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On certain double coset spaces of algebraic groups

to Elaine

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Introduction.

Let k be an algebraic number field of finite degree and $\mathcal{O}_k$ its ring of integers. Let V be a finite dimensional vector space over k, W its proper subspace ($W \neq \{0\}$), and L an $\mathcal{O}_k$-lattice in V. For a subgroup G of $GL(V)$, we put $G_W = \{g \in G \mid g(W) = W\}$, $G_L = \{g \in G \mid g(L) = L\}$.

Let $G = Sp(V, A)$, $SO(V, B)$ or $SU(V, H)$, where A is a non-degenerate alternating form, B is a non-degenerate quadratic form, and H is a non-degenerate Hermitian form over a quadratic extension K/k. Let W be a proper totally isotropic subspace of V. Then $G_W = P$ is a maximal k-parabolic subgroup of G, and the arithmetic subgroup G_L may be regarded as a discrete subgroup of the Lie group $\underline{G} = (\mathcal{R}_{k/Q}(G))_R$. Suppose that there exists a maximal compact subgroup $\mathcal{K}$ of $\underline{G}$ such that $D = \underline{G}/\mathcal{K}$ has the structure of a symmetric bounded domain. Then the subspace W corresponds with a rational boundary component X of $\bar{D}$ and the double coset space $G_L \backslash G/P$ is in bijective correspondence with the set of G_L-orbits among $\{g(X) \mid g \in G\}$.

The finiteness of such double coset spaces was proved by Godement [**5**] and Borel (cf. [**2**]) for the cases where G is a connected matric group, $G_L = G_{\mathcal{O}_k}$ and P is any k-parabolic subgroup of G.

In [**6**], the author gave an estimation of the number of such double coset spaces for $G = SU(V, H)$ under a certain condition. The purpose of the present paper is to generalize the results obtained in [**6**].

If $G = SO(V, B)$ (except when $\dim V = 2 \cdot \dim W$) or $SU(V, H)$, the problem to determine the above numbers is reduced to the problem of determining the order $|\tilde{G}_L \backslash \tilde{G}/\tilde{G}_W|$ where $\tilde{G} = O(V, B)$ or $U(V, H)$ respectively. (1.3, 4.18.)

To determine the order $|\tilde{G}_L \backslash \tilde{G}/\tilde{G}_W|$ we use a certain decomposition of the lattice L ('W-decomposition'). To explain, suppose, for example, that the space V is supplied with a non-degenerate quadratic form B, and that W is a totally isotropic subspace of V. Then there exists a basis $\{w_1, \cdots, w_s; w'_1, \cdots, w'_s; u_1, \cdots, u_r\}$ of V such that $W = \sum_{i=1}^{s} kw_i$, $B(w_i, w'_j) = \delta_{ij}$, $B(w'_i, w'_j) = 0$ for $i, j = 1, \cdots, s$; $B(u_i, w_j) = B(u_i, w'_j) = 0$ for $i = 1, \cdots, r$; $j = 1, \cdots, s$. If the lattice L is modular, then we can show that a similar decomposition of L

exists and it will be called a 'W-decomposition' of L. Proposition 3.3 shows the existence of W-decomposition for a modular lattice in a general situation.

Main results are given in 3.9, 4.15, 4.19, 5.7 and 5.8. In particular, we get:

1) If $G=Sp(V, A)$ and L is modular, then $|G_L\backslash G/G_W|=h(k)=$ the class number of k,

2) Suppose that $G=O(V, B)$ or $U(V, H)$, L is modular, and that any modular lattice of the same modulus in V is G-equivalent to L. Suppose, moreover, that we are in an 'elementary case' (cf. 5.1.). Then we have:

$$|G_L\backslash G/G_W|=\begin{cases} h(k), \text{ if } G=O(V, B), \\ h(K), \text{ if } G=U(V, H) \text{ and } \dim V=2\cdot\dim W, \\ h(K)\cdot[N(K^*)\cap U_k : N(U_K)], \text{ if } G=U(V, H) \text{ and } \dim V=2\dim W+1. \end{cases}$$

where U_K (resp. U_k) is the group of units in $\mathcal{O}_K$ (resp. $\mathcal{O}_k$).

Also, applying the general theory of lattices, similar problems of determining the order $|G_L\backslash G/G_W|$ for $G=GL(V)$ or $SL(V)$, L an arbitrary lattice in V, and W an arbitrary proper non-zero subspace of V can be solved and we get $|G_L\backslash G/G_W|=h(k)$. (2.5, 2.7.)

The author wishes to dedicate this paper to Elaine Iyanaga who has been the source of his inspirations.

§ 1. Generalities.

1.1. Let k be the quotient field of a Dedekind domain $\mathcal{O}$; and let V be a finite dimensional vector space over k. Let W be a proper subspace of V ($W\neq\{0\}$), and L be an $\mathcal{O}$-lattice in V (i.e. L is a finitely generated $\mathcal{O}$-submodule of V which contains a basis of V). Let $GL(V)$ be the group of regular linear transformations of V, and $SL(V)$ be its subgroup consisting of the linear transformations of determinant 1. For a subgroup G of $GL(V)$ we put $G_L=\{g\in G \mid gL=L\}$, $G_W=\{g\in G \mid gW=W\}$. When G and W are given we often write P instead of G_W. Our purpose in this paper is to estimate the order $|G_L\backslash G/P|$ for certain G and W.

Suppose that we are given a subgroup H of $GL(V)$ and $\mathcal{O}$-submodules L and M of V. We denote $L\underset{H}{\sim}M$ if and only if there exists an element h of H such that $hL=M$. And in this case we call L and M to be H-equivalent. We put $(L)_H=$ the set of $\mathcal{O}$-submodules of V which are H-equivalent to L.

We now have the following bijections: $G_L\backslash G/P\approx(W)_G/\underset{G_L}{\sim}\approx(L)_G/\underset{P}{\sim}$.

1.2. Given a subgroup H of $GL(V)$, we put $C_L(H)=\{\det(g) \mid g\in H_L\}$. $C_L(H)$ is a subgroup of $U_k=$ the group of units of $\mathcal{O}$. If the order $|C_L(G)/C_L(P)|$ is finite we denote the order by $c(G, L, W)$.

1.3. PROPOSITION. *Let* $G^1=G\cap SL(V)$. *Suppose that* $(W)_G=(W)_{G^1}$, $|G_L\backslash G/P|$ *is finite and that* $|C_L(G)/C_L(G_{W'})|$ *is finite for any element* W' *in* $(W)_G$. *Let* $W_1, \cdots, W_r$ *be a system of representatives of* $(W)_G/\underset{G_L}{\sim}$. *Then we have:*

$$|G_L^1\backslash G^1/G_W^1|=\sum_{i=1}^{r} c(G, L, W_i)\,.$$

PROOF. Firstly, let us show that $c(G, L, W_i)$ are independent of the choice of the representatives $W_1, \cdots, W_r$. In fact, if $W_i'\underset{G_L}{\sim} W_i$, then there exists an element g in G_L such that $gW_i=W_i'$, hence $C_L(G_{W_i'})=C_L(g^{-1}G_{W_i}g)=C_L(G_{W_i})$. This implies that $c(G, L, W_i)=c(G, L, W_i')$.

Suppose now that W' is an element of $(W)_G=(W)_{G^1}$. To show the proposition, it is sufficient to show that $(W')_{G_L}$ is decomposed into $c(G, L, W')$ G_L^1-equivalence classes. Let $X=gW'$ $(g\in G_L)$ be an element of $(W')_{G_L}$. We put $\varphi(X)=\det(g)\cdot C_L(G_{W'})$. This determines a mapping from $(W')_{G_L}$ onto the coset space $C_L(G)/C_L(G_{W'})$. Now let $Y=hW'$ $(h\in G_L)$ be another element of $(W')_{G_L}$. Suppose $\varphi(X)=\varphi(Y)$. Then there exists an element u in $G_{W'}\cap G_L$ such that $\det(h\cdot g^{-1})=\det(u)$. We now have $Y=h\cdot u^{-1}\cdot g^{-1}(X)$ and $\det(h\cdot u^{-1}\cdot g^{-1})=1$; i.e. $X\underset{G_L^1}{\sim} Y$. Conversely it is easy to see that if $X\underset{G_L^1}{\sim} Y$ then $\varphi(X)=\varphi(Y)$. Thus, the mapping φ gives a bijection of the space of G_L^1-equivalence classes in $(W')_{G_L}$ onto $C_L(G)/C_L(G_{W'})$.

1.4. From now on we assume that our group G is the set of k-rational points of an algebraic group defined over k, and in this section, we assume that k is an algebraic number field of finite degree. We denote the adele ring of k by A and the adelization of V and G by V_A, G_A. Given a place v of k, we denote by $\mathcal{O}_v$, k_v the completion of $\mathcal{O}$ and k; $L_v=L\otimes\mathcal{O}_v$, $V_v=V\otimes k_v$, $G_v=v$-completion of G. We put $L_A=\prod_{\mathfrak{p}} L_{\mathfrak{p}}\times\prod_{v} V_v$, where $\mathfrak{p}$ runs over all the prime ideals in k and v over all the infinite places of k. G_A operates naturally on V_A and L_A is a submodule of V_A. Let $G_{A,L}$ be the subgroup of G_A consisting of the elements which stabilize L_A. Then we have the following proposition:

1.5. PROPOSITION. *Suppose that* $|G_{L_{\mathfrak{p}}}\backslash G_{\mathfrak{p}}/P_{\mathfrak{p}}|=1$ *for all but finite* $\mathfrak{p}$, *then we have:*

$$|G_{A,L}\backslash G_A/P_A|=\prod_{\mathfrak{p}}|G_{L_{\mathfrak{p}}}\backslash G_{\mathfrak{p}}/P_{\mathfrak{p}}|\,.$$

PROOF. Given a lattice M and its adelization $M_A=\prod_{\mathfrak{p}} M_{\mathfrak{p}}\times\prod_{v} V_v$, we set $\varphi(M_A)=\prod_{\mathfrak{p}} M_{\mathfrak{p}}$. φ gives rise to a mapping (also written φ) from $(L_A)_{G_A}/\underset{P_A}{\sim}$ into $\prod_{\mathfrak{p}}(L_{\mathfrak{p}})_{G_{\mathfrak{p}}}/\underset{P_{\mathfrak{p}}}{\sim}$. To show the proposition, it is sufficient to prove that the mapping φ is bijective. It is obvious that φ is injective. To show that φ is

surjective, take an element X in $\prod_{\mathfrak{p}}(L_{\mathfrak{p}})_{G_{\mathfrak{p}}}/\underset{P_{\mathfrak{p}}}{\sim}$ whose $\mathfrak{p}$-component is represented by $M^{\mathfrak{p}}$. By our assumption, we may put $M^{\mathfrak{p}}=L_{\mathfrak{p}}$ for all but finite $\mathfrak{p}$'s. Then using the Lemma 5.2 in [**10**], we obtain a lattice $M=\bigcap_{\mathfrak{p}}(M^{\mathfrak{p}}\cap V)$ which satisfies that $M_A\in(L_A)_{G_A}$ and $\varphi(M_A)=X$. This completes the proof.

1.6. Two lattices L and M are said to be of the same G-genus if $(L_A)_{G_A}=(M_A)_{G_A}$. The order $|G_{A,L}\backslash G_A/G|=h(L)$ is the order of G-classes in the G-genus of L. If $h(L)$ is finite, it is called the class number of L (with respect to G).

1.7. PROPOSITION. *Suppose that there exist $c_1,\cdots,c_m$ in G_A such that $G_A=\bigcup_{i=1}^{m}G_{A,L}c_iP_A$ (disjoint) and that the following condition (A) is satisfied:*

$$(A)\qquad c_i\cdot P_{A,L}\cdot c_i^{-1}\subset G_{A,L}\,,\qquad \textit{for}\quad i=1,\cdots,m\,.$$

Suppose also that $P_A=\bigcup_{j=1}^{h}P_{A,L}\cdot p_j\cdot P$ (disjoint). Then we have:

$$|G_L\backslash G/P|\leqq m\cdot h\,.$$

In particular, if $m=1$, and $G_A=G_{A,L}\cdot G$, then $|G_L\backslash G/P|=h$.

(This proposition is a generalization of the proposition 7.5 in [**1**].)

PROOF. We have:

$$\begin{aligned}G_A&=\cup G_{A,L}\cdot c_i\cdot P_A\\&=\cup G_{A,L}\cdot c_i\cdot P_{A,L}\cdot p_j\cdot P\\&=\cup G_{A,L}\cdot c_i\cdot p_j\cdot P\,.\end{aligned}$$

We put $d_{ij}=c_i\cdot p_j$. Suppose $G_{A,L}\cdot G\cap G_{A,L}\cdot d_{ij}\cdot P\neq\emptyset$ for as many as t pairs (ij) $(t\leqq m\cdot h)$. We put those d_{ij}'s in an order and write $d_{ij}=e_k$ $(k=1,\cdots,t)$. We may assume that e_k belongs to G. Given any element g in G, we can write $g=x\cdot e_k\cdot p$, for some x in $G_{A,L}$, k, and p in P. As we have $G\cap G_{A,L}=G_L$, we obtain:

$$G=\bigcup_{k=1}^{t}G_L\cdot e_k\cdot P\quad\text{(disjoint)}\,.$$

This completes the proof.

§ 2. The case $G=GL(V)$.

2.1. k is the quotient field of a Dedekind domain $\mathcal{O}$, $G=GL(V)$. An $\mathcal{O}$-lattice L (not necessarily in V) is a finitely generated $\mathcal{O}$-module which is torsion-free (hence, it is a lattice in the vector space kL). Given such a lattice L, it is known that there exist elements $v_1,\cdots,v_n$ in kL and fractional ideals $\mathfrak{A}_1,\cdots,\mathfrak{A}_n$ in k such that:

$$L=\mathfrak{A}_1v_1\oplus\cdots\oplus\mathfrak{A}_nv_n\quad\text{(direct sum) [3]}\,.$$

n is called the rank of L. The ideal class of the product $\mathfrak{A}_1 \cdots \mathfrak{A}_n$ is determined by L and denoted by $c(L)$. Two lattices L and M are isomorphic as $\mathcal{O}$-modules if and only if they have the same rank and $c(L)=c(M)$ [3].

2.2. An $\mathcal{O}$-lattice is a projective $\mathcal{O}$-module. Hence, if we have an exact sequence $0 \to L_1 \to L_2 \to L_3 \to 0$ of $\mathcal{O}$-lattices, then L_2 is isomorphic with $L_1 \oplus L_3$, and $c(L_2)=c(L_1)\cdot c(L_3)$.

2.3. PROPOSITION. *Let L and M be lattices in V. Then we have: $L \underset{P}{\sim} M$ if and only if $c(L)=c(M)$ and $c(L\cap W)=c(M\cap W)$.*

PROOF. The 'only if' part is obvious. So let us assume that $c(L)=c(M)$ and $c(L\cap W)=c(M\cap W)$. We have the following exact sequences of lattices:

$$0 \longrightarrow L\cap W \longrightarrow L \longrightarrow L/L\cap W \longrightarrow 0$$

$$0 \longrightarrow M\cap W \longrightarrow M \longrightarrow M/M\cap W \longrightarrow 0\,.$$

By our assumptions we now have $c(L/L\cap W)=c(M/M\cap W)$. We also have

$$L=L\cap W\oplus L'\,, \qquad L'\cong L/L\cap W\,,$$

$$M=M\cap W\oplus M'\,, \qquad M'\cong M/M\cap W\,,$$

$$V=W\oplus W'\,, \qquad W'=kL'\,.$$

There exist elements g, g' in $GL(V)$ such that $g|W'=1$, $g(L\cap W)=M\cap W$, and $g'|W=1$, $g'(L')=M'$. Then $g'\cdot g(L)=M$ and $g'\cdot g\in P$.

2.4. Given a lattice $L=L_1\oplus L_2$ (rank $L_1=$ rank L_2) and a fractional ideal $\mathfrak{A}$, we get another lattice $M=\mathfrak{A}L_1\oplus\mathfrak{A}^{-1}L_2$ which is isomorphic to L. This, together with the previous proposition leads to the following theorem:

2.5. THEOREM. *For $G=GL(V)$, we have:*

$$|G_L\backslash G/P|=h(k)\,,$$

where $h(k)$ is the order of the ideal classes in k.

2.6. Using the decomposition in 2.1, we now have: $C_L(G)=U_k$. Using the notions in 2.3, we have: $C_L(P)\supset C_{L'}(GL(W'))=U_k$. Hence, $C_L(G)=C_L(P)$. Using the proposition 1.3, we get the following:

2.7. THEOREM. *For $G^1=SL(V)$, we have:*

$$|G^1_L\backslash G^1/G^1_W|=h(k)\,.$$

§ 3. F-spaces.

3.1. As before, k is the quotient field of Dedekind domain $\mathcal{O}$. We assume *ch.* $k\neq 2$. We let K be a semi-simple algebra over k such that $[K:k]\leqq 2$; i.e. K is either a quadratic extension of k or the direct sum of two copies of k or k itself. Let σ be the involution of K fixing the elements of k; σ is either the Galois involution of K/k, or $(x, y)^\sigma=(y, x)$, for $(x, y)\in K=k\times k$, or

$\sigma=1$ if $K=k$. For an element x in K, we write $N(x)=x^{\sigma}\cdot x$, $\mathrm{Tr}(x)=x^{\sigma}+x$. By $\mathcal{O}_K$, we denote the ring of all $\mathcal{O}$-integral elements in K; $\mathcal{O}_K$ is a Dedekind domain if K is a field, while if $K=k\times k$, then $\mathcal{O}_K=\mathcal{O}_k\times\mathcal{O}_k$ ($\mathcal{O}_k=\mathcal{O}$). By an $\mathcal{O}_K$-ideal we mean an $\mathcal{O}$-lattice in K which has a structure of an $\mathcal{O}_K$-module.

We assume that the vector spaces V and W are free K-modules. We denote by $GL_K(V)$ the group of all regular K-endomorphisms of V. We also assume now that our lattices in V have structure of $\mathcal{O}_K$-modules.

Let F be a form on V such that:

1) F is a k-bilinear form on V into K,
2) $F(ax, by)=a^{\sigma}\cdot F(x, y)\cdot b$, for $a, b\in K$ and $x, y\in V$,
3) $F(x, y)^{\sigma}=\varepsilon F(y, x)$, $\varepsilon=\pm 1$, $x, y\in V$.

We assume that the form F is non-degenerate (i.e. $F(x, V)=0$ implies $x=0$). Such a form F is called an $\varepsilon-K/k$ form, and the space V supplied with F will be called an F-space. We note that if there exists an element w in K such that $w^{\sigma}=-w$ then we may assume that $\varepsilon=1$. Because in such a case we may replace F by F' defined by $F'(x, y)=w\cdot F(x, y)$ for $x, y\in V$. An element w as above exists except when $K=k$. So we assume that $\varepsilon=-1$ only when $K=k$.

We put $G=\{g\in GL_K(V)\mid F(gx, gy)=F(x, y)$ for all $x, y\in V\}$ and call it the F-group on V.

For any place v of k, the space V_v is supplied with $\varepsilon-K_v/k_v$ form which will be denoted by F_v ($K_v=K\otimes k_v$).

3.2. Concerning with a lattice L in V, we define the following $\mathcal{O}_K$-ideals:

$$\mu(L)=\text{the } \mathcal{O}_K\text{-ideal generated by } F(x)=F(x, x), \text{ for } x\in L,$$

$$\mu_0(L)=\text{the } \mathcal{O}_K\text{-ideal generated by } F(x, y), \text{ for } x, y\in L.$$

If $\varepsilon=-1$, then we, of course, have $\mu(L)=\{0\}$. If $\varepsilon=1$, then it is known and easy to show that:

$$\mathcal{O}_K\cdot \mathrm{Tr}\,\mu_0(L)\subset\mu(L)\subset\mu_0(L)\subset\mu(L)\delta^{-1},$$

where

$$\delta^{-1}=\{x\in K\mid \mathrm{Tr}(x\mathcal{O}_K)\subset\mathcal{O}_K\}. \quad \text{(Cf. [6], [10].)}$$

L is called to be 'normal' if $\mu_0(L)=\mu(L)$.

If $K=k\times k$, then $\delta^{-1}=\mathcal{O}_K$; if $K=k$, then $\delta=2\mathcal{O}_k$.

We put $L^{\#}=\{x\in V\mid F(L, x)\subset\mathcal{O}_K\}$. $L^{\#}$ becomes an $\mathcal{O}_K$-lattice in V and is called the 'dual lattice' of L. L is called to be '($\mu_0(L)$-) modular' if $L=\mu_0(L)\,L^{\#}$. (Cf. [6], [8].)

Given a prime ideal $\mathfrak{p}$ in k, it is known that $(L_{\mathfrak{p}})^{\#}=(L^{\#})_{\mathfrak{p}}$ (Cf. [6].). Hence, L is $\mathcal{J}$-modular if and only if $L_{\mathfrak{p}}$ is $\mathcal{J}_{\mathfrak{p}}$-modular for all prime ideals $\mathfrak{p}$.

Also, if $L=L'\oplus L''$ (orthogonal sum) then $L^{\#}=L'^{\#}\oplus L''^{\#}$. Hence we can

see readily that such an L is $\mathcal{J}$-modular if and only if both L' and L'' are $\mathcal{J}$-modular.

3.3. PROPOSITION. *Let L be an $\mathcal{J}$-modular lattice in V, and W be a proper totally isotropic subspace of V. Then there exist a system of $\mathcal{O}_K$-ideals $\mathfrak{A}_1, \cdots, \mathfrak{A}_s$ and a basis $\{w_1, \cdots, w_s\}$ of W, and a system of elements $\{w'_1, \cdots, w'_s\}$ of V such that:*

$$L = \Big(\sum_{i=1}^{s} \mathfrak{A}_i^{-\sigma}\mathcal{J} \cdot w_i + \sum_{i=1}^{s} \mathfrak{A}_i \cdot w'_i\Big) \oplus L', \quad \mathfrak{A}_1 \supset \mathfrak{A}_2 \supset \cdots \supset \mathfrak{A}_s, \qquad w_i \in L,$$

$$L' = L \cap \{w_1, \cdots, w_s, w'_1, \cdots, w'_s\}^{\perp},$$

$$F(w_i, w'_j) = \delta_{ij} \text{ for all } i, j; \qquad F(w'_i, w'_j) = m_i\delta_{ij} \text{ for all } i, j.$$

PROOF. Let w be an isotropic element of V. Let $F(w, L) = \mathfrak{A}$. Choose an element w' of $\mathfrak{A}^{-1}L$ such that $F(w, w') = 1$. (When $K = k \times k$, then the $\mathcal{O}_K$-ideal $\mathfrak{A}$ has the form $(\mathfrak{A}_1, \mathfrak{A}_2)$ where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are $\mathcal{O}$-ideals in k. We put $\mathfrak{A}^{-1} = (\mathfrak{A}_1^{-1}, \mathfrak{A}_2^{-1})$.) Then we can show that $L = (\mathfrak{A}^{-\sigma}\mathcal{J} \cdot w + \mathfrak{A}w') \oplus L'$, where $L' = L \cap \{w, w'\}^{\perp}$. To show this, let us first show that $\mathfrak{A}^{-\sigma} \cdot \mathcal{J} \cdot w \subset L$. We have $F(\mathfrak{A}^{-\sigma} \cdot \mathcal{J} \cdot w, L) = \mathfrak{A}^{-1} \cdot \mathcal{J}^{\sigma} \cdot F(w, L) = \mathcal{J}^{\sigma} = \mathcal{J}$. This implies that $\mathfrak{A}^{-\sigma} \cdot \mathcal{J} \cdot w \subset \mathcal{J} \cdot L^{\#} = L$ because L is modular. Hence, we have $\mathfrak{A}^{-\sigma} \cdot \mathcal{J} \cdot w + \mathfrak{A}w' + L' \subset L$. To see the converse, let x be an element of L. Then there exist elements $a, b \in K$ and an element $x' \in \{w, w'\}^{\perp}$, such that $x = aw + bw' + x'$. And we have $F(w, x) = b \in \mathfrak{A}$, $F(x, w') = a^{\sigma} + b^{\sigma} \cdot F(w') \in \mathfrak{A}^{-1} \cdot \mathcal{J}$. But, $F(w') \in F(\mathfrak{A}^{-1}L, \mathfrak{A}^{-1}L) = \mathfrak{A}^{-\sigma-1} \cdot \mathcal{J}$, hence, $b^{\sigma} \cdot F(w') \in \mathfrak{A}^{-1} \cdot \mathcal{J}$, so $a \in \mathfrak{A}^{-\sigma} \cdot \mathcal{J}$. Thus, $L \subset \mathfrak{A}^{-\sigma} \cdot \mathcal{J} \cdot w + \mathfrak{A}w' + L'$. To show the proposition, let $\mathfrak{A}_1$ be a maximal element among $\mathcal{O}_K$-ideals $\mathfrak{A} = F(w, L)$ for $w \in W \cap L$. By the above, we have a decomposition:

$$L = (\mathfrak{A}_1^{-\sigma} \cdot \mathcal{J} \cdot w_1 + \mathfrak{A}_1 \cdot w'_1) \oplus L \cap \{w_1, w'_1\}^{\perp},$$

for an element $w_1 \in W$ and an element $w'_1 \in \mathfrak{A}_1^{-1} \cdot L$ such that $F(w_1, w'_1) = 1$. In view of 3.2, the lattice $L_1 = L \cap \{w_1, w'_1\}^{\perp}$ is again $\mathcal{J}$-modular. If $\dim W \geqq 1$, then $W \cap \{w_1, w'_1\}^{\perp} \neq \{0\}$. Repeating the above process, we get:

$$L_1 = (\mathfrak{A}_2^{-\sigma} \cdot \mathcal{J} \cdot w_2 + \mathfrak{A}_2 \cdot w'_2) \oplus L_1 \cap \{w_2, w'_2\}^{\perp},$$

where $\mathfrak{A}_2$ is a maximal $\mathcal{O}_K$-ideal among $F(w, L_1)$ for $w \in W \cap L_1$, $w_2 \in W \cap L_1$, $w'_2 \in \mathfrak{A}_2^{-1} \cdot L_1$ such that $F(w_2, w'_2) = 1$. We now have $F(w_1 + w_2, L) = \mathfrak{A}_1 + \mathfrak{A}_2$. Hence, by the maximality of $\mathfrak{A}_1$, we have $\mathfrak{A}_1 \supset \mathfrak{A}_2$. Repeating the process, we get the desired result.

3.4. Decomposition of L as above will be called a 'W-decomposition' of L. A W-decomposition is determined by the ideals $\mathfrak{A}_1, \cdots, \mathfrak{A}_s$; $F(w'_1), \cdots, F(w'_s)$; L'. We note that if $\mathcal{O}_K$ is a principal ideal ring then we may put $\mathfrak{A}_i = \mathcal{J}$ for all i because it is known that for an $\mathcal{J}$-modular lattice L, and a primitive element w in L, we have $F(w, L) = \mathcal{J}$. (Cf. [**6**]. For the case $K = k \times k$, see [**4**].) Given a W-decomposition of L, the system of $\mathcal{O}_K$-ideals $\mathfrak{A}_i$ $(i = 1, \cdots, s)$ will

be called 'the ideals of the W-decomposition'.

We note also that, in the above, the space $W^{\perp}/W$ is supplied with a $\varepsilon-K/k$ form determined naturally by F, and that the lattice L' can be identified with the lattice $L\cap W^{\perp}/L\cap W$. We will denote the F-group on $W^{\perp}/W$ by G'.

3.5. PROPOSITION. *Let L and M be $\mathfrak{J}$-modular lattices in V, and W be a proper totally isotropic subspace of V. Suppose that $c(L\cap W)=c(M\cap W)$. Then there exist W-decompositions for L and M such that the ideals of decomposition for L and M coincide with each other.*

PROOF. Choose W-decompositions for L and M:

$$L=\Big(\sum_{i=1}^{s}\mathfrak{A}_i^{-\sigma}\cdot\mathfrak{J}\cdot w_i+\sum_{i=1}^{s}\mathfrak{A}_i\cdot w_i'\Big)\oplus L',$$

$$M=\Big(\sum_{i=1}^{s}\mathfrak{B}_i^{-\sigma}\cdot\mathfrak{J}\cdot u_i+\sum_{i=1}^{s}\mathfrak{B}_i\cdot u_i'\Big)\oplus M'.$$

We put $L_1=\sum_{i=1}^{s}\mathfrak{A}_i^{-\sigma}\cdot w_i$, $L_2=\sum_{i=1}^{s}\mathfrak{A}_i\cdot w_i'$, $M_1=\sum_{i=1}^{s}\mathfrak{B}_i^{-\sigma}\cdot u_i$, $M_2=\sum_{i=1}^{s}\mathfrak{B}_i\cdot u_i'$. Then $L\cap W=\mathfrak{J}L_1$, $M\cap W=\mathfrak{J}M_1$. Hence there exists an element $g\in GL_K(W)$ sending L_1 onto M_1.

Let $L_1^*=\{f\mid f$ is an $\mathcal{O}_K$-morphism sending L_1 into $\mathcal{O}_K\}$. L_1^* has a structure of an $\mathcal{O}_K$-lattice. We define M_1^* in the same manner. We now have ${}^tg(M_1^*)=L_1^*$. Associating any given element w' of L_2 with an $\mathcal{O}_K$-morphism $f_{w'}$ defined by $f_{w'}(w)=F(w,w')$ for each element w in L_1, we can identify L_2 with L_1^*. Likewise, M_2 can be identified with M_1^*. Hence we may write ${}^tg^{-1}(L_2)=M_2$.

Setting $g(w_i)=v_i$, ${}^tg^{-1}(w_i')=v_i'$, we now have:

$$\mathfrak{J}M_1+M_2=\sum\mathfrak{A}_i^{-\sigma}\cdot\mathfrak{J}\cdot v_i+\sum\mathfrak{A}_i\cdot v_i'.$$

For elements w, w' in L_1 and L_2 respectively, we have $F(g(w),{}^tg^{-1}(w'))=F(w,w')$. Hence we get a W-decomposition of M whose ideals are $\mathfrak{A}_1,\cdots,\mathfrak{A}_s$, provided that $F(v_i',v_j')=0$ for $i\neq j$.

Suppose for instance that $F(v_2',v_1')=a\neq 0$. Put $v_2''=v_2'-a^{\sigma}\cdot v_1$. Then $F(v_2'',v_1')=0$ and $\mathfrak{A}_2\cdot a^{\sigma}\cdot v_1\subset\mathfrak{A}_1^{-\sigma}\cdot\mathfrak{J}\cdot v_1$ (the latter is implied by $F(\mathfrak{A}_2v_2',\mathfrak{A}_1v_1')=\mathfrak{A}_2^{\sigma}\mathfrak{A}_1F(v_2',v_1')\subset\mathfrak{J}$). Likewise, we can replace v_i' by a suitable element such that $F(v_i',v_j')=0$ holds for $i\neq j$. This completes the proof.

3.6. From now on we let W be a proper totally isotropic subspace of V. Suppose that there exists an $\mathfrak{J}$-modular lattice L and its W-decomposition with $\{\mathfrak{A}_i\}_{i=1,\cdots,s}$ as its ideals. If $\{\mathfrak{B}_i\}_{i=1,\cdots,s}$ is a system of $\mathcal{O}_K$-ideals such that $\mathcal{O}_K\supset\mathfrak{B}_i\supset\mathfrak{B}_{i+1}$ for $i=1,\cdots,s-1$, then there exists an $\mathfrak{J}$-modular lattice M which has a W-decomposition with $\{\mathfrak{A}_i\cdot\mathfrak{B}_i\}_{i=1,\cdots,s}$ as its ideals. In fact if

$$L=(\sum\mathfrak{A}_i^{-\sigma}\cdot\mathfrak{J}\cdot w_i+\sum\mathfrak{A}_i\cdot w_i')\oplus L'$$

is a W-decomposition of L, then the lattice

$$M=(\sum \mathfrak{A}_i^{-\sigma}\mathfrak{B}_i^{-\sigma}\cdot\mathcal{J}\cdot w_i+\sum \mathfrak{A}_i\mathfrak{B}_i\cdot w_i')\oplus L'$$

satisfies the condition. To see this, it is enough to see that the lattice $\mathfrak{A}_i^{-\sigma}\mathfrak{B}_i^{-\sigma}\cdot\mathcal{J}\cdot w_i+\mathfrak{A}_i\mathfrak{B}_i\cdot w_i'=N$ is $\mathcal{J}$-modular. $\mu_0(\mathfrak{A}_i^{-\sigma}\cdot\mathcal{J}\cdot w_i+\mathfrak{A}_i\cdot w_i')=\mathcal{J}$ implies that $\mu_0(N)=\mathcal{J}$. Now let $x=aw_i+bw_i'$ be an element of $N^{\#}$ $(a, b\in K)$. $F(x, N)\in\mathcal{O}_K$ implies that $a^\sigma\mathfrak{A}_i\mathfrak{B}_i\subset\mathcal{O}_K$, $b^\sigma\mathfrak{A}_i^{-\sigma}\mathfrak{B}_i^{-\sigma}\cdot\mathcal{J}\subset\mathcal{O}_K$. Hence we have $\mathcal{J}N^{\#}\subset N\subset\mathcal{J}N^{\#}$; i.e. N is $\mathcal{J}$-modular.

3.7. Situation being the same as in 3.3 and 3.4, we look at the elements $F(w_i')$. They are 0 if $\varepsilon=-1$, so we assume now that $\varepsilon=1$. In this case, $F(w_i')$'s are not unique. For instance, we may replace w_i' by $w_i''=w_i'+aw_i+x$ where $a\in\mathfrak{A}_i^{-1-\sigma}\mathcal{J}$, $x\in\mathfrak{A}_i^{-1}L'$. We then have $F(w_i, w_i'')=1$, $F(w_i'')=F(w_i')+\mathrm{Tr}(a)+F(x)$; $F(w_i')\in F(\mathfrak{A}_i^{-1}\cdot L)$, $\mathrm{Tr}(a)\in\mathrm{Tr}(\mathfrak{A}_i^{-1-\sigma}\mathcal{J})$, $F(x)\in F(\mathfrak{A}_i^{-1}\cdot L')$.

Generally, given an $\mathcal{O}_K$-ideal $\mathfrak{A}$, we put

$$S(L, W, \mathfrak{A})=F(\mathfrak{A}^{-1}\cdot L')+\mathrm{Tr}(\mathfrak{A}^{-1-\sigma}\mathcal{J}),$$

$$S(L, \mathfrak{A})=F(\mathfrak{A}^{-1}\cdot L)+\mathrm{Tr}(\mathfrak{A}^{-1-\sigma}\mathcal{J}).$$

They are modules because we have $F(ax+by)=F(ax)+F(by)+\mathrm{Tr}(a^\sigma F(x, y)b)$ for $x, y\in V$; $a, b\in K$. And we have

$$S(L, W, \mathfrak{A})\subset S(L, \mathfrak{A})\subset\mu(\mathfrak{A}^{-1}\cdot L).$$

We set $s(L, W, \mathfrak{A})=|S(L, \mathfrak{A})/S(L, W, \mathfrak{A})|$. For the sake of convenience, we set $s(L, W, \mathfrak{A})=1$ for any $\mathfrak{A}$, if $\varepsilon=-1$. Also if we have $\mathrm{Tr}(\mathfrak{A}^{-1-\sigma}\mathcal{J})\mathcal{O}_K=\mathfrak{A}^{-1-\sigma}\mathcal{J}$, then we have $s(L, W, \mathfrak{A})=1$. Hence, if $K=k\times k$ or $K=k$ and $2\in U_k$, then $s(L, W, \mathfrak{A})=1$ for any $\mathfrak{A}$.

On the other hand, if $s(L, W, \mathfrak{A}_i)=1$ for $i=1, \cdots, s$, then we may put $F(w_i')=0$ for $i=1, \cdots, s$. (We note here that $S(L, W, \mathfrak{A})$ and $S(L, \mathfrak{A})$ can be defined for a general lattice L by putting $L'=L\cap W^\perp/L\cap W$, $\mathcal{J}=\mu_0(L)$.)

3.8. PROPOSITION. *Let L be an $\mathcal{J}$-modular lattice in V, $h(K)$ be the order of $\mathcal{O}_K$-ideal classes, and $h(\mathcal{J}, V)$ be the order of G-equivalent classes among $\mathcal{J}$-modular lattices in V. We assume now that both $h(K)$ and $h(\mathcal{J}, W^\perp/W)$ are finite. Then we have:*

1) *if $h(\mathcal{J}, V)=1$, then $h(K)\cdot h(\mathcal{J}, W^\perp/W)\leqq|G_L\backslash G/P|$,*

2) *if $s(M, W, \mathfrak{A})=1$ for any $M\in(L)_G$ and $\mathfrak{A}$, then $|G_L\backslash G/P|\leqq h(K)\cdot h(\mathcal{J}, W^\perp/W)$.*

PROOF. Let $M\underset{G}{\sim}L$, and suppose that their W-decompositions are given by:

$$L=(\sum \mathfrak{A}_i^{-\sigma}\mathcal{J}w_i+\sum \mathfrak{A}_iw_i')\oplus L',$$

$$M=(\sum \mathfrak{B}_i^{-\sigma}\mathcal{J}u_i+\sum \mathfrak{B}_iu_i')\oplus M'.$$

If $L \underset{P}{\sim} M$, then we have $c(L \cap W) = c(M \cap W)$, $L' \underset{G'}{\sim} M'$. The rest is now clear from 3.2, 3.6 and 3.7.

3.9. COROLLARY. *Situation being the same as above, we have:*

1) *If* $\varepsilon = -1$, *then* $|G_L \backslash G/P| = h(K)$,

2) *If* $K = k \times k$, *then* $|G_L \backslash G/P| = h(K)$ *or* $h(k)$ *according as* $\dim V > 2 \dim W$ *or* $= 2 \dim W$.

3) *If* $K = k =$ *a non-dyadic local field whose residue class field is finite, then* $|G_L \backslash G/P| = 1$.

4) *If* k *is a* $\mathfrak{p}$*-adic number field and* K *is its unramified quadratic extension, then* $|G_L \backslash G/P| = 1$.

PROOF. Let U be any non-degenerate F space with $\varepsilon = -1$. Then it was shown by G. Shimura [**9**] that two modular lattices M and N in U are G-equivalent if and only if $\mu_0(M) = \mu_0(N)$. Hence we get (1), because we have $s(M, W, \mathfrak{A}) = 1$ for any $M \in (L)_G$, and any $\mathcal{O}$-ideal $\mathfrak{A}$.

Now let $K = k \times k$, $e = (1, 0)$. We put $eV = V_1$ and $e^\sigma V = V_2$; for any lattice L, let $eL = L_1$ and $e^\sigma L = L_2$. Then $V = V_1 + V_2$, $L = L_1 + L_2$ and L_i is an $\mathcal{O}$-lattice in the k-space V_i $(i = 1, 2)$. Let

$$L_1^* = \{x \in V_2 \mid F(L_1, x) \subset e\mathcal{O}_K\}, \qquad L_2^* = \{y \in V_1 \mid F(y, L_2) \in e\mathcal{O}_K\}.$$

Then $L^* = L_1^* + L_2^*$ [**4**]. Using theorem of elementary divisors, we can find a basis $\{y_i\}$ of V_2 over k, $\mathcal{O}$-ideals $\mathfrak{A}_i$ and $\mathfrak{B}_i$ such that $\mathfrak{B}_i \supset \mathfrak{B}_{i+1}$, and

$$L_1^* = \sum \mathfrak{A}_i y_i, \qquad L_2 = \sum \mathfrak{A}_i \mathfrak{B}_i y_i. \qquad \text{(Cf. [10].)}$$

Further, there exists a basis $\{x_i\}$ of V_1 over k such that

$$L_1 = \sum \mathfrak{A}_i^{-1} x_i \quad \text{and} \quad F(x_i, y_j) = \delta_{ij}. \qquad \text{(Cf. [10].)}$$

From the above, L is $\mathcal{J}$-modular if and only if $\mathfrak{B}_i = \mathcal{J}$ for all i. Further, if L and M are $\mathcal{J}$-modular, then $L \underset{G}{\sim} M$ if and only if $c(L_1) = c(M_1)$ (Cf. Theorem 1.5 of [**4**].) Suppose that L has a W-decomposition:

$$L = (\sum \mathfrak{A}_i^{-\sigma} \mathcal{J} w_i + \sum \mathfrak{A}_i w_i') \oplus L'.$$

We put $N = \sum \mathfrak{A}_i^{-\sigma} \mathcal{J} w_i$, $N' = \mathfrak{A}_i w_i'$. Then we have

$$\begin{aligned} c(eL) &= c(eN) \cdot c(eN') \cdot c(eL') \\ &= \prod c(e\mathfrak{A}_i^{-\sigma}) \cdot c(e\mathcal{J})^s \cdot \prod c(e\mathfrak{A}_i) \cdot c(eL') \\ &= \prod c(\mathfrak{C}_i^{-1}) \cdot c(e\mathcal{J})^s \cdot \prod c(\mathfrak{B}_i) \cdot c(eL'), \quad \text{where} \quad \mathfrak{A}_i = (\mathfrak{B}_i, \mathfrak{C}_i). \end{aligned}$$

It is clear that given any non-degenerate F-space U over K and any element α in the ideal class group of k, there exists an $\mathcal{J}$-modular lattice L in U such that $c(eL) = \alpha$. Now in view of 3.6, our assertion (2) is clear.

(3) is deduced immediately from Corollary 92:2a of [**8**], because now we

have $s(M, W, \mathfrak{A})=1$ for any $M\in(L)_G$ and any $\mathcal{O}$-ideal $\mathfrak{A}$.

(4) follows from proposition 5.2 of [6], because in this case L is normal and $\mathrm{Tr}(\mathfrak{A})\mathcal{O}_K=\mathfrak{A}$ for any $\mathcal{O}_K$-ideal $\mathfrak{A}$.

§ 4. F-spaces over an algebraic number field.

4.1. In this section, we let k be an algebraic number field of finite degree, K be either a quadratic extension of k or k itself, and the F-space V be either a quadratic space over k or a Hermitian vector space over K/k. In virtue of Corollary 3.9 and Proposition 1.5, the order $|G_{A,L}\backslash G_A/P_A|$ is equal to that of $\prod_{\mathfrak{p}}|G_{L_\mathfrak{p}}\backslash G_\mathfrak{p}/P_\mathfrak{p}|$ if L is a modular lattice in V. To proceed further, we need to study the properties of $s(L, W, \mathfrak{A})$.

4.2. PROPOSITION. *Let L be an $\mathcal{O}_K$-lattice in V. Then $s(L, W, \mathfrak{A})$ is finite and does not depend on the choice of $\mathfrak{A}$.*

To prove the above, we need several lemmas.

4.3. LEMMA. *Let L be any $\mathcal{O}_K$-lattice in V, and U be a proper non-zero subspace of V. Then for any prime ideal $\mathfrak{p}$ in k, we have $(L\cap U)_\mathfrak{p}=L_\mathfrak{p}\cap U_\mathfrak{p}$.*

PROOF. It is clear that we have $(L\cap U)_\mathfrak{p}\subset L_\mathfrak{p}\cap U_\mathfrak{p}$. To show the converse, we note that $L\cap U$ is an $\mathcal{O}_K$-lattice in U and at the same time, a direct summand of L; $L=L\cap U\oplus L'$. Hence, we have $L_\mathfrak{p}=(L\cap U)_\mathfrak{p}\oplus L'_\mathfrak{p}$. $(L\cap U)_\mathfrak{p}$ contains a basis of $U_\mathfrak{p}$ over $K_\mathfrak{p}$. Thus it is now clear that $L_\mathfrak{p}\cap U_\mathfrak{p}\subset(L\cap U)_\mathfrak{p}$.

4.4. Returning to a general situation, where k is the quotient field of a Dedekind domain $\mathcal{O}$, and K is either a quadratic extension of k, $k\times k$, or k itself, we define $S(\mathcal{O}_K)=\{N(x)+\mathrm{Tr}(y)\mid x, y\in\mathcal{O}_K\}$. $S(\mathcal{O}_K)$ is a submodule of $\mathcal{O}$.

4.5. LEMMA. *Let k be an algebraic number field of finite degree again, and K be either a quadratic extension of k or k itself. Then we have: $S(\mathcal{O}_K)=\mathcal{O}$ if and only if $S(\mathcal{O}_{K_\mathfrak{p}})=\mathcal{O}_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$ in k.*

PROOF. Let $\mathrm{Tr}(\mathcal{O}_K)=\mathfrak{q}_i^{e_i}\cdots\mathfrak{q}_r^{e_r}$.

To show 'only if' part, it is enough to prove it for $\mathfrak{p}=\mathfrak{q}_i$ $(i=1,\cdots,r)$. Let $N_{k/Q}(\mathfrak{q}_i^{e_i})=q$, and $\{a_j\}$ $(j=1,\cdots,q)$ be a complete system of representatives of $\mathcal{O}/\mathfrak{q}_i^{e_i}$. The hypothesis $S(\mathcal{O}_K)=\mathcal{O}$ implies that we may write $a_j=N(x_j)$ for x_j in $\mathcal{O}_K$ $(j=1,\cdots,q)$. $\{a_j\}$ now forms a complete system of representatives of $\mathcal{O}_\mathfrak{q}/\mathrm{Tr}(\mathcal{O}_{K_\mathfrak{q}})=\mathcal{O}_\mathfrak{q}/\mathfrak{q}_i^{e_i}$, which proves that $S(\mathcal{O}_{K_\mathfrak{q}})=\mathcal{O}_\mathfrak{q}$.

Conversely, suppose that $S(\mathcal{O}_{K_\mathfrak{p}})=\mathcal{O}_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$ in k. Let a be an arbitrary element of $\mathcal{O}$. Then, for $i=1,\cdots,r$, there exist $x_i\in\mathcal{O}_{K_\mathfrak{q}}$ and $y_i\in\mathfrak{q}_i^{e_i}$ such that $a=N(x_i)+y_i$. Let z_i be an element of $\mathcal{O}_K$ such that $z_i\equiv x_i \bmod \mathfrak{Q}_i^{e_i}$ $(\mathfrak{Q}_i=\mathfrak{q}_i\cdot\mathcal{O}_K)$. By Chinese remainder theorem, there exists an element z in $\mathcal{O}_K$ such that $z\equiv z_i \bmod \mathfrak{Q}_i^{e_i}$ for all $i=1,\cdots,r$. Hence, $a\equiv N(z)$ mod $\mathfrak{q}_i^{e_i}$ for all i. This means that $a\equiv N(z) \bmod \mathrm{Tr}(\mathcal{O}_K)$.

4.6. LEMMA. *Let K be a quadratic extension of an algebraic number field k. Then $S(\mathcal{O}_K)=\mathcal{O}$.*

PROOF. In view of the previous lemma, we only have to prove that $S(\mathcal{O}_{K_\mathfrak{p}})=\mathcal{O}_\mathfrak{p}$ for all prime ideal $\mathfrak{p}$ in k. If either $\mathfrak{p}$ does not ramify or 2 is not divisible by $\mathfrak{p}$, we have $\mathrm{Tr}(\mathcal{O}_{K_\mathfrak{p}})=\mathcal{O}_\mathfrak{p}$. Thus we may assume that $\mathfrak{p}$ divides 2 and ramifies in K/k. Let $\mathfrak{p}\mathcal{O}_K=\mathfrak{P}^2$. Then we have $\mathfrak{p}=\mathfrak{P}^2=N(\mathfrak{P})$, $K_\mathfrak{p}=K_\mathfrak{P}$, and the residue class field $k=\mathcal{O}_\mathfrak{p}/\mathfrak{p}$ has the characteristic 2. We put $\mathfrak{P}=(\Pi)$, $N(\Pi)=\pi$. Thus, $\mathfrak{p}=(\pi)$. Let x be an element of $\mathcal{O}_\mathfrak{p}$ such that $\deg_\mathfrak{p}(x)=t$. We then have $x=x_t\pi^t+x_{t+1}\pi^{t+1}+\cdots$, where x_t is an element of $\mathcal{O}_\mathfrak{p}$ which cannot be divided by $\mathfrak{p}$. There exists an element y of $\mathcal{O}_\mathfrak{p}$ such that $y^2\equiv x_t \bmod \mathfrak{p}$. Thus, $x=y^2\pi^t+\cdots=N(y\Pi^t)+\cdots$. Suppose now that $\mathrm{Tr}(\mathcal{O}_\mathfrak{P})=\mathfrak{p}^r$. Then, by what was said above, we can find a suitable set of elements $z_1,\cdots,z_u$ in $\mathcal{O}_{K_\mathfrak{P}}$ such that $x-(N(z_1)+\cdots+N(z_u))\in\mathfrak{p}^r$. This completes the proof.

4.7. PROOF OF 4.2. From the definitions it is clear that $S(\mathcal{O}_K)S(L, W, \mathfrak{A})\subset S(L, W, \mathfrak{A})$ and $S(L, W, \mathfrak{A})S(\mathcal{O}_{K_\mathfrak{p}})\subset S(L_\mathfrak{p}, W_\mathfrak{p}, \mathfrak{A}_\mathfrak{p})$. (To get the latter, we use Lemma 4.3 which allows us to establish $(L\cap W^\perp/L\cap W)_\mathfrak{p}=L_\mathfrak{p}\cap W_\mathfrak{p}^\perp/L_\mathfrak{p}\cap W_\mathfrak{p}$.)

We start with the case where $K=k$. The modules $S(L, \mathfrak{A})$ and $S(L, W, \mathfrak{A})$ are both Z-lattices in k which is now considered as a vector space over the field of rational numbers Q. This implies that $s(L, W, \mathfrak{A})=\prod[S(L, \mathfrak{A})_p/S(L, W, \mathfrak{A})_p]<\infty$ (p runs over the prime numbers). In virtue of Lemma 4.3, we have $(L\cap W^\perp/L\cap W)_p=L_p\cap(W^\perp)_p/L_p\cap W_p$. Hence, $S(L_p, W_p, \mathfrak{A}_p)\subset S(L, W, \mathfrak{A})_p$. While $S(Z)=Z$ implies that $S(L, W, \mathfrak{A})_p=S(L, W, \mathfrak{A})\otimes_Z Z_p=S(L, W, \mathfrak{A})\otimes S(Z_p)\subset S(L_p, W_p, \mathfrak{A}_p)$. Thus we get $s(L, W, \mathfrak{A})=\prod s(L_p, W_p, \mathfrak{A}_p)=s(L_2, W_2, \mathfrak{A}_2)$. Let $k_2=K'\times\cdots\times K''$, where $K',\cdots,K''$ are finite extensions of 2-adic number field. Then we have $\mathfrak{A}_2=(a')\times\cdots\times(a'')$ and $S(L_2, W_2, \mathfrak{A}_2)=(a'^2,\cdots,a''^2)S(L_2, W_2, \mathcal{O}_2)$, $S(L_2, \mathfrak{A}_2)=(a'^2,\cdots,a''^2)S(L_2, \mathcal{O}_2)$. Hence, $s(L_2, W_2, \mathfrak{A}_2)=s(L_2, W_2, \mathcal{O}_2)$. This completes the proof for the case $K=k$.

When K is a quadratic extension of k, $S(L, W, \mathfrak{A})$ is an $\mathcal{O}$-ideal in k and we further have $S(L, W, \mathfrak{A})_\mathfrak{p}=S(L_\mathfrak{p}, W_\mathfrak{p}, \mathfrak{A}_\mathfrak{p})$ for any prime ideal $\mathfrak{p}$ in k. Hence, we now have $s(L, W, \mathfrak{A})=\prod s(L_\mathfrak{p}, W_\mathfrak{p}, \mathfrak{A}_\mathfrak{p})$ where $\mathfrak{p}$ runs over the ramifying prime ideals dividing 2 (see 3.9 and the proof of 4.6). For such a prime ideal $\mathfrak{p}$, let $\mathfrak{A}_\mathfrak{p}=(a)$. Then we have $S(L_\mathfrak{p}, W_\mathfrak{p}, \mathfrak{A}_\mathfrak{p})=N(a)S(L_\mathfrak{p}, W_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})$, $S(L_\mathfrak{p}, \mathfrak{A}_\mathfrak{p})=N(a)S(L_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})$. Hence, $s(L_\mathfrak{p}, W_\mathfrak{p}, \mathfrak{A}_\mathfrak{p})=s(L_\mathfrak{p}, W_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})$ which implies that $s(L, W, \mathfrak{A})=s(L, W, \mathcal{O}_K)$.

4.8. From now on, we write $s(L, W)$ instead of $s(L, W, \mathfrak{A})$, and $s(L_\mathfrak{p}, W_\mathfrak{p})$ instead of $s(L_\mathfrak{p}, W_\mathfrak{p}, \mathfrak{A}_\mathfrak{p})$.

We note here that $S(L, W, \mathfrak{A})=S(L', \mathfrak{A})$, where $L'=L\cap W^\perp/L\cap W$ (provided that $\mu_0(L)=\mu_0(L')$).

We note also that in case $S(\mathcal{O}_K)=\mathcal{O}$, the $\mathcal{O}$-ideal $S(L, \mathfrak{A})$ may be identified with $\mu(L)N(\mathfrak{A}^{-1})$.

It is known that two modular lattices $L_\mathfrak{p}$ and $M_\mathfrak{p}$ in $V_\mathfrak{p}$ are $G_\mathfrak{p}$-equivalent if and only if $\mu_0(L_\mathfrak{p})=\mu_0(M_\mathfrak{p})$ and $S(L_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})=S(M_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})$. (Cf. Corollary 92:2a and Theorem 93:28 of [8], Proposition 5.2 of [6].)

4.9. PROPOSITION. *Let L be an $\mathcal{O}_K$-lattice in V. Then*

1) $s(L, W)\leqq |\mu(L)\cap k/\mathrm{Tr}(\mu_0(L))|$,

2) $s(L, W)\begin{cases}\leqq 2^d, \text{ if } K=k \ (d=\text{the degree of } k),\\ \leqq (N_{K/Q}(\delta))^{\frac{1}{2}}, \text{ if } K \text{ is a quadratic extension of } k \ (\delta=\text{the differente of } K/k).\end{cases}$

PROOF. (1) is obvious from definitions. Now, we generally have:

$$S(L, W, \mathcal{O}_K)\mathcal{O}_K\supset \mathrm{Tr}(\mu_0(L))\mathcal{O}_K\supset \delta\cdot\mu_0(L)\supset\delta\cdot S(L, \mathcal{O}_K).$$

If $K=k$, this implies (2). In the case K is a quadratic extension of k, we have:

$$\begin{aligned}&[S(L, \mathcal{O}_K)\mathcal{O}_K:\ S(L, W, \mathcal{O}_K)\mathcal{O}_K][S(L, W, \mathcal{O}_K)\mathcal{O}_K:\ \delta S(L, \mathcal{O}_K)]\\ &\quad=[S(L, \mathcal{O}_K)\mathcal{O}_K:\ \delta S(L, \mathcal{O}_K)]\\ &\quad=N_{K/Q}(\delta).\end{aligned}$$

While we have

$$[S(L, \mathcal{O}_K)\mathcal{O}_K:\ S(L, W, \mathcal{O}_K)\mathcal{O}_K]=s(L, W)^2.$$

Hence we get the desired result.

4.10. It is known that the order $h(\mathcal{J}, W^\perp/W)$ is finite. (Cf. Corollary 92:2a, Theorem 93:28 and Theorem 103:4 of [8]; 6.5 of [6].) Hence, in virtue of Proposition 3.8, we have:

$$|G_L\backslash G/P|\leqq h(K)\cdot h(\mathcal{J}, W^\perp/W)\cdot|\mu(L)\cap k/\mathrm{Tr}(\mathcal{J})|^s, \qquad s=\dim W.$$

Moreover, if we have $h(\mathcal{J}, V)=1$, then

$$h(K)\cdot h(\mathcal{J}, W^\perp/W)\leqq |G_L\backslash G/P|.$$

4.11. Let us evaluate the order $|G_{A,L}\backslash G_A/P_A|$ for an $\mathcal{J}$-modular lattice L in V. For this purpose it is enough to evaluate the order $|G_{L_\mathfrak{p}}\backslash G_\mathfrak{p}/P_\mathfrak{p}|$ for all the prime ideals $\mathfrak{p}$ in k.

Let us fix a prime ideal $\mathfrak{p}$ in k. Firstly, let us assume that $s(M_\mathfrak{p}, W_\mathfrak{p})=1$ for any lattice $M_\mathfrak{p}$ such that $L_\mathfrak{p}\underset{G_\mathfrak{p}}{\sim}M_\mathfrak{p}$. We then have $S(L_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})=S(M_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})$ $=S(L'_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})=S(M'_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})$ $(L'_\mathfrak{p}=L_\mathfrak{p}\cap W_\mathfrak{p}^\perp/L_\mathfrak{p}\cap W_\mathfrak{p},\ M'_\mathfrak{p}=M_\mathfrak{p}\cap W_\mathfrak{p}^\perp/M_\mathfrak{p}\cap W_\mathfrak{p})$. (Cf. 4.8.) This implies that $L'_\mathfrak{p}\underset{G'_\mathfrak{p}}{\sim}M'_\mathfrak{p}$. Hence, in this case, we have $|G_{L_\mathfrak{p}}\backslash G_\mathfrak{p}/P_\mathfrak{p}|=1$. In general, if $M_\mathfrak{p}$ is $G_\mathfrak{p}$-equivalent to $L_\mathfrak{p}$, we have $S(M_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})=S(L_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})\supset\mathrm{Tr}(\mathcal{J}_\mathfrak{p})$. Hence, as in 4.10, we have: $|G_{L_\mathfrak{p}}\backslash G_\mathfrak{p}/P_\mathfrak{p}|\leqq h(\mathcal{J}_\mathfrak{p}, W_\mathfrak{p}^\perp/W_\mathfrak{p})\cdot t^s$, where $t=|S(L_\mathfrak{p}, \mathcal{O}_{K_\mathfrak{p}})/\mathrm{Tr}(\mathcal{J}_\mathfrak{p})|$ $(h(\mathcal{J}_\mathfrak{p}, W_\mathfrak{p}^\perp/W_\mathfrak{p})$ is finite.). Thus we have:

1) $|G_{A,L}\backslash G_A/P_A|$ is finite,

2) $|G_{A,L}\backslash G_A/P_A| \leqq \prod_{\mathfrak{p}} h(\mathcal{J}_{\mathfrak{p}}, W_{\mathfrak{p}}^{\perp}/W_{\mathfrak{p}}) \cdot |S(L_{\mathfrak{p}}, \mathcal{O}_{K_{\mathfrak{p}}})/\mathrm{Tr}(\mathcal{J}_{\mathfrak{p}})|^s$, where $\mathfrak{p}$ runs over the prime ideals for which $s(L_{\mathfrak{p}}, W_{\mathfrak{p}}) \neq 1$ (the number of such prime ideals are finite (cf. 3.9)).

3) $|G_{A,L}\backslash G_A/P_A| = 1$ if $s(M_{\mathfrak{p}}, W_{\mathfrak{p}}) = 1$ for all lattice M belonging to the same genus as L and for all prime ideals $\mathfrak{p}$ in k.

4.12. We proceed to evaluate the order $|P_{A,L}\backslash P_A/P|$ for an $\mathcal{J}$-modular lattice L. Firstly, we will show that given any ideal class α, there exists an $\mathcal{J}$-modular lattice M such that $c(M \cap W) = \alpha$, and $(M_A)_{P_A} = (L_A)_{P_A}$. Let $L = \left(\sum_{i=1}^{s} \mathfrak{A}_i^{-\sigma}\mathcal{J} \cdot w_i + \sum_{i=1}^{s} \mathfrak{A}_i w_i'\right) \oplus L'$ be a W-decomposition of L. There exists, by Theorem of Dirichlet, a prime ideal $\mathfrak{P}_0$ in K such that $\mathfrak{P}_0$ does not divide δ and $c(\mathfrak{P}_0^{-\sigma}) \cdot c(L \cap W) = \alpha$. Put $M = \mathfrak{A}_1^{-\sigma}\mathfrak{P}_0^{-\sigma}\mathcal{J}w_1 + \mathfrak{A}_1\mathfrak{P}_0 w_1' + \sum_{i=2}^{s}(\mathfrak{A}_i^{-\sigma}\mathcal{J} \cdot w_i + \mathfrak{A}_i w_i') \oplus L' = M'' \oplus L'$. Then $c(M \cap W) = \alpha$; M'' is an $\mathcal{J}$-modular lattice and $S(M_{\mathfrak{p}}'', \mathcal{O}_{K_{\mathfrak{p}}}) = S(L_{\mathfrak{p}}'', \mathcal{O}_{K_{\mathfrak{p}}}) = \mathcal{J}_{\mathfrak{p}} \cap \mathcal{O}\left(L'' = \sum_{i=1}^{s}(\mathfrak{A}_i^{-\sigma}\mathcal{J}w_i + \mathfrak{A}_i w_i')\right)$ for any prime ideal $\mathfrak{p}$ dividing $\mathfrak{P}_0$. For a prime ideal $\mathfrak{q}$ which does not divide $\mathfrak{P}_0$, we have $M_{\mathfrak{q}}'' = L_{\mathfrak{q}}''$. Hence we have $M_{\mathfrak{p}} \underset{P_{\mathfrak{p}}}{\sim} L_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p}$ in k, which means that $(M_A)_{P_A} = (L_A)_{P_A}$.

On the other hand, if $(L_A)_{P_A} = (M_A)_{P_A}$, then the lattices $L \cap W^{\perp}/L \cap W$ and $M \cap W^{\perp}/M \cap W$ belong to the same G'-genus. From this, we get the following proposition:

4.13. PROPOSITION. *Let L be an $\mathcal{J}$-modular lattice in V, and $L' = L \cap W^{\perp}/L \cap W$. Then we have:*

1) *$h(K) \cdot h(L') \leqq |P_{A,L}\backslash P_A/P|$, where $h(L')$ is the class number of L' with respect to G'.*

2) *Suppose that we have the following condition:*

(B) *If $(L_A)_{P_A} = (M_A)_{P_A}$, then $S(L, \mathfrak{A}) = S(M, \mathfrak{A})$ and $S(L, W, \mathfrak{A}) = S(M, W, \mathfrak{A})$ for any $\mathcal{O}_K$-ideal $\mathfrak{A}$.*

Then we have:

$$|P_{A,L}\backslash P_A/P| \leqq h(K) \cdot h(L') \cdot s(L, W)^s,$$

where $s = \dim W$.

4.14. We note that the above condition (B) is automatically satisfied if $S(\mathcal{O}_K) = \mathcal{O}$.

Combining 1.7, 4.9, 4.10, 4.11 and 4.13, we get the following theorem:

4.15. THEOREM. *Let L be an $\mathcal{J}$-modular lattice in V and $L' = L \cap W^{\perp}/L \cap W$. Let $m = |G_{A,L}\backslash G_A/P_A|$. Then we have:*

1) *Suppose that the conditions (A) and (B) hold (cf. 1.7 and 4.13). Then*

$$|G_L\backslash G/P| \leqq m \cdot h(K) \cdot h(L') \cdot s(L, W)^s \begin{cases} \leqq m \cdot h(K) \cdot h(L') \cdot 2^{sd}, & \text{if } K = k,\ d = \deg k, \\ \leqq m \cdot h(K) \cdot h(L') \cdot N_{K/Q}(\delta)^{s/2}, & \text{if } [K:k] = 2, \end{cases}$$

where $s=\dim W$.

2) *Suppose that* $S(\mathcal{O}_K)=\mathcal{O}$ *and that* $s(M, W)=1$ *for all* M *belonging to the same genus as* L. *Then we have* $m=1$ *and*

$$|G_L\backslash G/P|\leqq h(K)\cdot h(L')\,.$$

If, moreover we have $h(\mathcal{J}, V)=1$, *then we have*

$$|G_L\backslash G/P|=h(K)\cdot h(L')\,.$$

4.16. The situation described in (2) of the above theorem exists, for example, in the following cases:

1) $K=k$: in this case, if $\dim V$ is odd, $S(\mathcal{O})=\mathcal{O}$, and $h(k)=1$, then $s(M, W)=1$ for all M belonging to the same genus as L, and $h(\mathcal{J}, V)=1$.

To see this, we note that the lattice L is normal (cf. 91C of [8]). Hence, $S(L, \mathcal{O})=\mu(L)=\mathcal{J}$. While we have $S(L, W, \mathcal{O})=S(L', \mathcal{O})=\mu(L')=\mathcal{J}$, because L' is again normal ($\dim W^{\perp}/W$ is odd). Hence, $s(L, W)=1$. The same argument is applicable for any modular lattice M, and we get $s(M, W)=1$. On the other hand, the existence of a proper totally isotropic subspace W implies that V is indefinite and $\dim V\geqq 3$. Hence, in virtue of 4.8, the statement $h(\mathcal{J}, V)=1$ is a direct result of Theorem 104:9 and Example 102:10 of [8].

2) $[K:k]=2$: in this case, if $\dim V$ is odd and any ideal class in K can be represented by a σ-invariant ideal, then $s(M, W)=1$ for all M belonging to the same genus as L, and $h(\mathcal{J}, V)=1$.

Again, we use the fact that both L and L' are normal (cf. 4.3 of [6]). Then the above statement is a direct result of 5.2 and 6.7 of [6].

4.17. For an arbitrary subgroup H of $GL(V)$, we let H^1 be the group $H\cap SL(V)$. We are now going to evaluate the order $|G_L^1\backslash G^1/P^1|$ for a modular lattice L in V. We will start by looking at the relation of $(W)_G$ and $(W)_{G^1}$.

Given an arbitrary subgroup H of $GL(V)$, we put $C(H)=\{\det(g)\,|\,g\in H\}$. If W' is an element of $(W)_G$, there exists an element g of G such that $W'=g(W)$. And now, the coset $\det(g)\cdot C(P)$ does not depend on the choice of g. We set $f(W')=\det(g)\cdot C(P)$. Then it is easy to see that elements W' and W'' of $(W)_G$ are G^1-equivalent if and only if $f(W')=f(W'')$. Thus there exists a bijective correspondence between the family of G^1-equivalence classes of subspaces belonging to $(W)_G$ and the coset space $C(G)/C(P)$.

If $\dim V>2\cdot\dim W$, then it is clear that $C(P)=C(G)$. If, on the other hand, $\dim V=2\cdot\dim W$, then V has a basis $\{w_1, \cdots, w_s;\ w_1', \cdots, w_s'\}$ such that $W=\sum_{i=1}^{s}Kw_i$, $F(w_i, w_j')=\delta_{ij}$, $F(w_i')=0$. With respect to this basis, the element g of P has the following form:

$g=\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$; A, B, and C are $s\times s$ matrices such that ${}^tA^\sigma\cdot C=1$, ${}^tC^\sigma\cdot B+{}^tB^\sigma\cdot C=0$. In particular, the element of the form $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-\sigma} \end{pmatrix}$ belongs to P. Hence, $C(P)=\{1\}$ if $K=k$; while $C(P)=\{x^{1-\sigma}\mid x\in K^*\}$ if $[K:k]=2$. In the latter case, $C(G)=\{x\in K\mid N(x)=1\}$ and it coincides with $C(P)$, because the first Galois cohomology group of K^* is trivial.

4.18. The order $|C_L(G)/C_L(P)|$ is finite. In fact, if $K=k$, then $C_L(G)\subset C(G)=\{\pm 1\}$, and $c(G, L, W)\leqq 2$. Likewise, in this case, we have $c(G, L, W')\leqq 2$ for any element W' of $(W)_G$.

Next, let $[K:k]=2$, and W' be any element of $(W)_G$. Then we have $C_L(G)\subset\{u\in U_K\mid N(u)=1\}$; and $C_L(G)\supset C_L(G_{W'})\supset C_L(G)^n$, where $n=\dim V$. (Because, if u is an element of $C_L(G)$, then the scalar transformation $u\cdot 1$ belongs to $G_{W'}\cap G_L$.) Hence, we have an onto mapping $\varphi: C_L(G)/C_L(G)^n\to C_L(G)/C_L(G_{W'})$. While the group $C_L(G)/C_L(G)^n$ may be regarded as a subgroup of a finite group U_K/U_K^n. Hence, the order $|C_L(G)/C_L(G_{W'})|=c(G, L, W')$ is finite.

Suppose that L has a W'-decomposition:

$$L=\Big(\sum_{i=1}^{s}\mathfrak{A}_i^{-\sigma}\mathfrak{g}w_i+\sum_{i=1}^{s}\mathfrak{A}_i w_i'\Big)\oplus L'.$$

If L' has an orthogonal summand of rank 1, then it is obvious that $C_L(G_{W'})=C_L(G)$. If, on the other hand, $[K:k]=2$ and we have $F(w_1')=0$ in the above decomposition, then, for any element u in U_K, there exists an element g of G with determinant $u^{1-\sigma}$, defined by $g(w_1)=u\cdot w_1$, $g(w_1')=u^{-\sigma}\cdot w_1'$, $g\mid\{w_1, w_1'\}^\perp=1$. Thus, in this case, the group $C_L(G)/C_L(G_{W'})$ is a subgroup of the first Galois cohomology group $H^1(G(K/k), U_K)$, where $G(K/k)$ is the Galois group of K/k. The order of the group $H^1(G(K/k), U_K)$ is known to be finite and will be denoted by $h(K/k)$ (cf. 1.7 of [6]). Thus, if we have $s(L, W')=1$, then the order $c(G, L, W')$ divides $h(K/k)$.

4.19. THEOREM. *Let L be a modular lattice in V, and $W_1,\cdots, W_r$ be a system of representatives of $(W)_G/\underset{G_L}{\sim}$. Then we have:*

1) $(W)_G=(W)_{G^1}$ *except when* $K=k$, $\dim V=2\cdot\dim W$; *and in the latter case,* $(W)_G=(W)_{G^1}\cup(g_0(W))_{G^1}$ *(disjoint), where g_0 is any element of G whose determinant is* -1.

2) *In general, we have* $|G_L^1\backslash G^1/P^1|\leqq\sum_{i=1}^{r}c(G, L, W_i)$; *in particular, if* $(W)_G=(W)_{G^1}$, *then* $|G_L^1\backslash G^1/P^1|=\sum_{i=1}^{r}c(G, L, W_i)$.

3) *If* $G=G_L\cdot G^1\cdot P$, *then we have* $|G_L^1\backslash G^1/P^1|\cdot|C(G)/C(P)|=\sum_{i=1}^{r}c(G, L, W_i)$.

4) *If* $[K:k]=2$, *and* $\dim V$ *is odd, then* $r\leqq|G_L^1\backslash G^1/P^1|\leqq r\cdot h(K/k)$.

PROOF. (1) was shown in 4.17. (2) is clear from Proposition 1.3 and its

proof. To see (3), we notice that our assumption $G=G_L\cdot G^1\cdot P$ implies that $C(G)=C(G_L)\cdot C(P)$. Hence, each G^1-equivalence class of subspaces belonging to $(W)_G$ can be written as $(g(W))_{G^1}$ with an element g of G_L. Fixing such an element g of G_L, we note that each element of $(g(W))_{G^1}$ can be written as $g(W')$ where W' is an element of $(W)_{G^1}$. Let W' and W'' be elements of $(W)_{G^1}$. Then $W'\underset{G_L^1}{\sim}W''$ if and only if $g(W')\underset{G_L^1}{\sim}g(W'')$ (because, $g\in G_L$). This implies (3). (4) is a direct outcome of 4.17, because the lattice L is now normal (cf. 4.8).

4.20. In the above, if we have $C(G)=C(G_L)$, then we have $G=G_L\cdot G^1$; hence, in this case, the assumption of (3) of the above theorem is satisfied.

Suppose that $K=k$. If the lattice L has an orthogonal summand of rank 1, then it is clear that $C(G)=C(G_L)=\{\pm 1\}$. Similarly, if L has an orthogonal summand M such that $M=\mathfrak{A}^{-1}w+\mathfrak{A}w'$; $\mathfrak{A}=(a)$, $F(w, w')=1$, $F(w)=F(w')=0$, then we can show that $C(G)=C(G_L)$. In fact, let $v=a^{-1}w+aw'$, and let g be an element of $GL(V)$ defined by $g(x)=x-2\dfrac{F(x, v)}{F(v)}v$, for an arbitrary element x of V. Then we have $v\in L$, $F(v)=2$, and $g\in G$. The element g is a reflection with respect to v, hence, $\det(g)=-1$. Moreover, if x is an element of L, then $F(x, v)\in\mathcal{O}$, hence, $g(x)\in L$. As $g^2=1$, we now have $g(L)=L$. Hence, $C(G)=C(G_L)$.

§ 5. Elementary cases.

5.1. In this section, we consider special cases of the situation studied in the previous section. So, k is an algebraic number field of finite degree, and K is either a quadratic extension of k or k itself. The F-space V is either a quadratic vector space over k, or a Hermitian vector space over K/k; W is a proper totally isotropic subspace of V, and L is an $\mathcal{O}_K$-lattice in V.

The 4-tuple $(K/k, V, W, L)$ is said to satisfy the condition (C) if it satisfies either one of the following conditions:

$(C\ 1)$ $\qquad \dim V=2\dim W+1,\ S(\mathcal{O}_K)=\mathcal{O}$,

$(C\ 2)$ $\qquad \dim V=2\dim W,\ \mu(L)=\mathcal{O}_K\operatorname{Tr}\mu_0(L)$.

If the condition $(C\ 1)$ (resp. $(C\ 2)$) is satisfied, we say that we are in the case 1 (resp. case 2). We call the cases 1 and 2 the 'elementary cases'.

We note that if lattices L and M belong to the same G-genus, then $(K/k, V, W, L)$ satisfies (C) if and only if $(K/k, V, W, M)$ satisfies the same condition (C). Also, if $[K:k]=2$, and $\dim V=2\cdot\dim W+1$, then we are automatically in the case 1 (cf. 4.6).

5.2. LEMMA. *Suppose we are in an elementary case. Then* $s(L, W)=1$.

PROOF. Case 1: In this case, as L is normal, we have $\mu(L)=\mu_0(L)$. Identifying $\mu(L)$ with $\mu(L)\cap k$, we may put $S(L, \mathfrak{A})=\mu(L)N(\mathfrak{A}^{-1})=\mu(L')N(\mathfrak{A}^{-1})$ $=S(L, W, \mathfrak{A})$, where $L'=L\cap W^{\perp}/L\cap W$ (cf. 4.8).

Case 2: By definition, we have

$$\mathrm{Tr}\,(\mathfrak{A}^{-1-\sigma}\mu_0(L))\subset S(L, W, \mathfrak{A})\subset S(L, \mathfrak{A})\subset \mathrm{Tr}\,(\mathfrak{A}^{-1-\sigma}\mu_0(L))\,.$$

This completes the proof.

5.3. To proceed further, it is necessary to look at the order $h(L')$ for a lattice L' of rank 1. Let U be an F-space of dimension 1, L' an $\mathcal{O}_K$-lattice in U, and G' the F-group on U. The space U can be identified with K, and under this identification, the lattice L' can be regarded as an $\mathcal{O}_K$-ideal and the group G' can be regarded as the set $\{x\in K\,|\,N(x)=1\}$. Thus, if $K=k$, we may put $G'=\{\pm 1\}$. Hence, in this case, two lattices L' and M' are in the same G'-genus if and only if $L'=M'$. Thus, in this case, we have $h(L')$ $=1$.

Next, suppose that $[K:k]=2$. $\mathcal{O}_K$-ideals $\mathfrak{A}$ and $\mathfrak{B}$ will be called G'-equivalent if the corresponding lattices are G'-equivalent; $\mathfrak{A}$ and $\mathfrak{B}$ are in the same G'-genus if the corresponding lattices are in the same G'-genus. If $\mathfrak{A}$ and $\mathfrak{B}$ are G'-equivalent, then $N(\mathfrak{A})=N(\mathfrak{B})$.

5.4. LEMMA. *Suppose $[K:k]=2$. Then, $\mathcal{O}_K$-ideals $\mathfrak{A}$ and $\mathfrak{B}$ belong to the same G'-genus if and only if $N(\mathfrak{A})=N(\mathfrak{B})$.*

PROOF. 'Only if' part is obvious. To prove 'if' part, suppose that $N(\mathfrak{A})=N(\mathfrak{B})$. Given any prime ideal $\mathfrak{p}$ in k, there exists an element $x_{\mathfrak{p}}$ in $K_{\mathfrak{p}}$ such that $\mathfrak{A}_{\mathfrak{p}}=x_{\mathfrak{p}}\mathfrak{B}_{\mathfrak{p}}$. Our hypothesis implies that $N(x_{\mathfrak{p}})\in U_{\mathfrak{p}}=$ the group of units in $\mathcal{O}_{\mathfrak{p}}$. Now, it is easy to see that $N(K_{\mathfrak{p}})\cap U_{\mathfrak{p}}=N(U_{K_{\mathfrak{p}}})$, where $U_{K_{\mathfrak{p}}}=$ the group of units in $\mathcal{O}_{K_{\mathfrak{p}}}$. Hence, there exists an element $a_{\mathfrak{p}}\in U_{K_{\mathfrak{p}}}$ such that $N(x_{\mathfrak{p}})=N(a_{\mathfrak{p}})$. Therefore, $\mathfrak{A}_{\mathfrak{p}}=x_{\mathfrak{p}}a_{\mathfrak{p}}^{-1}\mathfrak{B}_{\mathfrak{p}}$, $N(x_{\mathfrak{p}}a_{\mathfrak{p}}^{-1})=1$. Thus $\mathfrak{A}$ and $\mathfrak{B}$ belong to the same G'-genus.

5.5. LEMMA. *Suppose $[K:k]=2$. Let $C(K)$ be the group of ideal classes in K, and let $C_0(K)$ be the subgroup of $C(K)$ consisting of σ-invariant elements. Let $\mathfrak{A}$ be an $\mathcal{O}_K$-ideal in K. Then the order $g(K/k)$ of G'-equivalence classes among the ideals belonging to the same G'-genus as $\mathfrak{A}$ is finite, and it is equal to $[N(K^*)\cap U_k: N(U_k)]\cdot[C(K):C_0(K)]$. In particular, $g(K/k)$ does not depend on the choice of the ideal $\mathfrak{A}$.*

PROOF. Firstly, let us note that $\mathcal{O}_K$-ideals $\mathfrak{A}$ and $\mathfrak{B}$ are G'-equivalent if and only if there exists an element x in K such that $\mathfrak{A}=x\mathfrak{B}$, and $N(x)\in N(U_K)$.

In fact, if that is the case, then there exists an element y of U_K such that $N(x)=N(y)$. We have $\mathfrak{A}=x\cdot y^{-1}\mathfrak{B}$, and $N(x\cdot y^{-1})=1$.

Next, we put $A=$ the set of ideals belonging to the same G'-genus as $\mathfrak{A}$; and for an arbitrary element α of $C(K)$, we put $A_\alpha=$ the subset of A consisting of the ideals whose class is α. We shall prove that 1) if A_α is not

empty, then the order of G'-equivalence classes among the ideals belonging to A_α is finite and equal to $[N(K^*)\cap U_k : N(U_k)]$; 2) the number of ideal classes such that A_α is not empty is equal to $[C(K) : C_0(K)]$.

1) Suppose that A_α is not empty. We take an element $\mathfrak{A}_0$ of A_α and fix it. For any element $\mathfrak{B}$ of A_α, there exists an element c of K^* such that $\mathfrak{B}=c\mathfrak{A}_0$. We put $f(\mathfrak{B})=N(c)\cdot N(U_K)$. It is easy to see that f is well-defined, and we get a mapping $f: A_\alpha \to N(K^*)\cap U_k/N(U_K)$. From the remark made in the beginning of this proof, it is easy to see that two ideals $\mathfrak{B}_1$ and $\mathfrak{B}_2$ belonging to A_α are G'-equivalent if and only if $f(\mathfrak{B}_1)=f(\mathfrak{B}_2)$. It is also easy to see that the mapping f is onto. Hence, there exists a bijection between the set of G'-equivalence classes of ideals belonging to A_α and the quotient set $N(K^*)\cap U_k/N(U_K)$. The latter quotient set is a finite abelian group of type $(2, 2, \cdots, 2)$ because the square of any element belonging to $N(K^*)\cap U_k$ belongs to $N(U_K)$.

2) For ideals $\mathfrak{A}$ and $\mathfrak{B}$, we have $N(\mathfrak{A})=N(\mathfrak{B})$ if and only if there exists an ideal $\mathfrak{C}$ such that $\mathfrak{A}\cdot\mathfrak{B}^{-1}=\mathfrak{C}^{1-\sigma}$. The number of ideal classes α such that A_α is not empty is equal to the number of the set $\{c(\mathfrak{A}\cdot\mathfrak{B}^{-1})\,|\,N(\mathfrak{A})=N(\mathfrak{B})\}$. The latter set is equal to $\{c(\mathfrak{C}^{1-\sigma})\,|\,\mathfrak{C}$ is an arbitrary $\mathcal{O}_K$-ideal$\}=\{x^{1-\sigma}\,|\,x\in C(K)\}$, whose order is equal to $[C(K) : C_0(K)]$. This completes the proof.

5.6. In view of 4.8 and the proof of 5.2, the number of G-genus among $\mathcal{J}$-modular lattices in a space satisfying the condition (C) is one. Hence, if L is an $\mathcal{J}$-modular lattice in such a space V, then $h(L)=h(\mathcal{J}, V)$.

In particular, if $[K : k]=2$, then it is known that

$$h(\mathcal{J}, V)=[C(K) : C'(K)] \quad \text{or} \quad [C(K) : C'(K)]\cdot[\underline{C}(L) : \underline{U}(L)]$$

according as $\dim V$ is odd or even, where $C'(K)$ is the subgroup of $C(K)$ consisting of the classes containing ideals which are σ-invariant; and

$$\underline{C}(L)=\prod_{\mathfrak{q}} C(G_{\mathfrak{q}})/C_{L_{\mathfrak{q}}}(G_{\mathfrak{q}}),$$

where $\mathfrak{q}$ runs over the ramifying prime ideals in k;

$$\underline{U}(L)=\{(a_{\mathfrak{q}})\in\underline{C}(L)\,|\,a_{\mathfrak{q}}=u\cdot C_{L_{\mathfrak{q}}}(G_{\mathfrak{q}}),\ \text{for } u\in U_K,\ N(u)=1\}.$$

(Cf. [6], [10]).

Hence, if $h(\mathcal{J}, V)=1$, then $g(K/k)=[N(K^*)\cap U_k : N(U_K)]$. Also it is known that if $h(K)=1$, then $N(K^*)\cap U_k=N(U_K)$ (cf. Lemma 2.7 of [7]). Thus, if $h(K)=1$, then $g(K/k)=1$.

5.7. THEOREM. *Suppose that we are in an elementary case. Then, for an $\mathcal{J}$-modular lattice L in V,*

1) $|G_{A,L}\backslash G_A/P_A|=1$,

2) $|P_{A,L}\backslash P_A/P|=\begin{cases} h(K), & \textit{if we are in the case 2, or if } K=k, \\ h(K)\cdot g(K/k), & \textit{otherwise.}\end{cases}$

3) *Generally, we have* $|G_L\backslash G/P| \leqq \begin{cases} h(K), & \text{if we are in the case 2, or } K=k, \\ h(K)\cdot g(K/k), & \text{otherwise.} \end{cases}$

In particular, if $h(\mathcal{J}, V)=1$, *then* $|G_L\backslash G/P|$

$$= \begin{cases} h(K), & \text{if we are in the case 2 or } K=k, \\ h(K)\cdot[N(K^*)\cap U_k : N(U_K)], & \text{otherwise.} \end{cases}$$

If, furthermore, $h(K)=1$, *then* $|G_L\backslash G/P|=1$.

PROOF. 1), 2) and 3) are direct results of 4.11, 4.13, and 3.8, 5.6 respectively.

5.8. THEOREM. *Suppose that we are in an elementary case. Let* L *be an* $\mathcal{J}$*-modular lattice in* V, *and let* $|G_L\backslash G/P|=r$. *Then we have,*

1) *if we are in the case* 1, *then* $|G_L^1\backslash G^1/P^1|=r$,

2) *if we are in the case* 2, *then we have,*

$$|G_L^1\backslash G^1/P^1| \leqq \begin{cases} 2r, & \text{if } K=k, \\ r\cdot h(K/k), & \text{if } [K:k]=2. \end{cases}$$

PROOF. This is a direct result of 4.18, 4.19, and 5.2.

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