J. Math. Soc. Japan
Vol. 23, No. 2, 1971

On D-dimensions of algebraic varieties*)

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(Received June 29, 1970)

§ 1. Introduction.

Let k be an algebraically closed field of characteristic zero. We shall work in the category of schemes over k. Let V be a complete algebraic variety, and let D be a divisor on V. In this paper, we shall introduce the notion of the D-dimension of V which we denote by $\kappa(D, V)$, and prove some theorems (Theorems 1, 2, 3 and 4) about $\kappa(D, V)$. Furthermore, when V is non-singular, we define the *Kodaira dimension* (or the canonical dimension) $\kappa(V)$ of V, to be $\kappa(K_V, V)$, where K_V denotes a canonical divisor of V. The Kodaira dimension would seem to be the most fundamental invariant in the theory of birational classification of algebraic varieties. Our theorems concerning $\kappa(D, V)$ and $\kappa(V)$ establish fundamental results in the theory of birational classification. In particular, Theorem 5 shows that it would be enough to consider algebraic varieties of Kodaira co-dimension zero[1], of Kodaira dimension zero and of Kodaira dimension $-\infty$, in order to classify algebraic varieties to the extent that Italian algebraic geometers did for algebraic surfaces about sixty years ago.

The main results of this paper have been announced in [**9**].

The author would like to express the deepest appreciation to Professor K. Kodaira for his thoughtfull and critical guidance given during the completion of this paper.

§ 2. Statement of the results.

Letting V be a complete algebraic variety of dimension n and D a divisor on V, we denote by $l(D)-1$ the dimension of the complete linear system $|D|$ associated with D. We consider the set of all positive integers m satisfying $l(mD)>0$, which we indicate by $N(D)$. Assume that $N(D)$ is not empty. Then $N(D)$ forms a sub-semigroup of the additive group of all integers. Hence,

*) This was presented as a doctoral thesis to the Faculty of Science, University of Tokyo.

1) The Kodaira co-dimension of an algebraic variety V of dimension n is defined to be $n-\kappa(V)$.

letting $m_0(D)$ be the g. c. d. of the integers belonging to $\boldsymbol{N}(D)$, we can find a positive integer $N(D)$ such that m belongs to $\boldsymbol{N}(D)$ provided that $m\equiv 0$ mod $m_0(D)$ and $m\geqq N(D)$.

THEOREM 1. *There exist positive numbers α, β and a non-negative integer κ such that the following inequality holds for every sufficiently large integer m:*

$$\alpha m^{\kappa}\leqq l(mm_0(D)D)\leqq \beta m^{\kappa}.$$

It is easy to check that κ is independent of the choice of α and β. *We define the D-dimension of V to be the integer κ*, provided that $l(mD)>0$ for at least one positive integer m. We denote the D-dimension of V by $\kappa(D, V)$. In the case in which $l(mD)=0$ for every positive integer m, *we define the D-dimension of V to be* $-\infty$: $\kappa(D, V)=-\infty$.

THEOREM 2. *Assume that $\kappa(D, V)>0$. For any positive integer p, there exists a positive number γ such that the following inequality holds for every sufficiently large integer m:*

$$l(mm_0(D)D)-l(\{mm_0(D)-pm_0(D)\}D)\leqq \gamma m^{\kappa-1},$$
$$\kappa=\kappa(D, V).$$

We recall that, in classical algebraic geometry, *the index of an algebraic system on an algebraic variety of dimension n is defined to be the number of those distinct members of the system which pass through r independent generic points of V*, where $r=$ the dimension of the system$+$the dimension of its member$-n+1$.

THEOREM 3. *Suppose that $\kappa=\kappa(D, V)$ is positive. Then there exists a κ dimensional irreducible algebraic system of algebraic sub-varieties of dimension $n-\kappa$ with index 1, such that $\kappa(D_w, V_w)=0$, where V_w denotes a general member of the algebraic system and D_w the induced divisor on V_w of D. Moreover, such an algebraic system is unique up to birational equivalence.*

We introduce the notion of the *co-D-dimension* of V, which we write $c\kappa(D, V)$, *by setting* $c\kappa(D, V)=n-\kappa(D, V)$.

THEOREM 4. *Let $\tilde{V}$, V be complete algebraic varieties and let f be a proper surjective morphism from $\tilde{V}$ to V. For any divisor D on V, we have $\kappa(f^*D, \tilde{V})=\kappa(D, V)$. Moreover, if a general fiber $\tilde{V}_v=f^{-1}(v)$ is irreducible, then for any divisor $\tilde{D}$ on $\tilde{V}$, we have $c\kappa(\tilde{D}, \tilde{V})\geqq c\kappa(\tilde{D}_v, \tilde{V}_v)$.*

In order to define the Kodaira dimension of an arbitrary algebraic variety V, we take a non-singular projective model V^* of V, whose existence is assured by a celebrated theorem of Hironaka (see [5]). Then *we define the Kodaira dimension $\kappa(V)$ of V to be $\kappa(K^*, V^*)$*, where K^* denotes a canonical divisor of V^*. $\kappa(V)$ is well defined and is a birational invariant.

THEOREM 5. *If $\kappa=\kappa(V)$ is positive, then there exists a fiber space f: $V^*\rightarrow W$ of non-singular projective algebraic varieties such that*

i) *V^* is birationally equivalent to V,*
ii) *W is of dimension κ,*
iii) *f is surjective and proper,*
iv) *any general fiber $V^*_w = f^{-1}(w)$ is irreducible,*
v) *V^*_w has the Kodaira dimension 0.*

Moreover, such a fiber space is unique up to birational equivalence.

The former part of this theorem is a direct generalization of a theorem[2] which states that a minimal surface S with $K_S^2=0$ and a plurigenus $\geqq 2$ is elliptic. Moreover, the latter part is a generalization of Proposition 7 in [8, II].

THEOREM 6. *Let $\tilde{V}$, V be non-singular projective algebraic varieties and f a proper surjective morphism from $\tilde{V}$ to V. In the case in which $\tilde{V}$ is étale over V, we have $\kappa(\tilde{V})=\kappa(V)$. On the other hand, in the case in which any general fiber $f^{-1}(v)=\tilde{V}_v$ is irreducible, we have $c\kappa(\tilde{V})\geqq c\kappa(\tilde{V}_v)$.*

The former assertion is a generalization of a theorem in the theory of algebraic surfaces to the effect that every unramified covering manifold of an elliptic surface is also elliptic. The latter is a generalization of a theorem[3] saying that every algebraic surface of general type cannot contain a pencil of elliptic curves.

We note that *the above theorems have counterparts in the category of complex spaces*[4].

§ 3. Notation and preliminary propositions.

In this section, we let V denote a normal complete algebraic variety of dimension n, and let D be a Cartier divisor on V. We shall use the notation listed below:

$k(V)$ = the field of rational functions on V,
$[D]$ = the line bundle associated with D,
$\boldsymbol{L}(D)$ = the vector space consisting of all regular sections of $[D]$,
$l(D)$ = the dimension of $\boldsymbol{L}(D)$,
$\boldsymbol{L}^*(D)$ = the vector space consisting of all rational sections of $[D]$,
(ω) = the divisor corresponding to a non-zero element $\omega \in \boldsymbol{L}^*(D)$ (Note that, if $\eta \in \boldsymbol{L}(D)$, $\neq 0$, then (η) is positive),
$|D| = \{(\omega)\,;\ \omega \in \boldsymbol{L}(D),\ \neq 0\}$; $|D|$ is called the complete linear system associated with D.

" $\sim$ " indicates the linear equivalence of divisors.

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2) Lemma 7 in [**11**].
3) Lemma 5 in Chapter 6 in [**10**].
4) The existence of a non-singular model of any compact complex variety was recently proved by Hironaka (see [**7**]).

In what follows in this section we fix a divisor D such that $l(D)=N+1>0$. Let $\{\varphi_0, \varphi_1, \cdots, \varphi_N\}$ be a basis of $\boldsymbol{L}(D)$. We define a rational map Φ_D by

$$V \ni z \mapsto \Phi_D(z)=\varphi_0(z):\varphi_1(z):\cdots:\varphi_N(z) \in \boldsymbol{P}^N ,$$

where z is a general point of V. We denote by W_D the rational transform of V by Φ_D which is a closed sub-variety of $\boldsymbol{P}^N$. Moreover, for every integer $m>0$, we abbreviate Φ_{mD}, W_{mD}, and $\boldsymbol{L}(mD)$ to Φ_m, W_m, and $\boldsymbol{L}_m$, respectively. We let $\{\psi_0, \psi_1, \cdots, \psi_l\}$ be a basis of $\boldsymbol{L}_m$ and we choose a basis of $\boldsymbol{L}_{m+1}$ of the form $\{\varphi_0\psi_0, \varphi_0\psi_1, \cdots, \varphi_0\psi_l, \cdots\}$. Then, for a general point $z \in V$, we define a generically surjective rational map $\rho_m: W_{m+1} \to W_m$ by

$$\rho_m(\varphi_0(z)\psi_0(z):\cdots:\varphi_0(z)\psi_l(z):\cdots)=\psi_0(z):\cdots:\psi_l(z) .$$

Obviously, we have $\Phi_m=\rho_m \cdot \Phi_{m+1}$. Therefore, we have a sequence of fields:

$$k(W_1) \subset k(W_2) \subset \cdots \subset k(W_m) \subset \cdots \subset k(V) .$$

Since $k(V)$ is finitely generated over $k(W_1)$, there is an integer m_1 such that $k(W_m)=k(W_{m_1})$ for all $m \geqq m_1$. Hence ρ_m is birational for $m \geqq m_1$. From the following proposition we infer that $k(W_{m_1})$ is algebraically closed in $k(V)$.

PROPOSITION 1. *Let z be an element of $k(V)$ which is algebraic over $k(W_D)$. Then there exists an integer $\delta \geqq 1$ such that z belongs to $k(W_{\delta D})$.*

PROOF. Let $\{\varphi_0, \varphi_1, \cdots, \varphi_N\}$ be a basis of $\boldsymbol{L}(D)$, and let z satisfy the following equation:

$$z^r+a_1z^{r-1}+\cdots+a_r=0 , \tag{1}$$

where $a_1, \cdots, a_r \in k(W_D)$. Since $k(W_D)=k(\varphi_1/\varphi_0, \cdots, \varphi_N/\varphi_0)$, we have homogeneous polynomials $F_0, F_1, \cdots, F_r$ of the same degree δ such that $a_i=F_i(\varphi_0, \varphi_1, \cdots, \varphi_N)/F_0(\varphi_0, \varphi_1, \cdots, \varphi_N)$ for $1 \leqq i \leqq r$. The equation (1) leads to the following equation:

$$(zF_0(\varphi))^r+F_1(\varphi)(zF_0(\varphi))^{r-1}+ \cdots +F_0(\varphi)^{r-1}F_r(\varphi)=0 , \tag{2}$$

where we abbreviate $F_j(\varphi_0, \varphi_1, \cdots, \varphi_N)$ to $F_j(\varphi)$ for $0 \leqq j \leqq r$. Note that $zF_0(\varphi)$, $F_1(\varphi), \cdots, F_0(\varphi)^{r-1}F_r(\varphi)$ are elements of $\boldsymbol{L}^*(\delta D), \boldsymbol{L}(\delta D), \cdots, \boldsymbol{L}(r\delta D)$, respectively. Now we take a covering of V by affine open sets $\{U_\lambda\}_{\lambda \in \Lambda}$ such that $[D]$ is trivial on U_λ for every $\lambda \in \Lambda$. We indicate the restriction of any entity $\sharp$ to U_λ by the symbol $\sharp_\lambda$. It is clear that $F_1(\varphi)_\lambda, \cdots, (F_0(\varphi)^{r-1}F_r(\varphi))_\lambda \in H^0(U_\lambda, \mathcal{O}_V)$. Since the ring $H^0(U_\lambda, \mathcal{O}_V)$ is integrally closed, we infer from the equation (1) that $zF_0(\varphi)_\lambda \in H^0(U_\lambda, \mathcal{O}_V)$. Therefore, we have $zF_0(\varphi) \in \boldsymbol{L}(\delta D)$. This implies that $z \in k(W_{\delta D})$.

PROPOSITION 2. *Let D be a divisor on V. Then there exists a number β such that $l(mD) \leqq \beta m^n$ for all $m \gg 0$. Furthermore, when D is ample, there exists a positive number α such that $\alpha m^n \leqq l(mD)$ for all $m \gg 0$.*

PROOF. When D is ample, $l(mD)$ is a polynomial of degree n for all $m \gg 0$.

Hence we have an estimate:

$$\alpha m^n \leqq l(mD) \leqq \beta m^n \qquad \text{for all} \quad m \gg 0\,,$$

where α, β are positive numbers depending only on D. In the case in which D may not be ample, we take a projective model V^* of V such that a birational map $T: V^* \to V$ is regular. Note that $l(mT^*D) = l(mD)$. Hence we may assume that V is projective. We take an ample divisor D^* such that $D^* \sim D+H$, where H is a suitably chosen ample divisor. Then we have

$$l(mD) \leqq l(mD+mH) = l(mD^*) \leqq \beta^* m^n \qquad \text{for a constant } \beta^*$$

and for all $m \gg 0$.

PROPOSITION 3. *Let f: $V \to W$ be a fiber space of complete normal algebraic varieties such that $f^*(k(W))$ is algebraically closed in $k(V)$. Then, for any divisor D on W, $\boldsymbol{L}(D)$ is isomorphic to $\boldsymbol{L}(f^*(D))$ by the map induced by f.*

PROOF. Let ϕ be a rational function on V such that $(\phi) \geqq -f^*(D)$. Since $\phi | V_w$ has no pole on V_w, V_w being the generic fiber over the generic point w of W, $\phi | V_w = \phi$ belongs to $k(w) = k(W)$. From this observation, Proposition 3 follows at once.

§ 4. Proofs of Theorems 1 and 3.

First, we note that it is sufficient to prove these theorems for a normal algebraic variety V and for an effective divisor D. In fact, taking the normalization V^* of V, we define $l(R^*D, V^*) = l(D, V)$ where $R: V^* \to V$ is a birational morphism. We fix an integer $\bar{m}_0$ satisfying $\bar{m}_0 m_0(D) \in \boldsymbol{N}(D)$ and an effective divisor D' which is linearly equivalent to $\bar{m}_0 m_0(D) D$. We wish to prove the inequalities in Theorem 1 under the assumption that the following inequality holds for all $\mu \gg 0$:

$$\alpha \mu^\kappa \leqq l(\mu D') \leqq \beta \mu^\kappa\,.$$

For this purpose, let m be any given large integer. We divide m by $\bar{m}_0$ with a sufficiently large residue, i.e., we let $m = \mu \cdot \bar{m}_0 + q$, where $q \cdot m_0(D) \in \boldsymbol{N}(D)$ and q is bounded when m grows to infinity. Then we have

$$l(mm_0(D)D) = l(\{\bar{m}_0 \mu m_0(D) + q m_0(D)\} D) \geqq l(\mu \bar{m}_0 m_0(D) D) = l(\mu D') \geqq \alpha \mu^\kappa\,.$$

Moreover, we divide m by $\bar{m}_0$ with a sufficiently small residue, i.e., we let $m = \mu \bar{m}_0 - q'$, where $q' \cdot m_0(D) \in \boldsymbol{N}(D)$ and q' is bounded. Then we have

$$l(mm_0(D)D) = l(\mu \bar{m}_0 m_0(D) D - q' m_0(D) D) \leqq l(\mu \bar{m}_0 m_0(D) D) = l(\mu D') \leqq \beta \mu^\kappa\,.$$

Thus, we may assume that V is normal and D effective. By the consideration in § 3, we have a fiber space of algebraic varieties Φ_{m_1}: $V \to W_{m_1} \subset \boldsymbol{P}^N$ which has the following properties:

1) Φ_{m_1} is a generically surjective map,
2) $k(W_m)=k(W_{m_1})$ for all integer $m \geqq m_1$,
3) $k(W_{m_1})$ is algebraically closed in $k(V)$,

where Φ_{m_1} denotes Φ_{m_1D}, etc. By taking the normal graph, we have a birational morphism $T: V^* \to V$ such that the rational map $\Phi_{m_1} \circ T$ is regular. In view of the isomorphism:

$$T^*: \boldsymbol{L}(m_1D) \simeq \boldsymbol{L}(m_1T^*D),$$

we can replace V, D by V^*, T^*D, respectively. Hence we may assume that Φ_{m_1} is a morphism. For simplicity, we abbreviate m_1D, Φ_{m_1}, W_{m_1} and $l(m_1D)-1$ to E, f, W and N, respectively. Note: we can assume that W is normal.

We fix a basis $\{\varphi_0, \cdots, \varphi_N\}$ of $\boldsymbol{L}(E)$ such that f is defined by means of this basis. Let F be the maximal fixed component of $|E|$, and let H denote a hyperplane section of W in $\boldsymbol{P}^N$. Then we have a member of $|E|$ of the form: $F+f^*(H)$, where $f^*(H)$ indicates the divisor induced from H by f. Hence, by Proposition 3, we have

$$l(mm_1D)=l(mE)=l(mF+f^*(mH)) \geqq l(mf^*(H))=l(mH).$$

From Proposition 2, we infer the existence of a positive number α such that $l(mH) \geqq \alpha m^{\kappa}$ for all $m \gg 0$, where κ denotes the dimension of W. Thus we have

$$l(\mu m_1D) \geqq \alpha\mu^{\kappa} \qquad \text{for all} \quad \mu \gg 0. \tag{3}$$

We represent the divisor F as a sum: $F=\sum n_\nu A_\nu$, where the A_ν denote the irreducible components of F, and define

$$L=\sum_{f(A_\nu)=W} n_\nu A_\nu, \qquad F^*=\sum_{f(A_\nu)\neq W} n_\nu A_\nu.$$

Then, for any integer $m>0$, we have

$$|mE| \ni mL+mF^*+f^*(mH).$$

Furthermore, we take a general member $\sum n_\nu B_\nu$ of $|mE|$, where the B_ν denote its irreducible components, and let

$$L_m=\sum_{f(B_\nu)=W} n_\nu B_\nu, \qquad F_m^*=\sum_{f(B_\nu)\neq W} n_\nu B_\nu.$$

Hence we have

$$L_m+F_m^* \sim mL+mF^*+f^*(mH). \tag{4}$$

Restricting both divisors to a general fiber V_w of f, we have

$$L_m | V_w=(L_m+F_m^*) | V_w \sim (mL+mF^*+f^*(mH)) | V_w=mL | V_w.$$

Moreover, we shall prove

$$L_m | V_w=mL | V_w. \tag{5}$$

Assuming the equality (5), we proceed with the proof of Theorem 1. From the equality (5), we infer that $L_m = mL$. This implies that L_m is one of the fixed components of $|mE|$. Hence, we have

$$l(mm_1D) = l(L_m + F_m^*) = l(F_m^*) = l(mF^* + f^*(mH)). \tag{6}$$

On the other hand, we can take a positive divisor H^* on W such that $F^* \leqq f^*(H^*)$. Therefore, by Proposition 3, we have

$$l(mF^* + f^*(mH)) \leqq l(mf^*(H^*) + f^*(mH)) = l(m(H^* + H)). \tag{7}$$

By Proposition 2 we can choose a number β which satisfies

$$l(m(H^* + H)) \leqq \beta m^\kappa \quad \text{for all} \quad m \gg 0.$$

Combining this with (6) and (7), we have

$$l(\mu m_1 D) \leqq \beta \mu^\kappa \quad \text{for all} \quad \mu \gg 0. \tag{8}$$

By a similar inference as before, we derive from (4) and (8) the inequality in Theorem 1.

PROOF OF THE EQUALITY $L_m | V_w = mL | V_w$. We denote by $\mathcal{L}$ the sheaf of germs of regular sections of the bundle $[mm_1D]$. Then we have the homomorphism: $\sigma = \sigma_{\mathcal{L}} : f^* f_*(\mathcal{L}) \to \mathcal{L}$ (see [2, 0_{I}. 4.4.3.3]). Let C, Σ, V_1 and f_1 be, respectively, the cokernel of σ, the support of C, $V - \Sigma$ and $f | V_1$. Then the restriction of σ to V_1: $f_1^* f_*(\mathcal{L}) \to \mathcal{L} | V_1$ is surjective. Hence by a theory of Grothendieck (see [2, II. 4.2.3]) we have a fiber space $g: \boldsymbol{P}(f_*(\mathcal{L})) \to W$ and a morphism $h_1: V_1 \to \boldsymbol{P}(f_*(\mathcal{L}))$ over W such that $\mathcal{L}_1 = \mathcal{L} | V_1$ is isomorphic to $h_1^* \mathcal{O}_{\boldsymbol{P}}(1)$. In the above we abbreviate $\boldsymbol{P}(f_*(\mathcal{L}))$ to $\boldsymbol{P}$. Let Z be an algebraic variety of which the underlying space is the closure of $h_1(V_1)$ in $\boldsymbol{P}$. A hyperplane defined by $\lambda_0 X_0 + \cdots + \lambda_N X_N = 0$ in $\boldsymbol{P}^N$ cuts off on W a positive divisor H_λ. Let W_λ denote an affine open set $W - H_\lambda$. Then $h | f^{-1}(W_\lambda)$ is described as follows. Recalling that the sheaf $f_*(\mathcal{L})$ is coherent, we can take $\psi_0, \psi_1, \cdots, \psi_N \in H^0(W_\lambda, f_*(\mathcal{L}))$ such that $H^0(W_\lambda, f_*(\mathcal{L}))$ is generated by $\psi_0, \psi_1, \cdots, \psi_N$ as an $H^0(W_\lambda, \mathcal{O}_W)$-module (see [2, I. 1.5.5]). Regarded as a rational map, h_1 coincides with the rational map defined by

$$V \supset f^{-1}(W_\lambda) \ni z \mapsto \psi_0(z) : \cdots : \psi_N(z) \in \boldsymbol{P}^N_{H^0(W_\lambda, \mathcal{O}_W)}$$

for a general point z of V. On the other hand, we have

$$f^{-1}(W_\lambda) = V - E_\lambda^* \supset V - E_\lambda$$

where E_λ and E_λ^* denote $(\lambda_0 \psi_0 + \cdots + \lambda_N \psi_N) \in |E|$ and $E_\lambda - F$, respectively. Moreover, we have

$$H^0(W_\lambda, f_*(\mathcal{L})) = H^0(f^{-1}(W_\lambda), \mathcal{L}) \subset H^0(V - E_\lambda, \mathcal{L})$$

$$= \bigcup_{e=1}^{\infty} H^0(V, \mathcal{L}(emE_\lambda)),$$

where we denote by $H^0(V, \mathcal{L}(emE_\lambda))$ the space of rational sections ω of $\mathcal{L}$ on V such that the corresponding divisor $(\omega) \geqq -emE_\lambda$.

Fix an element $\eta \in \boldsymbol{L}(emE_\lambda)$ such that $(\eta) = emE_\lambda$. Then $H^0(V, \mathcal{L}(emE_\lambda))$ is isomorphic to $H^0(V, \mathcal{L}^{\otimes(e+1)})$ by the map $\phi \mapsto \phi\eta$. Now we can find an integer ε such that $\phi_0, \phi_1, \cdots, \phi_N \in H^0(V, \mathcal{L}(\varepsilon mE_\lambda))$. Therefore, considering the function fields of $h_1(V_1)$, W and $W_{(\varepsilon+1)m}$, we have the relations of inclusions:

$$k(V) = k(V_1) = k(W_{(\varepsilon+1)m}) \supset k(h_1(V_1)) \supset k(W)\,.$$

In view of the equalities $k(W_{(\varepsilon+1)m}) = k(W)$ and $k(h_1(V_1)) = k(Z)$, we conclude that the morphism $g: Z \rightarrow W$ is birational. Applying the theorem of upper semi-continuity to the function $l(mm_1D_w)$ of $w \in W$, we infer that $l(mm_1D_w) = \dim H^0(V_w, \mathcal{L}_w)$ is constant on a certain dense open subset W^* of W. Hence, we have

$$f_*(\mathcal{L}) \underset{\mathcal{O}_W}{\otimes} k(w) \simeq H^0(V_w, \mathcal{L}_w) \qquad \text{for} \quad w \in W^*\,,$$

where $k(w)$ denotes $\mathcal{O}_{W,w}/\mathfrak{m}\mathcal{O}_{W,w} \simeq k$.

Finally we wish to show that Φ_θ is the morphism $h \underset{W}{\times} \operatorname{Spec} k(w)$ from V_w to $Z_w \subset \boldsymbol{P}(f_*(\mathcal{L})) \underset{W}{\times} \operatorname{Spec} k(w)$, where θ denotes $mE|V_w$. For this it is sufficient to note that

$$\boldsymbol{P}(f_*(\mathcal{L})) \underset{W}{\times} \operatorname{Spec} k(w) = \boldsymbol{P}(f_*(\mathcal{L}) \underset{\mathcal{O}_W}{\otimes} k(w)) \simeq \boldsymbol{P}(H^0(V_w, \mathcal{L}_w)) \qquad \text{for} \quad w \in W^*\,,$$

and that $h_w^*(\mathcal{O}_{\boldsymbol{P}}(1))$ is isomorphic to $\mathcal{L}_w$, where we write h_w instead of $h \underset{W}{\times} \operatorname{Spec} k(w)$. Recalling that Z is birationally equivalent to W, we conclude that h_w is a constant morphism. Therefore we have

$$\dim H^0(V_w, \mathcal{L}_w) = l(mm_1D_w) = 1$$

and also $\dim |mL|V_w| = 0$. This establishes the equality (5).

Furthermore, we see that, for any integer $i > 0$,

$$l(iD_w) \leqq l(im_1D_w) = 1\,.$$

From this we infer the existence of the algebraic system in Theorem 3.

Now we shall prove the uniqueness of the algebraic system in Theorem 3 in the following form: *Let $f^1: V^1 \rightarrow W^1$ be a fiber space of complete algebraic varieties which has the following properties:*

1) *V^1 is birationally equivalent to V,*
2) *W^1 has dimension $\kappa = \kappa(D, V)$,*
3) *f^1 is proper and surjective,*
4) *any general fiber $f^{1-1}(w) = V_w^1$ is irreducible,*
5) *the D_w^1-dimension of V_w^1 is zero,*

where D^1 is a divisor corresponding to D by the birational map from V^1 to V.

Then this fiber space is birationally equivalent to the fiber space $f: V \to W$ *constructed in* § 3, i. e., *there exist two birational maps* $\tau: V^{!} \to V$ *and* $\rho: W^{!} \to W$ *such that* $f \cdot \tau = \rho \cdot f^{!}$.

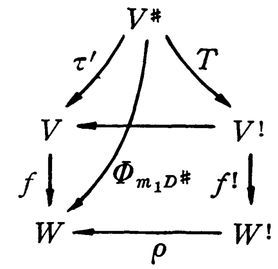
By the consideration in § 3, we have a generically surjective rational map $\Phi_{m_1 D^{!}}$ from $V^{!}$ to $W_{m_1 D}$ such that

$$k(W_{m_1 D^{!}}) = k(W_{(m_1+1)D^{!}}) = \cdots \subset k(V^{!}) = k(V) \quad \text{for an integer } m > 0\,.$$

Note that $W_{m_1 D^{!}}$ is birationally equivalent to W. We take a monoidal transformation $T: V^{\#} \to V^{!}$ such that $\Phi_{m_1 D^{!}} \cdot T$ is everywhere defined. Moreover, we have the isomorphism $\boldsymbol{L}(mD^{!}) \simeq \boldsymbol{L}(mD^{\#})$ and so $\Phi_{m_1 D^{!}} \cdot T = \Phi_{m_1 D^{\#}}$, where by $D^{\#}$ we denote $T^{*}D^{!}$. By the property 5), say $l(m_1 D^{!} \mid V^{!}_{w}) = l(m_1 D^{!}_{w}) = l(m_1 D^{\#}_{w}) = 1$, we have a generically surjective rational map ρ from $W^{!}$ to W such that $\rho \cdot f^{!} \cdot T = \Phi_{m_1 D^{\#}}$. Hence, we have

$$k(V) = k(V^{!}) = k(V^{\#}) \supset k(W^{!}) \supset k(W)\,.$$

The equality $\dim W^{!} = \kappa = \dim W$ implies that $k(W^{!})$ is algebraic over $k(W)$. Therefore, the equality $k(W^{!}) = k(W)$ follows from the property 4), i. e., ρ is birational. Recalling that f is defined to be Φ_{mD}, we have a birational map τ' such that $\Phi_{mD^{\#}} = f \cdot \tau'$. Let τ be $\tau' \cdot T^{-1}$. Then $f \cdot \tau = \rho \cdot f^{!}$ (see the diagram (9)). This completes the proof of the uniqueness.



(9)

§ 5. Proof of Theorem 2.

We use the same notation as in the proof of Theorem 1. A similar argument as at the beginning of the proof of Theorem 1 shows that we can replace D by an effective divisor D. Now we make the following observation: For $m \geqq 1$, the maximal fixed component of the complete linear system $|mD|$ can be described as a sum of divisors L_m, Ξ_m, Θ_m and $f^{*}(\Gamma_m)$. These are defined as follows: Letting $\sum n_\nu C_\nu$ be a general member of $|mD|$, where $n_\nu > 0$, the C_ν are irreducible curves and $C_\mu \neq C_\nu$ for $\mu \neq \nu$, we set

$L_m = \sum_{f(C_\nu)=W} n_\nu C_\nu\,,$

$H_m =$ the largest of all positive divisors H on W such that $\sum n_\nu C_\nu \geqq f^{*}(H)$,

$\Xi_m =$ the largest of all positive divisors of the form $\sum a_\nu \bar{f}(A_\nu)$ satisfying

$\sum n_\nu C_\nu - f^*(H_m) \geqq \sum a_\nu \bar{f}(A_\nu)$, where the A_ν denote prime divisors on W, $e(A_\nu)$ is the g.c.d. of all the multiplicities of the irreducible components of $f^*(A_\nu)$ and $\bar{f}(A_\nu) = f^*(A_\nu)/e(A_\nu)$,

$\Theta_m = \sum n_\nu C_\nu - L_m - f^*(H_m) - \Xi_m$,

$\Gamma_m =$ the maximal fixed component of $|H_m|$.

In fact, by the results in the proof of Theorem 1, we have $m_1 L_m = m L_{m_1}$. Hence it follows that L_m is one of the fixed components of $|mD|$. From this we infer that any element of $H^0(V, \mathcal{O}_V(mD))$ is derived from the rational function on W.

With this observation in mind, we proceed with the proof. First, we note that we can replace p by pm_2 for any integer $m_2 > 0$ because

$$l(mD) - l(mD - pD) \leqq l(mD) - l(mD - pm_2 D).$$

Moreover, we can replace m by mm_2. To see this, we let $m = \mu m_2 - q$, where $0 \leqq q < m_2$. Then we have

$$l(mD) - l(mD - pD) = l(\mu m_2 D) - l(\mu m_2 D - (q+p)D).$$

From this inequality our assertion follows.

We fix m_2 to be l.c.m. of all $e(f(C_\nu))$ such that $\Xi_1 \geqq \bar{f}(f(C_\nu)) > 0$. Then we infer immediately that $\Xi_{\bar{m}}$ and $\Xi_{\bar{p}}$ vanish, where we abbreviate mm_2 and pm_2 to $\bar{m}$ and $\bar{p}$, respectively. Now, for $m > p$, we have

$$l(\bar{m}D - \bar{p}D) = l((\bar{m} - \bar{p})D) = l(f^*(H_{\bar{m}-\bar{p}})) = l(H_{\bar{m}} - H_{\bar{p}}), \tag{10}$$

and also

$$l(mD) = l(f^*(H_{\bar{m}})) = l(H_{\bar{m}}). \tag{11}$$

Adding a suitable positive divisor J to $H_{\bar{p}}$ such that $J + H_{\bar{p}}$ is ample, we fix a prime divisor $\bar{H}$ which is linearly equivalent to $J + H_{\bar{p}}$. Then we have

$$l(H_{\bar{m}} - H_{\bar{p}}) \leqq l(H_{\bar{m}} - \bar{H}). \tag{12}$$

Using a sequence of cohomology groups, we have

$$l(H_{\bar{m}}) - l(H_m - \bar{H}) \leqq l(H_{\bar{m}} | \bar{H}), \tag{13}$$

where we denote by $H_{\bar{m}} | \bar{H}$ the induced divisor on the variety $\bar{H}$. Since $L_{\bar{m}} = \bar{m} L_1$ and $\Theta_{\bar{m}} = \bar{m}\Theta_1$, we have $H_{\bar{m}} \sim m H_{\bar{1}}$. Hence, we have

$$l(H_{\bar{m}} | \bar{H}) = l(m H_{\bar{1}} \mid \bar{H}). \tag{14}$$

By Proposition 2, the right hand side is smaller than $\gamma m^{\kappa-1}$ for a constant γ. Combining this with (10), (11), (12), (13) and (14) we obtain the inequality in Theorem 2.

§ 6. Proof of Theorem 4.

First, we shall give a proof of the first assertion of Theorem 4. By Proposition 3, we can assume that $k(\tilde{V})/k(V)$ is finite. Taking the Galois closure of $k(\tilde{V})/k(V)$ and constructing a projective model of it, we see that it is sufficient to prove the assertion in the case in which $k(\tilde{V})/k(V)$ is a Galois extension. Let G denote its Galois group. Replacing $\bar{m}_0 m_0(D)$ by D in case $\kappa(D, V) \geqq 0$, we assume that D is effective. By the natural injection: $\boldsymbol{L}(mD) \rightarrow \boldsymbol{L}(mf^*(D))$, we have the generically surjective map $f_m: W_{mf^*(D)} \rightarrow W_{mD}$ such that $\Phi_{mf^*(D)} \cdot f = f_m \cdot \Phi_{mD}$. We wish to prove $k(W_{mf^*(D)})/k(W_{mD})$ is finite algebraic for $m \gg 0$. For this it is sufficient to prove that any element a of $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(mf^*(D)))$ is algebraic over $k(W_{mD})$ for $m \gg 0$, because $k(W_{mf^*(D)})$ is the fractional field of the ring generated by $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(mf^*(D)))$ in $k(\tilde{V})$. We have r fundamental symmetric functions $S_1(a), \cdots, S_r(a)$ of $\sigma_1(a), \cdots, \sigma_r(a)$, where r is the order of G and $\sigma_1, \cdots, \sigma_r$ are the elements of G. Clearly $S_j(a)$ belongs to $H^0(\tilde{V}, \mathcal{O}_{\tilde{V}}(rmf^*(D)))$ for every $1 \leqq j \leqq r$. Hence, $S_j(a)$ can be described as $f^*(b_j)$, where $b_j \in H^0(V, \mathcal{O}_V(rmD))$. From this we can derive an algebraic equation:

$$a^r + b_1 a^{r-1} + \cdots + b_r = 0 .$$

This proves that a is algebraic over $k(W_{rmD})$. Moreover, it is easy to check that $\kappa(D, V) = -\infty$ if and only if

$$\kappa(f^*D, \tilde{V}) = -\infty . \qquad \text{Q. E. D.}$$

To prove the latter assertion of Theorem 4, we let $\mathcal{L}$ be the invertible sheaf associated with the divisor $m_0(\tilde{D})\tilde{D}$ under the assumption $\boldsymbol{N}(\tilde{D}) \neq \phi$. We consider the rational map $h = \sigma_{\mathcal{L}^{\otimes m_2}}: V \rightarrow \boldsymbol{P}(f_*(\mathcal{L}^{\otimes m_2}))$ for an integer $m_2 \gg 0$ over V and denote by Z the image of V by h which is the closed subvariety of $\boldsymbol{P}(f_*(\mathcal{L}^{\otimes m_2}))$. Then we have $\dim Z_v = \kappa(\tilde{D}_v, \tilde{V}_v)$ for a general point v of V, because $h_v = h \mid \tilde{V}_v = \Phi_{m_2 m_0(\tilde{D})\tilde{D}_v}$. Moreover, by Theorem 3 we conclude that $\tilde{V}_z = h^{-1}(z)$ is irreducible for a general point z of Z and that $\kappa(\tilde{D}_z, \tilde{V}_z) = 0$, where $\tilde{D}_z$ denotes the restriction of $\tilde{D}$ to $\tilde{V}_z$. On the other hand, we let $g: V \rightarrow W$ denote the fiber space $\Phi_{m_1 m_0(\tilde{D})\tilde{D}}: V \rightarrow W_{m_1 m_0(\tilde{D})\tilde{D}}$ constructed in § 3. Owing to the vanishing of $\kappa(\tilde{D}_z, \tilde{V}_z)$, we obtain a generically surjective rational map $t: Z \rightarrow W$ such that $t \cdot h = g$. Hence, we see that $\dim Z \geqq \dim W = \kappa(\tilde{D}, \tilde{V})$. Recalling that $\dim Z = \dim Z_v + \dim V = \kappa(\tilde{D}_v, \tilde{V}_v) + \dim V$, we conclude that $\kappa(\tilde{D}, \tilde{V}) \leqq \kappa(\tilde{D}_v, \tilde{V}_v) + \dim V$. This implies $c\kappa(\tilde{D}, \tilde{V}) \geqq c\kappa(\tilde{D}_v, \tilde{V}_v)$. In case $\boldsymbol{N}(\tilde{D}) = \phi$, we have by definition, $\kappa(\tilde{D}, \tilde{V}) = -\infty$ and so $c\kappa(\tilde{D}, \tilde{V}) = +\infty \geqq c\kappa(\tilde{D}_v, \tilde{V}_v)$.

REMARK 1. The proof above suggests a generalization of Theorem 3 in the following form: *Let $f: \tilde{V} \rightarrow V$ be a fiber space of algebraic varieties such that f is a proper and surjective morphism and let $\tilde{D}$ be a divisor on $\tilde{V}$. Suppose* *that* $\kappa(\tilde{D}_v, \tilde{V}_v) \geqq 0$ *for a general point* v *of* V. *Then there exists a fiber space* $h: \tilde{V}^* \to W$ *over* V *satisfying*

1) $\tilde{V}^*$ *is birationally equivalent to* $\tilde{V}$,
2) *the structure map from* W *to* V *is surjective, proper and* $\kappa(\tilde{D}_v, \tilde{V}_v) = \dim W/V$ (*where* $\dim W/V$ *denotes* $\dim W - \dim V$),
3) h *is surjective and proper,*
4) *any general fiber* $V_z^* = h^{-1}(z)$ *is irreducible,*
5) $\kappa(\tilde{D}_z^*, \tilde{V}_z^*) = 0$ (*where* $\tilde{D}^*$ *is the complete inverse image of the divisor* $\tilde{D}$ *by the birational map from* $\tilde{V}^*$ *to* $\tilde{V}$).

Furthermore, such a fiber space is unique up to birational equivalence over V.

REMARK 2. By using Theorem 3 we can prove the following result concerning $m_0(D)$: *D can be uniquely described as a sum of divisors D_0 and D^* such that*

1) $m_0(D) = m_0(D_0)$, $m_0(D^*) = 1$,
2) $\kappa(D, V) = \kappa(D^*, V)$, $\kappa(D_0, V) = \kappa(D_0 | V_w, V_w) = 0$, *where* V_w *is a general member of the algebraic system introduced in the statement of Theorem* 3,
3) *the number of the irreducible components of* D_0 *is the least of those of the divisors* D_0 *satisfying the conditions* 1) *and* 2).

Moreover, we note that $\boldsymbol{N}(D_0) = m_0(D)\boldsymbol{N}$ *and* $\boldsymbol{N}(D^*) = \{n \in \boldsymbol{N}$ *such that* $n > N(D)\}$. *In particular,* $c\kappa(D, V) = 0$ *implies* $m_0(D) = 1$.

§ 7. Proofs of Theorems 5 and 6.

Applying Theorem 3 to the case in which V is non-singular and D a canonical divisor K of V, we obtain a fiber space of non-singular projective algebraic varieties $f: V^* \to W$ which satisfies the conditions 1), 2), 3) and 4) in the statement of Theorem 5 and the condition 5*) $K | V_w^*$-dimension of V is zero. Hence, in order to prove Theorem 5 it is sufficient to show that $K | V_w^*$ *is a canonical divisor of* V_w^*. For this, let W_1 be an open dense subscheme of V such that $f | f^{-1}(W_1)$ is smooth. We abbreviate $f^{-1}(W_1)$ and $f | V_1$ by V_1 and f_1, respectively. Referring to [3, II. 4. 3] we have an exact sequence:

$$0 \longrightarrow f_1^*(\Omega_{W_1}^1) \longrightarrow \Omega_{V_1}^1 \longrightarrow \Omega_{V_1/W_1}^1 \longrightarrow 0. \tag{15}$$

From this an isomorphism: $\Omega_{V_1}^n \simeq \Omega_{V_1/W_1}^{n-\kappa} \otimes f_1^* \Omega_{W_1}^\kappa$ follows (see [12]). Restricting these sheaves to a general fiber V_w^* we obtain

$$\Omega_{V_1}^n | V_w^* \simeq \Omega_{V_1/W_1}^{n-\kappa} | V_w^* \otimes f_1^*(\Omega_{W_1}^\kappa) | V_w^*. \tag{16}$$

Since $f_1^*(\Omega_{W_1}^\kappa) | V_w^* \simeq \mathcal{O}_{V_w^*} \otimes \Omega_{W_1}^\kappa |_w \simeq \mathcal{O}_{V_w^*}$ and $\Omega_{V_1/W_1}^1 | V_w^* \simeq \Omega_{V_w^*}^1$, the isomorphism (16) leads to $\Omega_{V_1}^n | V_w^* \simeq \Omega_{V_w^*}^{n-\kappa}$. This implies that $K | V_w^*$ is a canonical divisor of V_w^*, as required.

Now, let $f: \tilde{V} \to V$ be a fiber space of complete non-singular algebraic varieties. Suppose that f is an étale morphism. Then $\Omega^1_{\tilde{V}/V} = 0$ (see [3, I. 3.1]). Hence, by the exact sequence (15) we have $f^*\Omega^1_V \simeq \Omega^1_{\tilde{V}}$. This leads to $f^*K_V \sim K_{\tilde{V}}$. Therefore, applying the former assertion of Theorem 4 with $D = K_V$, we can prove the former part of Theorem 6. As for the latter part of Theorem 6, using the linear equivalence $K_{\tilde{V}} | \tilde{V}_w \sim K_{\tilde{V}_w}$, we can prove it by a similar argument.

§ 8. Counterparts of Theorems 1, ···, 6 in the category of complex spaces.

Now let us consider in the category of complex spaces, which we denote by (An). Replacing a complete algebraic variety V, a morphism, a rational map, a non-singular algebraic variety, ···, in the statements of theorems in § 2, respectively, by a compact irreducible reduced complex space (such a space is called a complex variety), a holomorphic map, a meromorphic map, a complex manifold, ···, we obtain the statements of the corresponding theorems in (An). Let us refer to the theorem in (An) corresponding to Theorem x in § 2 as Theorem x^*. We note that Theorem 3* asserts the existence of an algebraic system of compact complex sub-spaces of M and that W in Theorem 5* admits a structure of an algebraic variety, since W_{mD} is a closed complex sub-space of $\boldsymbol{P}^N$.

Using the fact that $\kappa(D, M)$ is the largest of the dimensions of the varieties W_{mD}, $m \geqq 1$, we obtain the following Corollary to Theorem 1*.

COROLLARY. *If there exists a divisor D on M with $\kappa = \kappa(D, M)$, then the transcendental degree $a(M)$*[5] *of the field of meromorphic functions on M is not smaller than κ. In particular, the vanishing of $a(M)$ implies that $\kappa(M) \leqq 0$.*

This is a generalization of a theorem which says that if there exists on a compact complex surface S a divisor D with $D^2 > 0$, then S is algebraic.

PROOFS OF THEOREMS 1* AND 2*. Let M denote a compact complex variety. By the resolution theorem which was recently proved by Hironaka (see [7]) we can assume that M is a compact complex manifold and furthermore we have a fiber space $h: M^* \to V^*$ which has the following properties (we call $h: M^* \to V^*$ *an algebraic reduction of* M):

i) M^* is a compact complex manifold which is bimeromorphically equivalent to M,
ii) V^* is a compact complex manifold of dimension $a(M)$, which admits a structure of a projective algebraic variety,
iii) h is a proper surjective holomorphic map which induces an isomorphism between the fields of meromorphic functions on M^* and

5) We call $a(M)$ the algebraic dimension of M.

the field of meromorphic functions on V^*.
Then we find a number m_3 and a divisor C on V^* corresponding to the pair of the fiber space $h: M^* \to V^*$ and the divisor D such that there exists an isomorphism:

$$\boldsymbol{L}(mm_3D) \simeq \boldsymbol{L}(mf^*(C)) \simeq \boldsymbol{L}(mC) \qquad \text{for all} \quad m > 0 .$$

Hence, applying Theorems 1 and 2, we can prove Theorems 1* and 2*.

PROOF OF THEOREM 3*. In order to prove Theorem 3* in the same way as in the proof of Theorem 3, it is sufficient to prove that $H^0(W_\lambda, f_*(\mathcal{L}))$ is generated as an $H^0(W_\lambda, \mathcal{O}_W)$-module by elements of $H^0(M, \mathcal{L}(\varepsilon mE_\lambda))$ for an integer $\varepsilon > 0$. Applying GAGA technique to W we shall prove this assertion. In this proof, Gothic letters $\boldsymbol{W}, \boldsymbol{A}, \cdots$ denote an algebraic variety, a coherent algebraic sheaf, $\cdots$, respectively. Since $f_*(\mathcal{L})$ is a coherent analytic sheaf on W, there exists a coherent algebraic sheaf $\boldsymbol{A}$ such that $f_*(\mathcal{L}) = \boldsymbol{A}^{an}$ (The symbol $\boldsymbol{A}^{an}$ denotes an analytic sheaf canonically associated with $\boldsymbol{A}$). Then we have

$$H^0(W_\lambda, f_*(\mathcal{L})) = H^0(\boldsymbol{W}_\lambda, \boldsymbol{A})^{an} = H^0(\boldsymbol{W}_\lambda, \boldsymbol{A}) H^0(W_\lambda, \mathcal{O}_W) ,$$

where $\boldsymbol{W}$ is an algebraic variety canonically associated with W, i.e., $\boldsymbol{W}^{an} = W$. Since $\boldsymbol{A}$ is algebraic we have

$$H^0(\boldsymbol{W}_\lambda, \boldsymbol{A}) = \bigcup_{e=1}^{\infty} H^0(\boldsymbol{W}, \boldsymbol{A}(em\boldsymbol{H}_\lambda)) .$$

Moreover, we have

$$H^0(\boldsymbol{W}, \boldsymbol{A}(em\boldsymbol{H}_\lambda))^{an} = H^0(W, \boldsymbol{A}^{an}(emH_\lambda)) = H^0(W, f_*(\mathcal{L})(emH_\lambda)) .$$

By the projection formula, we obtain

$$\begin{aligned} H^0(W, f_*(\mathcal{L})(emH_\lambda)) &= H^0(W, f_*(\mathcal{L}[f^{-1}(emH_\lambda)])) \\ &= H^0(M, \mathcal{L}(emE_\lambda^*)) \subset H^0(M, \mathcal{L}(emE_\lambda)) . \end{aligned}$$

Recalling that $H^0(\boldsymbol{W}_\lambda, \boldsymbol{A})$ is a finite $H^0(\boldsymbol{W}_\lambda, \mathcal{O}_{\boldsymbol{W}})$-module, we thus prove the assertion. This proof was suggested by M. Kashiwara.

PROOFS OF PROPOSITION 3* AND THEOREM 4*. Using a theorem of Picard concerning the essential singularities of an analytic function, we may give a short proof of Proposition 3*. Before proving Theorem 4* we will construct an algebraic reduction of a fiber space of compact complex varieties $f: \tilde{M} \to M$. Let $\tilde{h}: \tilde{M}^* \to \tilde{V}$ and $h: M^* \to V$ be algebraic reductions of $\tilde{M}$ and M, respectively. Then the inclusion $k(M) = k(V) \subsetneq k(\tilde{M}) = k(\tilde{V})$ yields a rational map g from $\tilde{V}$ to V such that $h \cdot f = g \cdot \tilde{h}$. By means of monoidal transformations, we can assume that g is regular. We call the triple of the fiber space $g: \tilde{V} \to V$, $\tilde{h}$ and h, an algebraic reduction of the fiber space $f: \tilde{M} \to M$. Now we shall prove the former assertion of Theorem 4*. In doing this we can assume that the triple of $g: \tilde{V} \to V$, $\tilde{h}: \tilde{M} \to \tilde{V}$ and $h: M \to V$ is an algebraic reduction of $f: \tilde{M} \to M$. In the case in which $a(M) = \dim M$, h is bimeromorphic. Hence, $\boldsymbol{L}_M(D)$[6] $\simeq \boldsymbol{L}_V(h^{-1*}D)$. By Theorem 4 we have $\kappa(h^{-1*}D, V) = \kappa(g^*h^{-1*}D, \tilde{V})$. Combined with the natural isomorphism: $\boldsymbol{L}_{\tilde{V}}(g^*h^{-1*}(D)) \simeq \boldsymbol{L}_{\tilde{M}}(\tilde{h}^*g^*h^{-1*}(D)) = \boldsymbol{L}_{\tilde{M}}(f^*D)$, these give $\kappa(D, M) = \kappa(f^*D, \tilde{M})$. In the case in which $a(M) = 0$, V reduces to a point, and $\kappa(D, M) = 0$. We shall derive a contradiction under the assumption $l(f^*D) \geqq 2$. We choose two linearly independent sections from $\boldsymbol{L}_{\tilde{M}}(f^*D)$. Using them we construct a non-constant meromorphic map $s: \tilde{M} \to C = \boldsymbol{P}^1$. Clearly we can assume that s is holomorphic. By definition, we have $\tilde{M} \supsetneqq fs^{-1}(q)$ for a general point q of C. Hence, for a general point p of $\tilde{M}$, $sf^{-1}(p)$ is a finite set. Let Γ_s denote the graph of s. We define Γ to be an image of Γ_s by a proper holomorphic map:

$$f \times 1_W: \quad \tilde{M} \times C \longrightarrow M \times C\,.$$

By a theorem of Remmert, Γ is a complex subvariety of $M \times C$. Composing the injection $\Gamma \subsetneqq M \times C$ with canonical projections: $M \times C \to M$ and $M \times C \to C$, we have two holomorphic surjective maps $\xi: \Gamma \to M$ and $\eta: \Gamma \to C$. Since $\xi^{-1}(p) = sf^{-1}(p)$ is finite for a general point p of M, we see that $\dim \Gamma = \dim M$.

LEMMA 1. *Let $f: \tilde{M} \to M$ be a fiber space of compact complex varieties with the same dimension. Then $a(\tilde{M}) = a(M)$.*

PROOF. Let $g: \tilde{V} \to V$, $\tilde{h}: \tilde{M} \to \tilde{V}$ and $h: M \to V$ form an algebraic reduction of $f: \tilde{M} \to M$. For any function $x \in k(\tilde{M})$, there exists a function $y \in k(\tilde{V})$ such that $x = h^*(y)$. We wish to prove that the polar divisor $(y)_\infty$ of y cannot be mapped onto V by g. For this, we describe $(y)_\infty$ as a sum of positive divisors $L^\#$ and $D^\#$ where $D^\#$ is the largest of all positive divisors such that $h(D^\#) \subsetneqq V$. Then if $L^\#$ is not empty, we have $fh^{-1}(L^\#) = M$. This implies $\dim M \leqq \dim h^{-1}(L^\#)$. It is clear that $\dim h^{-1}(L^\#) = \dim \tilde{M} - 1 < \dim \tilde{M}$. This contradicts the condition $\dim \tilde{M} = \dim M$.

From this lemma, $a(\Gamma) = a(M) = 0$ follows. On the other hand, using η, we obtain $a(\Gamma) \geqq a(C) = 1$. Thus we have encountered the contradiction. Finally, in the case in which $a(M) > 0$, we are going to prove by induction with respect to the dimension of M. Let v be a general point of V. As usual we use the following notation: $M_v = h^{-1}(v)$, $\tilde{M}_v = (h \circ f)^{-1}(v)$, $f_v = f | M_v$ and $D_v = D | M_v$. Note that $\dim M_v < \dim M$ in this case. By applying the induction hypothesis to the fiber space $f_v: \tilde{M}_v \to M_v$ and the divisor D_v, we have $\kappa(D_v, M_v) = \kappa(f_v^*(D_v), \tilde{M}_v)$. It is no loss of generality to assume that D is positive. We describe $f^*(D)$ as a sum of positive divisors L' and D', where D' is the largest of all positive divisors D' such that $h \cdot f(D') \neq V$. Then we have

6) In order to avoid the confusion, we denote by $\boldsymbol{L}_M(D)$ the space of regular sections $\boldsymbol{L}(D)$ of a divisor D on M.

$f^*(D)|\tilde{M}_v = L'|\tilde{M}_v$.

LEMMA 2 (Hironaka). *Let M be a compact complex variety and D a divisor on M. Let $f: M^* \to V$ be an algebraic reduction of M and D^* a complete inverse image of D by the bimeromorphic map from M^* to V. Then $\kappa(D_v^*, M_v^*) \leqq 0$, where M_v^* is a general fiber $f^{-1}(v)$ of f.*

PROOF. This can be proved by the same argument as in the proof of Theorem 3* (see [6]).

By this lemma we have $0 = \kappa(D_v, M_v) = \kappa(f_v^* D_v, \tilde{M}_v)$. This implies that L' is one of the fixed components of $|mf^*(D)|$ for any $m > 0$. Thus we have

$$\kappa(f^*D, \tilde{M}) = \kappa(D', \tilde{M})\,. \tag{17}$$

As in the proof of Theorem 2, we can find a positive divisor E such that $D' \geqq (h \cdot f)^*E$ and $\kappa(D', M) = \kappa((h \cdot f)^*E, \tilde{M})$. Since V is algebraic, we can apply Theorem 4* to the fiber space $h \cdot f: \tilde{M} \to V$ and the divisor E on V. Then we have

$$\kappa((h \cdot f)^*E, \tilde{M}) = \kappa(E, V)\,. \tag{18}$$

From $f^*D \geqq D' \geqq (h \cdot f)^*E$, it follows that $D \geqq h^*E$. Hence, we have $\kappa(D, M) \geqq \kappa(h^*E, M) = \kappa(E, V)$. Combining this with (17) and (18), we obtain $\kappa(D, M) = \kappa(f^*D, \tilde{M})$, as required.

Note that the same argument can be used to prove Proposition 3*. The latter part of Theorem 4* can be proved by the same argument as in the proof of Theorem 4, because Theorem 3* has already been proved.

REMARK 3. We give a stronger result than the latter part of Theorem 4* in the restricted case: *Let $f: \tilde{M} \to M$ be a fiber space of compact complex varieties and $\tilde{D}$ a divisor on M. Suppose that $a(M) = 0$. Then $\kappa(\tilde{D}, \tilde{M}) \leqq \kappa(\tilde{D}_q, \tilde{M}_q)$ for a general point q of M.* We shall prove this in the case $D > 0$, because in the other cases the proof is easy. We denote by $g: \tilde{M} \to W$ the fiber space $\Phi_{m_1 D}: \tilde{M} \to W_{m_1 D}$ for $m_1 \gg 0$. It is no loss of generality to assume that g is holomorphic. We wish to prove $fg^{-1}(p) = M$ for any general point p of W by induction with respect to the dimension of W. For this we let W_1 be a general hyperplane section of W which passes through the point p. We write $\tilde{M}_1$, f_1 instead of $g^{-1}(W_1)$, $f|\tilde{M}_1$, respectively. If $f_1(\tilde{M}_1) = M$, then $f_1 g^{-1}(p) = fg^{-1}(p) = M$ by our induction hypothesis. We shall derive a contradiction in the case in which $f(\tilde{M}_1) \neq M$. In this case, we can find a positive divisor G such that $f(\tilde{M}_1) \subset \operatorname{supp} G$ (the symbol $\operatorname{supp} G$ denotes the support of G). Then we have $M_1 \subset \operatorname{supp} f^*G$. By the former part of Theorem 4*, we obtain

$$0 < \kappa(W_1, W) = \kappa(g^{-1}(W_1), \tilde{M}) = \kappa(\tilde{M}_1, \tilde{M}) \tag{19}$$

and

$$\kappa(g^{-1}(W_1), M) \leqq \kappa(f^{-1}(G), \tilde{M}) = \kappa(G, M) = 0\,. \tag{20}$$

This contradicts the inequality (19). By $fg^{-1}(p) = M$ we have $gf^{-1}(q) = W$ for any general point q of M. Using the notation in the proof of Theorem 2, we recall that m_2D is described as $m_2L_1+m_2\Theta_1+g^*H_{m_2}$. Then by Proposition 3* we have

$$\kappa(H_{m_2}, W)=\kappa(g_q^*(H_{m_2}), \tilde{M}_q),$$

where $\tilde{M}_q=f^{-1}(q)$ is irreducible and reduced, $g_q=g|\tilde{M}_q$. Furthermore, we have

$$\kappa(g_q^*(H_{m_2}), \tilde{M}_q)\leqq\kappa((m_2L_1+m_2\Theta_1+g^*(H_{m_2}))|\tilde{M}_q, \tilde{M}_q)=\kappa(\tilde{D}_q, \tilde{M}_q),$$

because $\operatorname{supp} L_1 \not\supset M_q$ and $\operatorname{supp}\Theta_1 \not\supset M_q$ for a general point q of M. Hence, we obtain the inequality $\kappa(\tilde{D}, \tilde{M})\leqq\kappa(\tilde{D}_q, \tilde{M}_q)$.

REMARK 4. The following assertion is a generalization of Lemma 2 due to K. Ueno: *Let* $f:\tilde{M}\to M$ *be a fiber space of compact complex varieties. Then* $a(M)\leqq a(\tilde{M})\leqq a(M)+\dim f$, *where* $\dim f$ *denotes* $\dim\tilde{M}-\dim M$.

The left hand side inequality is easily proved and so we shall prove the right hand side. Consider first the case $a(M)=0$. In this case, we use the induction with respect to $\dim M$. Let x be a non-constant meromorphic function on $\tilde{M}$ and D the polar divisor $(x)_\infty$ of x. We define a meromorphic map g to be $g(p)=1:x(p)\in \boldsymbol{P}^1$ for a general point p of M (g can be assumed to be holomorphic). Apply Theorem 4* to the pair of the fiber space $g:\tilde{M}\to\boldsymbol{P}^1$ and the divisor $\tilde{H}$ on $\tilde{M}$ such that $\kappa(\tilde{H}, \tilde{M})=a(\tilde{M})$. Then we obtain

$$a(\tilde{M})=\kappa(\tilde{H}, \tilde{M})\leqq\kappa(\tilde{H}_1, \tilde{M}_1)+\dim\boldsymbol{P}^1\leqq a(\tilde{M}_1)+1, \tag{21}$$

where $\tilde{M}_1$ is a general fiber of g and so is a general irreducible component of $|D|$, and $\tilde{H}_1=\tilde{H}|\tilde{M}_1$. Suppose that $f(D)=M$. Then there exists an irreducible component $\tilde{M}_1$ of $|D|$ such that $f(\tilde{M}_1)=M$. By using our induction hypothesis in the case of the fiber space $f|M_1:\tilde{M}_1\to M$, we have

$$a(\tilde{M}_1)\leqq\dim\tilde{M}_1-\dim M=\dim\tilde{M}-\dim M-1.$$

Combining this with (21) we obtain the inequality $a(\tilde{M})\leqq\dim f$.

Now suppose that $f(D)\neq M$. Then from $a(M)=0$, we derive that x reduces to a constant, a contradiction. In the case in which $a(M)>0$, we use the induction with respect to $\dim\tilde{M}$. We let the triple consisting of a fiber space of algebraic varieties g: $\tilde{V}\to V$, $\tilde{h}:\tilde{M}\to\tilde{V}$ (an algebraic reduction of $\tilde{M}$) and $h:M\to V$ (an algebraic reduction of M) be an algebraic reduction of $f:\tilde{M}\to M$. For a general point v of V, we have a fiber space $f_v=f|M_v:\tilde{M}_v=f^{-1}(M_v)\to M_v=h^{-1}(v)$. By our induction hypothesis we have

$$a(\tilde{M}_v)\leqq a(M_v)+\dim f_v, \tag{22}$$

because $\dim\tilde{M}_v<\dim\tilde{M}$. Clearly it follows that $\dim f_v=\dim f$, $a(M_v)\geqq 0$ and

$$a(\tilde{M}_v)\geqq\dim\tilde{V}_v=\dim\tilde{V}-\dim V=a(\tilde{M})-a(M).$$

Combining these with (22), we obtain $a(\tilde{M})-a(M)\leqq \dim f$.

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