma\,;\,\sigma\in\mathcal{G}_s\}$ has the following property.

LEMMA 3.4.1.

$$f_{\sigma_2\sigma_1}(\omega)=f_{\sigma_1}(\omega)+f_{\sigma_2}(\sigma_1\omega) \qquad a.\,e.\ \omega\,.$$

PROOF.

$$\sigma_2\sigma_1\omega=T_{f_{\sigma_2\sigma_1}(\omega)}\omega=T_{f_{\sigma_2}(\sigma_1\omega)}\sigma_1\omega=T_{f_{\sigma_1}(\omega)+f_{\sigma_2}(\sigma_1\omega)}\omega\,.$$

Let us define a function on $\mathcal{G}_s\times\Omega$ with values in $\boldsymbol{R}$ by $f_{\mathfrak{T}}(\sigma, \omega)=f_\sigma(\omega)$, $\omega\in\Omega$, $\sigma\in\mathcal{G}_s$. Then it follows from the above lemma that the function $f_{\mathfrak{T}}=f_{\mathfrak{T}}(\sigma, \omega)$ is just a cocycle of the group $\mathcal{G}_s$ with respect to $\boldsymbol{R}$, namely, $f_{\mathfrak{T}}\in H^1(\mathcal{G}_s, \boldsymbol{R})$. Let $\varphi_\theta\in H^1(\mathfrak{T}, \boldsymbol{R})$ be a cocycle of the flow $\mathfrak{T}$ corresponding to $\theta\in\mathcal{G}_s$ and define a mapping $\varphi_\theta\circ f_{\mathfrak{T}}$ from $\mathcal{G}_s\times\Omega$ into $\boldsymbol{R}$ by

$$(\varphi_\theta\circ f_{\mathfrak{T}})(\sigma, \omega)=\varphi_\theta(f_{\mathfrak{T}}(\sigma, \omega), \omega)\,.$$

PROPOSITION 3.4.1. *$\varphi_\theta\circ f_{\mathfrak{T}}$ is a cocycle in $H^1(\mathcal{G}_s, \boldsymbol{R})$ and is homologous to $f_{\mathfrak{T}}$. Moreover if a cocycle $g=g(\sigma, \omega)\in H^1(\mathcal{G}_s, \boldsymbol{R})$ is homologous to the cocycle $f_{\mathfrak{T}}$ with respect to an admissible coboundary $h=h(\omega)$, then there exists a transformation $\theta\in\mathcal{G}_s$ such that*

$$\varphi_\theta\circ f_{\mathfrak{T}}=g \qquad a.\,e.\ \omega\,, \qquad \textit{for any } \sigma\in\mathcal{G}_s\,.$$

PROOF.

$$\begin{aligned}(\varphi_\theta \circ f_{\mathcal{T}})(\sigma_2\sigma_1, \omega) &= \varphi_\theta(f_{\mathcal{T}}(\sigma_2\sigma_1, \omega), \omega)\\ &= \varphi_\theta(f_{\mathcal{T}}(\sigma_1, \omega), \omega) + \varphi_\theta(f_{\mathcal{T}}(\sigma_2, \sigma_1\omega), T_{f_{\mathcal{T}}(\sigma_1,\omega)}\omega)\\ &= (\varphi_\theta \circ f_{\mathcal{T}})(\sigma_1, \omega) + (\varphi_\theta \circ f_{\mathcal{T}})(\sigma_2, \sigma_1\omega) .\end{aligned}$$

Since φ_θ has the form

$$\varphi_\theta(t, \omega) = f_{\theta^{-1}}(T_t\omega) - f_{\theta^{-1}}(\omega) + t ,$$

replacing t by $f_{\mathcal{T}}(\sigma, \omega)$ in the above formula, we get

$$\begin{aligned}(\varphi_\theta \circ f_{\mathcal{T}})(\sigma, \omega) &= f_{\theta^{-1}}(T_{f_{\mathcal{T}}(\sigma,\omega)}\omega) - f_{\theta^{-1}}(\omega) + f_{\mathcal{T}}(\sigma, \omega)\\ &= f_{\theta^{-1}}(\sigma\omega) - f_{\theta^{-1}}(\omega) + f_{\mathcal{T}}(\sigma, \omega)\end{aligned}$$

a. e. ω for any $\sigma \in \mathcal{G}_s$. This means that $\varphi_\theta \circ f_{\mathcal{T}}$ is homologous to $f_{\mathcal{T}}$ with respect to the coboundary $f_{\theta^{-1}}$ and this proves the first assertion. Put $\bar{\theta}\omega = T_{h(\omega)}\omega$, $\omega \in \Omega$. Then $\theta = \bar{\theta}^{-1} \in \mathcal{G}_s$ and $\varphi_\theta(t, \omega) = h(T_t\omega) - h(\omega) + t$ a. e. ω. It follows

$$\begin{aligned}(\varphi_\theta \circ f_{\mathcal{T}})(\sigma, \omega) &= h(T_{f_{\mathcal{T}}(\sigma,\omega)}\omega) - h(\omega) + f_{\mathcal{T}}(\sigma, \omega)\\ &= h(\sigma\omega) - h(\omega) + f_{\mathcal{T}}(\sigma, \omega)\\ &= g(\sigma, \omega) \qquad \text{a. e. } \omega .\end{aligned}$$

We shall give a condition that a flow with the same trajectories as $\mathcal{T}$ is isomorphic to the flow $\mathcal{T}$. Let $\tau_\theta \in \boldsymbol{F}(\mathcal{G}_s)$ and define $(\tau_\theta \circ g)(\sigma, \omega) = \tau_\theta(g(\sigma, \omega), \omega)$, where $g \in H^1(\mathcal{G}_s, \boldsymbol{R})$.

THEOREM 3.4.1. *Let $\mathcal{S} = (\Omega, \mathfrak{B}, Q, S_t)$ be a time changed flow of $\mathcal{T} = (\Omega, \mathfrak{B}, P, T_t)$ with respect to $\theta \in G_s$. Then $f_{\mathcal{T}} = \tau_\theta \circ f_{\mathcal{S}}$. Conversely if $\mathcal{S}$ is the 1-parameter group of bimeasurable transformations on $(\Omega, \mathfrak{B})$ with the same trajectories as $\mathcal{T}$ and $f_{\mathcal{T}} = \tau_\theta \circ f_{\mathcal{S}}$ with $\theta \in \mathcal{G}_s$, then $\mathcal{S}$ becomes a flow with the measure $Q(E) = P(\theta^{-1}E)$, $E \in \mathfrak{B}$ and is isomorphic to $\mathcal{T}$.*

PROOF. Since

$$\theta T_t \theta^{-1}\omega = S_t\omega ,$$
$$P(\theta^{-1}E) = Q(E) , \qquad E \in \mathfrak{B} ,$$

we get

$$\sigma\omega = T_{f_{\mathcal{T}}(\sigma,\omega)}\omega = S_{f_{\mathcal{S}}(\sigma,\omega)}\omega = \theta T_{f_{\mathcal{S}}(\sigma,\omega)}\theta^{-1}\omega = T_{\tau_\theta(f_{\mathcal{S}}(\sigma,\omega),\omega)}\omega ,$$

and then

$$f_{\mathcal{T}}(\sigma, \omega) = (\tau_\theta \circ f_{\mathcal{S}})(\sigma, \omega) \quad \text{a. e. } \omega , \qquad \text{for any } \sigma \in \mathcal{G}_s .$$

Since $f_{\mathcal{S}}(S_t, \omega) = t$ a. e. ω, it is easy to see the converse statement.

§ 4. The dynamical system $(\Omega, \mathfrak{B}, P, \mathcal{A})$.

4.1. The dynamical system $(\Omega, \mathfrak{B}, P, \mathcal{A})$ and the entropy of the flow $\mathcal{T}$.

Let $\boldsymbol{F}(\mathcal{A})$ be the set of all time change functions $\tau_\sigma = \tau_\sigma(t, \omega)$, $\sigma \in \mathfrak{A}$, each of which has the nonzero derivative at $t=0$ for a. e. ω. Let

$$\mathcal{A} = \{\sigma \in \mathfrak{A}\,;\ \tau_\sigma \in \boldsymbol{F}(\mathcal{A})\} .$$

If $\sigma_1, \sigma_2 \in \mathcal{A}$, we get

$$\lim_{t \to 0} \frac{\tau_{\sigma_1 \sigma_2}(t, \omega)}{t} = \lim_{\tau_{\sigma_2} \to 0} \frac{\tau_{\sigma_1}(\tau_{\sigma_2}(t, \sigma_1^{-1}\omega), \omega)}{\tau_{\sigma_2}(t, \sigma_1^{-1}\omega)} \lim_{t \to 0} \frac{\tau_{\sigma_2}(t, \sigma_1^{-1}\omega)}{t} .$$

Since $t = \tau_{\sigma^{-1}}(\tau_\sigma(t, \sigma\omega), \omega)$,

$$\lim_{s \to 0} \frac{\tau_{\sigma^{-1}}(s, \omega)}{s} = \lim_{t \to 0} \frac{t}{\tau_\sigma(t, \sigma\omega)} = \frac{1}{\tau'_\sigma(0, \omega)} \qquad \text{a. e. } \omega ,$$

so that the set $\mathcal{A}$ forms a subgroup of $\mathfrak{A}$. Since $\{T_t\} \subset \mathcal{A}$ the dynamical system $\mathcal{A} = (\Omega, \mathfrak{B}, P, \mathcal{A})$ is an extension of the dynamical system $\mathcal{T} = (\Omega, \mathfrak{B}, P, T_t)$.

We shall show the structure of $\mathcal{A}$ is related to the entropy of the flow $\mathcal{T}$. In the following, we assume the flow is ergodic.

LEMMA 4.1.1 (Ya. G. Sinai [8]). *Let $\mathcal{T}$ be an ergodic flow and let τ_σ be a measurable time change function of $\mathcal{T}$ with $\sigma \in \mathcal{A}$. Then we get*

$$\tau_\sigma(t, \omega) = \lambda t \qquad a.\,e.\ \omega .$$

Using the above lemma we get

THEOREM 4.1.1. *Suppose the entropy of the ergodic flow $\mathcal{T}$ is positive finite. Then*

$$\mathcal{A} = \mathfrak{A} \cap \mathcal{C} .$$

PROOF. Let $\mathcal{S} = (\Omega, \mathfrak{B}, P, S_t)$ be a time changed flow of the flow $\mathcal{T}$ by τ_σ, $\sigma \in \mathcal{A}$. Since S_t is P-measure invariant, we get

$$h(T_1) = h(S_1) = h(T_\lambda) = \lambda h(T_1) .$$

This implies $\lambda = 1$. Hence it follows

$$\sigma T_t \sigma^{-1}\omega = S_t\omega = T_t\omega ,$$

namely, $\sigma \in \mathcal{C}$. Clearly $\mathcal{A} \supset \mathfrak{A} \cap \mathcal{C}$. This completes the proof.

The following is a restatement of the above result.

COROLLARY 4.1.1. *Suppose that the flow $\mathcal{T}$ is ergodic. If there exists an automorphism $\sigma \in \mathcal{A}$ which does not commute with some $T_{t_0} \in \{T_t\}$, then the entropy $h(\mathcal{T})$ of the flow $\mathcal{T}$ is zero or infinite.*

It seems to me very interesting to show the converse assertion of the Theorem 4.1.1, but it is still open.

As to the group $\mathcal{A}\cap\mathcal{G}_s$, we get

THEOREM 4.1.2. *Let $\mathcal{T}$ be an ergodic flow. Then*

$$\{T_t\}=\mathcal{A}\cap\mathcal{G}_s .$$

PROOF. Let $\sigma\in\mathcal{A}\cap\mathcal{G}_s$. Then

$$\tau_\sigma(t, \sigma\omega)=f_\sigma(T_t\omega)-f_\sigma(\omega)+t=\lambda t, \qquad \text{a. e. } \omega .$$

Since $\mathcal{T}$ is ergodic, f_σ must be a constant, say $f_\sigma(\omega)=c$ a. e. ω. This implies $\sigma=T_c$ a. e. ω.

There exist, in fact, ergodic flows such that $\mathcal{A}$ is not included in C.

EXAMPLE 1. Let A be an ergodic continuous group automorphism on the 2-dimensional torus $\boldsymbol{M}_2$. Then the eigenvalues of A are the irrational algebraic numbers of degree 2; we identify the group automorphism A with the unimodular integral matrix associated with it. We denote one of them by λ and let $(1, \gamma)$ be the eigenvector of A with respect to λ. Put $g_t=(t, \gamma t)$ (mod 1). Then the family $\{g_t\}$ is the 1-parameter subgroup of $\boldsymbol{M}_2$, and then the flow defined by

$$T_t g=g+g_t , \qquad g\in\boldsymbol{M}_2 ,$$

is ergodic and $AT_tA^{-1}=T_{\lambda t}$. Thus $A\in\mathcal{A}$ and A does not commute with the flow.

Notice that $\mathcal{A}$ contains all continuous group automorphisms which have the proper direction $(1, \gamma)$.

Another example of the flow with zero entropy of which $\mathcal{A}$ is not included in C is the horocycle flow on the compact manifold with the constant negative curvature. Moreover it is known the flow induced from a Brownian motion has the infinite entropy and $\mathcal{A}$ is not included in C.

EXAMPLE 2. Until now we have discussed a flow with continuous time parameter. The notions of orbit-preserving transformation groups and other concepts introduced previously are available also to a flow with discrete parameter, i. e., an automorphism.

In this example, concerning to Theorem 4.1.1, we wish to mention to Bernoulli shift. This is the example of the dynamical system with positive finite entropy for which $\mathcal{A}\neq\{T_t\}$.

Let $X=\{x_1, x_2, \cdots, x_n\}$ be a set with a measure

$$p_k>0 \quad \text{and} \quad \sum p_k=1 .$$

Let $\mathcal{T}=(\Omega, \mathfrak{B}, P, T)$ be a two sided Bernoulli shift defined in a usual manner, where

$$\Omega=\prod_{-\infty}^{\infty}\otimes X_i , \qquad X_i=X$$

and

$$(T\omega)_i=\omega_{i+1} , \qquad \omega=(\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) .$$

Let L be a permutation of X and define a transformation σ_L by

$$\sigma_L(\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) = (\cdots, L\omega_{-1}, L\omega_0, L\omega_1, \cdots).$$

For convenience, we call σ_L a permutation of Ω.

G. A. Hedlund, M. L. Curtis, and R. C. Lyndon have determined a class of the continuous transformations commuting with shift on a symbolic dynamical system [1]. By using their results, we can determine the group $\mathcal{A} \cap \mathcal{K}$ associated with the Bernoulli shift, where $\mathcal{K}$ is the group of all continuous transformations on Ω.

We define the equivalence relation $\sim$ by

$$x_i \sim x_j, \quad \text{if} \quad p_i = p_j.$$

Let $\{X_1, X_2, \cdots, X_k\}$ be the partition of X into the equivalent classes, and let $L(X, P)$ be the set of all permutations which preserve each X_j.

THEOREM 4.1.3. *The group $\mathcal{A} \cap \mathcal{K}$ is generated by T and σ_L; $L \in L(X, P)$. Moreover, if p_j's are all different, the group $\mathcal{A} \cap \mathcal{K}$ coincides with the group $\mathcal{T} = \{T^m\}$.*

4.2. Ergodicity and spectrum of the flow $\mathcal{T}$ and the groups $\mathfrak{A}_s$ and $\mathcal{A}$.

We shall give a condition of ergodicity of the flow $\mathcal{T}$ appealing to $\mathfrak{A}_s$.

THEOREM 4.2.1. *The dynamical system $\mathfrak{A}_s = (\Omega, \mathfrak{B}, P, \mathfrak{A}_s)$ is ergodic if and only if the flow $\mathcal{T} = (\Omega, \mathfrak{B}, P, T_t)$ is ergodic.*

PROOF. Since $\{T_t\} \subset \mathfrak{A}_s$, 'if part' is trivial to see. Suppose $\mathfrak{A}_s$ is ergodic. Let E be a $\mathfrak{B}$-set which is invariant under $\{T_t\}$. It is enough to show that E is invariant under any $\sigma \in \mathfrak{A}_s$. Let $\omega \in E$. Then $\sigma\omega = T_{f_{\mathcal{T}}(\sigma,\omega)}\omega \in E$, namely, $\sigma E \subset E$. Put $\eta = T_{-f_{\mathcal{T}}(\sigma^{-1}\xi)}\xi$. Then $\eta \in E$ and moreover $\xi = \sigma\sigma^{-1}\xi = \sigma T_{-f_{\mathcal{T}}(\sigma^{-1}\xi)}\xi = \sigma\eta$. This shows $\sigma E = E$.

From the above theorem, we get the following criterion for the flow to be ergodic.

COROLLARY. *If $\mathfrak{A}_s$ contains an ergodic element, then the flow is ergodic.*

Now we shall give a characterization of the spectrum of the flow $\mathcal{T}$ appealing to the structure of the group $\mathcal{A}$.

THEOREM 4.2.2. *Suppose that the flow $\mathcal{T}$ is ergodic. If $\mathcal{A}$ contains a 1-parameter subgroup $\{\sigma_s;\ s \in \boldsymbol{R}\}$ such that $\sigma_s \overline{\in} C$ for any $s \in \boldsymbol{R}$. Then the flow $\mathcal{T}$ is weakly mixing.*

PROOF. Let $\{U_t\}$ and $\{V_t\}$ be the unitary operators induced from $\{\sigma_t\}$ and $\{T_t\}$; $U_t F(\omega) = F(\sigma_t\omega)$ and $V_t F(\omega) = F(T_t\omega)$, $F \in L^2(\Omega, P)$. Suppose that the flow $\mathcal{T}$ has an eigenvalue $\mu \neq 0$ and eigenfunction F_μ; $V_t F_\mu = \exp(2\pi i \mu t) F_\mu$, $t \in \boldsymbol{R}$. Since $\sigma_s \overline{\in} C$ and $\sigma_s \in \mathcal{A}$, there exists a function $\lambda(s)$ such that $\sigma_s T_t \sigma_s^{-1} = T_{\lambda(s)t}$ by Lemma 4.1.1. Then we see that there exists a time s_0 such that

$\lambda(s_0) \neq 1$, $\lambda(s)\lambda(t) = \lambda(s+t)$ and $\tau_{\sigma_s}(t, \omega) = \lambda(s)t$. It follows

$$\begin{aligned} V_t U_s F_\mu(\omega) &= F_\mu(\sigma_s T_t \omega) \\ &= F_\mu(T_{\lambda(s)t}\sigma_s\omega) \\ &= U_s V_{\lambda(s)t} F_\mu(\omega) \\ &= \exp(2\pi i \lambda(s)\mu t) U_s F_\mu(\omega) . \end{aligned}$$

Thus $\{U_s F_\mu ; -\infty < s < \infty\}$ is a family of eigenfunctions corresponding to the eigenvalues $\lambda(s)\mu$. This contradicts to the separability of $L^2(\Omega, P)$.

Appealing to the above theorem, we can see the horocycle flow on the manifold with a constant negative curvature and the flow induced from the Brownian motion are weakly mixing [4].

Department of Mathematics
Tsuda College

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