J. Math. Soc. Japan
Vol. 23, No. 1, 1971

Orbits of one-parameter groups III (Lie group case)

By Morikuni GOTO[1)]

(Received April 21, 1970)

§ 1. Introduction

This paper is a continuation of the previous paper, [5] M. Goto, *Orbits of one-parameter groups* II, which will be quoted as *Orbits* II, and the main purpose of this paper is to prove the following theorem.

THEOREM. *Let $\mathcal{G}$ be a Lie group. Let $\mathcal{L}$ be an analytic subgroup, and let $\mathcal{X}$ be a one-parameter subgroup, of $\mathcal{G}$. Then either*

(a) *$\mathcal{X}$ is a closed straight line and $\mathcal{X}\bar{\mathcal{L}}$ is topologically the same as the direct product $\mathcal{X}\times\bar{\mathcal{L}}$, or*

(b) *We can give a toral group structure to the set $\overline{\mathcal{X}\mathcal{L}}/\bar{\mathcal{L}}$ such that $\mathcal{X}\bar{\mathcal{L}}/\bar{\mathcal{L}}$ becomes an everywhere dense one-parameter subgroup in it.*

The theorem was proved for the general linear group $\mathcal{GL}(n, R)$, in a slightly weaker form (Theorem 1 in *Orbits* II), and it can be applied for all analytic subgroups of $\mathcal{GL}(n, R)$[2)]. However, in order to prove the theorem for a closed analytic subgroup $\mathcal{G}$ of $\mathcal{GL}(n, R)$, we need some groups which are not in $\mathcal{G}$, but in the algebraic hull of $\mathcal{G}$.

Hence in order to extend the method in *Orbits* II to general analytic groups, it was necessary to find a suitable analytic group $\mathcal{S}$ which contains the given $\mathcal{G}$ and all the groups which appear in the process of the proof. For the purpose, we introduce the notion of *semi-algebraic* subgroups of $\mathcal{GL}(n, R)$ and *adjoint semi-algebraic* analytic groups in § 2. For a given analytic group $\mathcal{G}$, we can find an adjoint semi-algebraic group $\mathcal{S}$ which contains $\mathcal{G}$ as a closed normal subgroup by (3.4). Thus, roughly speaking, by considering the adjoint representation of $\mathcal{S}$, we can reduce the problem into the case of linear groups. The proof of the Theorem is given in § 5 and § 6.

In § 4 we shall give some lemmas, which are based on "category argument" of locally compact groups, and which make the brute force part of

1) Research supported in part by NSF GP4503.

2) By a theorem in Goto [4], every analytic subgroup of $\mathcal{GL}(n, R)$ is isomorphic with a *closed* subgroup of $\mathcal{GL}(m, R)$ for a sufficiently large m.

the proof of Theorem 1 in *Orbits* II much more unified and shortened. Thus, assuming the preliminary parts in *Orbits* II, this paper is self-contained.

Unless specified otherwise, an analytic group and its corresponding Lie algebra will be denoted by the same capital script and capital Roman letter, respectively. For example, if $\mathcal{L}$ is an analytic subgroup of an analytic group $\mathcal{G}$, then G will denote the Lie algebra of $\mathcal{G}$ and L will denote the subalgebra of G corresponding to $\mathcal{L}$. If φ is a continuous homomorphism from an analytic group $\mathcal{G}$ into an analytic group $\mathcal{H}$, then we denote the corresponding Lie algebra homomorphism from G into H also by φ. All Lie algebras in this paper are finite-dimensional over the field R of real numbers. For a finite-dimensional vector space V over R, we let $M(V)$ denote the Lie algebra of all endomorphisms, and $\mathcal{GL}(V)$ the group of all automorphisms, of V.

§ 2. Semi-algebraic groups

Let V be a finite-dimensional vector space over R, and let $\mathcal{H}$ be an analytic subgroup of $\mathcal{GL}(V)$. We let $[\mathcal{H}]$ denote the *identity component group of the algebraic hull* of $\mathcal{H}$, in this paper.[3] The Lie algebra $[H]$ of $[\mathcal{H}]$ is the smallest algebraic Lie algebra containing H.

Definition. An analytic subgroup $\mathcal{S}$ of $\mathcal{GL}(V)$ is called *semi-algebraic* if $\mathcal{S}$ contains a maximal compact subgroup of $[\mathcal{S}]$. A subalgebra of $M(V)$ is said to be *semi-algebraic* if the corresponding analytic group is semi-algebraic.

Let $\mathcal{S}$ be a semi-algebraic group. Since $\mathcal{S}$ contains the commutator subgroup of $[\mathcal{S}]$ and since all maximal compact subgroups are conjugate to each other, all compact subgroups of $[\mathcal{S}]$ are contained in $\mathcal{S}$. Obviously, *a semi-algebraic group is closed.*

Let $\mathcal{G}$ be an analytic subgroup of $\mathcal{GL}(V)$. Let us pick up a maximal compact subgroup $\mathcal{K}$ of $[\mathcal{G}]$. Then $\mathcal{K}\mathcal{G}$ is the smallest semi-algebraic group containing $\mathcal{G}$. We denote $\mathcal{K}\mathcal{G}$ by $\{\mathcal{G}\}$ and call it the *semi-algebraic hull* of $\mathcal{G}$. Since the semi-simple part of $\mathcal{K}$ is contained in $\mathcal{G}$, we can find a toral subgroup $\mathcal{T}$ with $\{\mathcal{G}\}=\mathcal{T}\mathcal{G}$ and $T\cap G=0$. The Lie algebra $T+G$ of $\{\mathcal{G}\}$ will be denoted by $\{G\}$, and will be called the *semi-algebraic hull* of G.

Definition. Let G be a Lie algebra, and let ξ be a representation of G (into a suitable $M(n, R)$). ξ is said to be *minimal* if the center of $[\xi(G)]$ is contained in $\xi(G)$.

We note that for a minimal representation ξ of G, the center of $\xi(G)$, the center of $[\xi(G)]$, and the centralizer of $\xi(G)$ in $[\xi(G)]$ all coincide.

Let H be a Lie algebra. We let $I(H)$ denote the Lie algebra of all inner

3) In *Orbits* II, $[\mathcal{H}]$ denotes the algebraic hull of $\mathcal{H}$.

derivations of H. $I(H)$ is a subalgebra of $M(H)$ and the corresponding analytic group $\mathscr{I}(H)$ is the adjoint group of H. The adjoint group $\mathscr{I}(\mathscr{H})$, composed of all inner automorphisms of an analytic group $\mathscr{H}$ with the Lie algebra H, can be identified with $\mathscr{I}(H)$. If H is an algebraic subalgebra of $M(V)$, then $I(H)$ is also algebraic because the adjoint representation is rational. Conversely if $I(H)$ is algebraic, then it is known that there exists a faithful representation ξ of H such that $\xi(H)$ is algebraic.[4)]

(2.1) *Every Lie algebra has a faithful minimal representation.*

PROOF. Let ξ be a faithful representation of G, and let C_1 be the center of $[\xi(G)]$. If $C=C_1\cap\xi(G)\neq C_1$, then we can find an abelian subalgebra C_2 with $C_1=C_2+C$ and $C_2\cap C=0$. Because $\xi(G)$ contains the commutator subalgebra of $[\xi(G)]$, we can find an ideal G_2 of $[\xi(G)]$ such that $[\xi(G)]=C_2+G_2$, $G_2\supset\xi(G)$ and $C_2\cap G_2=0$. Since $I([\xi(G)])$ is algebraic and $I(G_2)$ is essentially the same as $I([\xi(G)])$, we see that $I(G_2)$ is algebraic. Hence G_2 has a faithful representation η such that $\eta(G_2)$ is algebraic. This implies that if ξ is not minimal, then we can find a faithful representation ζ of G with $\dim[\zeta(G)] < \dim[\xi(G)]$. Because G has a faithful representation,[5)] G has a faithful minimal representation. Q.E.D.

DEFINITION. A Lie algebra G is said to be *adjoint semi-algebraic* if $I(G)$ is semi-algebraic. An analytic group $\mathscr{G}$ is *adjoint semi-algebraic* if the Lie algebra G is adjoint semi-algebraic.

(2.2) *Let G be an adjoint semi-algebraic Lie algebra, and let ξ be a faithful minimal representation of G. Then $\xi(G)$ is semi-algebraic.*

PROOF. For x in $[\xi(G)]$ we let $\varphi(x)$ denote the restriction of ad x in $\xi(G)$: $\varphi(x)\in M(\xi(G))$. Since ξ is faithful we identify $\xi(G)$ with G. Then φ induces a rational homomorphism from the analytic subgroup corresponding to $[\xi(G)]$ onto $[\mathscr{I}(G)]$. If $\exp Rx$ is a circle, then so is $\exp R\varphi(x)$. Since $I(G)$ is semi-algebraic, we have that $\varphi(x)\in I(G)$. Because $\varphi(\xi(G))=I(G)$, we can find y in $\xi(G)$ with $\varphi(x)=\varphi(y)$. On the other hand, since ξ is minimal, the kernel of φ is the center of $\xi(G)$. Hence $x-y\in\xi(G)$, and so $x\in\xi(G)$. Q.E.D.

§ 3. Semi-algebraic hull of an adjoint group

(3.1) *Let $\tilde{\mathscr{G}}$ be a simply connected analytic group, and let C be the center of $\tilde{\mathscr{G}}$. Let $\mathscr{I}=\mathscr{I}(\tilde{\mathscr{G}})$ be the adjoint group of $\tilde{\mathscr{G}}$. Then for σ in $\{\mathscr{I}\}$ and c in C we have that $c^\sigma=c$.*

PROOF. Let C^0 denote the identity component group of C. Then C^0 is simply connected, and is elementwise fixed by $[\mathscr{I}]$. Let $\mathscr{K}$ be a maximal

4) See Goto [**2**], Matsushima [**8**], and Chevalley [**1**].

5) See e.g. Jacobson [**7**].

compact subgroup of $[\mathcal{J}]$. Since every element of the discrete factor group $\mathcal{C}/\mathcal{C}^0$ is fixed by $\mathcal{K}$, for a fixed element d in $\mathcal{C}$ and σ in $\mathcal{K}$, we have that $d^\sigma d^{-1}=c(\sigma)\in \mathcal{C}^0$. For τ also in $\mathcal{K}$, we have that

$$c(\sigma\tau)=d^{\sigma\tau}d^{-1}=(d^\sigma d^{-1})^\tau d^\tau d^{-1}=c(\sigma)^\tau c(\tau)=c(\sigma)c(\tau).$$

Hence $\mathcal{K}\ni\sigma\mapsto c(\sigma)\in \mathcal{C}^0$ is a continuous homomorphism. Since $\mathcal{C}^0$ contains no compact proper subgroup, $c(\sigma)$ must be the identity. Q. E. D.

Let $\mathcal{G}$ be an analytic group, locally isomorphic with $\tilde{\mathcal{G}}$. Then there exists a discrete subgroup $\mathcal{D}$ of $\mathcal{C}$ such that the factor group $\tilde{\mathcal{G}}/\mathcal{D}$ is isomorphic with $\mathcal{G}$. By (3.1), $\mathcal{D}$ is fixed by $\{\mathcal{J}\}$. Therefore, we have the following (3.2).

(3.2) *Let $\mathcal{G}$ be an analytic group, and $\mathcal{J}$ the adjoint group of $\mathcal{G}$. Then $\{\mathcal{J}\}$ is an automorphism group of $\mathcal{G}$.*

For elements g and h of a group we adopt the notation $h^{\mathrm{Ad}(g)}=g^{-1}hg$.

(3.3) *Let $\tilde{\mathcal{G}}$ be a simply connected analytic group, and let $\mathcal{M}$ be a compact connected subgroup of $\{\mathcal{J}(\tilde{\mathcal{G}})\}$. If g is an element of $\tilde{\mathcal{G}}$ such that* $\mathrm{Ad}(g)$ *commutes with every element of $\mathcal{M}$, then $g^\sigma=g$ for all σ in $\mathcal{M}$.*

PROOF. Let σ be an element of $\mathcal{M}$. The equalities $\sigma\circ\mathrm{Ad}(g)=\mathrm{Ad}(g)\circ\sigma$ and $\mathrm{Ad}(g^\sigma)=\sigma^{-1}\circ\mathrm{Ad}(g)\circ\sigma$ imply that $g^\sigma g^{-1}=c(\sigma)$ is in the center C of $\tilde{\mathcal{G}}$. By (3.1), $\mathcal{K}\ni\sigma\mapsto c(\sigma)\in \mathcal{C}$ is a homomorphism. On the other hand $\mathcal{C}$ contains no compact connected subgroup except the identity group. Q. E. D.

(3.4) *Let $\mathcal{G}$ be an analytic group. We can find an adjoint semi-algebraic group $\mathcal{S}$ which contains $\mathcal{G}$ as a closed normal subgroup such that $\mathcal{J}(\mathcal{S})|_G=\{\mathcal{J}(\mathcal{G})\}$, where $\mathcal{J}(\mathcal{S})|_G$ denotes the restriction of $\mathcal{J}(\mathcal{S})$ to the invariant subspace G.*

PROOF. We denote $\mathcal{J}(\mathcal{G})$ simply by $\mathcal{J}$, and take a toral subgroup $\mathcal{T}$ of $[\mathcal{J}]$ with $\{\mathcal{J}\}=\mathcal{T}\mathcal{J}$ and $T\cap I=0$. Since $\mathcal{T}$ is an automorphism group of $\mathcal{G}$, we can construct a semi-direct product $\mathcal{S}=\mathcal{T}\times\mathcal{G}$ by defining the multiplication

$$(\sigma, a)(\tau, b)=(\sigma\tau, a^\tau b)\qquad \sigma,\tau\in\mathcal{T},\qquad a, b\in\mathcal{G}.$$

Let us prove that this $\mathcal{S}$ satisfies the conditions.

Let $\tilde{\mathcal{G}}$ be the universal covering group of $\mathcal{G}$. Then we can construct the semi-direct product $\tilde{\mathcal{S}}=\mathcal{T}\times\tilde{\mathcal{G}}$ in a similar manner. Since $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are locally isomorphic to each other, $\mathcal{J}(\tilde{\mathcal{S}})$ can be identified with $\mathcal{J}(\mathcal{S})$. Hence after this without changing the notations, let us assume that $\mathcal{G}$ is simply connected.

We let φ denote the adjoint representation of $\mathcal{S}$, and ψ the restriction of φ into the subspace G. We denote the identity and the identity automorphism of $\mathcal{G}$ by e and ε, respectively. Then for g and h in $\mathcal{G}$, and σ and τ in $\mathcal{T}$, we have

$$(1)\qquad (\sigma, g)^{-1}(\varepsilon, h)(\sigma, g)=(\varepsilon, h^{\sigma\mathrm{Ad}(g)}),$$

$$(\sigma, g)^{-1}(\tau, e)(\sigma, g)=(\tau, (g^{-1})^{\tau}g)\,. \tag{2}$$

From (1) we see that the kernel of ψ is given by $\{(\mathrm{Ad}(g^{-1}), g);\ \mathrm{Ad}(g)\in\mathfrak{T}\}$. On the other hand, $\mathrm{Ad}(g)\in\mathfrak{T}$ implies that $g^{\tau}=g$ for all τ in $\mathfrak{T}$, by (3.3). Hence from (2) we can see that the kernel of ψ coincides with the center of $\mathcal{S}$, that is ψ is a faithful representation of $\mathcal{J}(\mathcal{S})$. Also (1) indicates that $\psi(\mathcal{S})=\mathfrak{T}\mathcal{J}=\{\mathcal{J}\}$.

Since $\varphi(s)\mapsto\psi(s)$ gives a one-one continuous homomorphism from $\varphi(\mathcal{S})$ onto $\psi(\mathcal{S})$ and $\psi(\mathcal{S})$ is a closed subgroup of $\mathcal{GL}(G)$, the homomorphism $\varphi(s)\mapsto\psi(s)$ must be a homeomorphism, and $\varphi(\mathcal{S})$ is a closed subgroup of $\mathcal{GL}(S)$.

Let N be the set of all elements of $[I(S)]$ which vanish on G. Then N is an ideal composed of nilpotent endomorphisms, and $N\cap I(S)=0$. Hence $[N, I(S)]=0$.

Let x be an element of $[I(S)]$ such that $\exp Rx$ is a circle. Let x_1 denote the restriction of x to G. Then $x_1\in[I]$ and $\exp Rx_1$ is a circle. Hence $x_1\in\{I\}=\psi(S)$. Hence we can find an element y of S with $x_1=\psi(y)$. Since $\varphi(s)\mapsto\psi(s)$ is a topological isomorphism, $\exp R\varphi(y)$ is also a circle group, and in particular, $\varphi(y)$ is a semi-simple endomorphism. On the other hand, $n=x-\varphi(y)=0$ on G, and so $n\in N$. Thus we have that $x=\varphi(y)+n$, $[\varphi(y), n]=0$, x and $\varphi(y)$ are semi-simple, and n is nilpotent. Hence $n=0$, and this proves that $I(S)$ is semi-algebraic. Q. E. D.

§ 4. Locally compact groups

First we shall generalize (2.2) of *Orbits* II into the following form.

(4.1) *Let $\mathcal{G}$ be a topological group, and let $\mathcal{A}$ and $\mathfrak{B}$ be locally compact groups with countable bases. Let α and β be continuous homomorphisms from $\mathcal{A}$ and $\mathfrak{B}$ into $\mathcal{G}$, respectively. Let $\mathcal{L}$ be a normal subgroup of $\mathfrak{B}$ such that $\beta(\mathcal{L})$ is closed. If $\alpha(\mathcal{A})\beta(\mathfrak{B})$ is a locally compact set, then the map ρ*

$$\mathcal{A}\times\mathfrak{B}\ni(a, b)\mapsto\rho(a, b)=\alpha(a)^{-1}\beta(b)\beta(\mathcal{L})\in\mathcal{G}/\beta(\mathcal{L})$$

is (continuous and) open. More precisely, setting

$$\mathcal{D}=\{(a, b)\in\mathcal{A}\times\mathfrak{B};\ \alpha(a)^{-1}\beta(b)\in\beta(\mathcal{L})\}\,,$$

$\mathcal{D}$ is a closed subgroup of $\mathcal{A}\times\mathfrak{B}$, and the map ρ induces a homeomorphism $\tilde{\rho}$ from the right coset space $\mathcal{D}\backslash\mathcal{A}\times\mathfrak{B}$ onto the locally compact set $\alpha(\mathcal{A})\beta(\mathfrak{B})/\beta(\mathcal{L})$.

PROOF. Let a_1 and a_2 be elements of $\mathcal{A}$, and let b_1 and b_2 be elements of $\mathfrak{B}$. If $\rho(a_1, b_1)=\rho(a_2, b_2)$, then $(a_1, b_1)\in\mathcal{D}(a_2, b_2)$, and conversely. Hence $\mathcal{D}$ is a closed subgroup of $\mathcal{A}\times\mathfrak{B}$ and ρ induces a continuous one-one map $\tilde{\rho}$ from the coset space $\mathcal{D}\backslash\mathcal{A}\times\mathfrak{B}$ onto $\alpha(\mathcal{A})\beta(\mathfrak{B})/\beta(\mathcal{L})=\mathcal{M}$. Thus $\mathcal{A}\times\mathfrak{B}$ is acting as a transitive transformation group on $\mathcal{M}$. On the other hand, $\mathcal{A}\times\mathfrak{B}$ is a locally compact group with a countable base, and $\mathcal{M}$ is locally compact.

Hence the map ρ is open.[6)] Q. E. D.

(4.2) *In* (4.1) *we assume moreover that* $\mathcal{A}$ *is an abelian group and* $\mathcal{L}$ *contains the commutator subgroup of* $\mathfrak{B}$. *Let* $g(\lambda)$, $a(\lambda)$, *and* $b(\lambda)$, $\lambda \in R$, *be one-parameter subgroups of* $\mathcal{G}$, $\mathcal{A}$, *and* $\mathfrak{B}$, *respectively. Suppose that the one-parameter subgroups* $\alpha(a(\lambda))$ *and* $\beta(b(\lambda))$ *are commutative to each other and* $g(\lambda) = \alpha(a(\lambda))\beta(b(\lambda))$. *We set* $\mathcal{X} = \{g(\lambda)\,;\, \lambda \in R\}$. *Then we have either*

(a) $\mathcal{X}\beta(\mathcal{L})$ *is a closed subset of* $\alpha(\mathcal{A})\beta(\mathfrak{B})$, *or*

(b) $\tilde{\rho}^{-1}(\overline{\mathcal{X}\beta(\mathcal{L})}/\beta(\mathcal{L})) = \mathcal{K}$ *is a compact connected abelian group and* $\tilde{\rho}^{-1}(g(\lambda)\beta(\mathcal{L}))$ *is an everywhere dense one-parameter subgroup in* $\mathcal{K}$.

PROOF. Since $\mathcal{D}$ contains $\mathcal{L}$, $\mathcal{D}$ is a normal subgroup and $\mathcal{A}\times\mathfrak{B}/\mathcal{D}$ is an abelian group. $h(\lambda) = (a(-\lambda), b(\lambda))\mathcal{D}$ is a one-parameter subgroup of $\mathcal{A}\times\mathfrak{B}/\mathcal{D}$. We set $\mathcal{Y} = \{(a(-\lambda), b(\lambda))\,;\, \lambda \in R\}$. If $h(\lambda)$ is a closed one-parameter subgroup, then $\mathcal{Y}\mathcal{D}$ is closed, and so is $\rho(\mathcal{Y}\mathcal{D}) = \mathcal{X}\beta(\mathcal{L})$ in $\mathcal{M}$, whence $\mathcal{X}\beta(\mathcal{L})$ is locally compact. If $h(\lambda)$ is not a closed one-parameter subgroup, then its closure $\mathcal{K} = \overline{\mathcal{Y}\mathcal{D}}/\mathcal{D}$ is a compact connected subgroup of $\mathcal{A}\times\mathfrak{B}/\mathcal{D}$. Q. E. D.

§ 5. Linear group case

Let $\mathcal{G}$ be a closed analytic subgroup of $\mathcal{GL}(n, R)$. Let L be a subalgebra of G and let x be an element of G. We set $[x]\cap[L] = D$ and decompose $[x]$ into a direct sum: $[x] = A' + D$, $A'\cap D = 0$, such that A' is an algebraic subalgebra of $[x]$. Then, we can find $y \in A'$ and $z \in D$ with $y + z = x$. It is obvious that $A' = [y]$ and $D = [z]$. We denote the one-parameter groups $\exp Rx$, $\exp Ry$ and $\exp Rz$ by $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Since z normalizes L, $\mathcal{Z}\mathcal{L}$ is an analytic subgroup of $\mathcal{G}$. We set $\overline{\mathcal{Y}} = \mathcal{A}$ and $\overline{\mathcal{Z}\mathcal{L}} = \mathfrak{B}$.

The product $[\mathcal{Y}][\mathcal{L}]$ is a locally compact set, and $[\mathcal{Y}]\cap[\mathcal{L}]$ is finite. Hence for the closed set $\mathcal{A}$ in $[\mathcal{Y}]$ and $\mathfrak{B}$ in $[\mathcal{L}]$, we have that $\mathcal{A}\mathfrak{B}$ is closed in $[\mathcal{Y}][\mathcal{L}]$ and is locally compact itself. (See *Orbits* II)

(5.1) *Under the above assumptions, either*

(a) $\mathcal{X}$ *is a closed straight line,* $\mathcal{X}\bar{\mathcal{L}}$ *is locally compact and the map* $\mathcal{X}\times\bar{\mathcal{L}} \ni (\exp \lambda x, l) \mapsto \exp \lambda x \cdot l \in \mathcal{X}\bar{\mathcal{L}}$ *is a homeomorphism, or*

(b) y *and* z *are contained in* $\{G\}$.

PROOF. We note that for a non-closed analytic subgroup Q of an analytic group $\mathcal{P}$, we can find a non-closed one-parameter subgroup $\mathcal{V}$ in Q with $\bar{Q} = \bar{\mathcal{V}}Q$.[7)] Because every compact subgroup of $[\mathcal{G}]$ is contained in $\{\mathcal{G}\}$, if $\overline{\mathcal{Y}}$ or $\bar{\mathcal{Z}}$ is a toral group then we have the case (b). Also if $\bar{\mathcal{L}} \supset \mathcal{Z}$, then we have the case (b). If $\mathcal{Z}\bar{\mathcal{L}}$ is not closed, then we can find a non-closed one-

6) See e. g. Helgason [**6**].

7) See Goto [**3**].

parameter subgroup $\mathcal{U}$ in $\mathcal{Z}\bar{\mathcal{L}}$ but not in $\bar{\mathcal{L}}$ such that $\overline{\mathcal{Z}\bar{\mathcal{L}}}=\overline{\mathcal{U}}\mathcal{Z}\bar{\mathcal{L}}=\overline{\mathcal{U}\bar{\mathcal{L}}}$, and so $z\in\{G\}$. Now, let us assume that $\mathcal{Z}$ is a closed straight line, $\mathcal{Z}\bar{\mathcal{L}}$ is closed and $\mathcal{Z}$ is not contained in $\bar{\mathcal{L}}$. If $\mathcal{Z}\cap\bar{\mathcal{L}}$ is not the identity only, then the factor group $\mathcal{Z}\bar{\mathcal{L}}/\bar{\mathcal{L}}$ is a circle, and $\mathcal{Z}\bar{\mathcal{L}}$ can be written as a product of a circle and $\bar{\mathcal{L}}$. Hence we have the case (b).

Now it remains only the case when $\mathcal{Y}$ and $\mathcal{Z}$ are closed straight lines, $\mathcal{Z}\bar{\mathcal{L}}$ is closed, and $\mathcal{Z}\cap\bar{\mathcal{L}}=e$ (the identity). Since $\mathcal{Y}\cap\mathcal{Z}\bar{\mathcal{L}}$ is finite and $\mathcal{Y}$ contains no finite subgroup except e, we have $\mathcal{Y}\cap\mathcal{Z}\bar{\mathcal{L}}=e$. It is easy to see that the locally compact set $\mathcal{Y}\mathcal{Z}\bar{\mathcal{L}}$ is topologically the direct product $\mathcal{Y}\times\mathcal{Z}\times\bar{\mathcal{L}}$. On the other hand, $\mathcal{X}$ is a one-parameter subgroup of the two-dimensional vector group $\mathcal{Y}\mathcal{Z}$. Hence we have the case (a). Q. E. D.

§ 6. Proof of the theorem

In virtue of (3.4), in order to prove the theorem we may assume that $\mathcal{G}$ is adjoint semi-algebraic, without loss of generality. We choose a fixed minimal faithful representation ξ of the adjoint semi-algebraic Lie algebra G, and for the sake of convenience, we identify G with $\xi(G)$.

Let us denote the adjoint representation of $\mathcal{G}$ (onto the adjoint group $\mathcal{I}=\mathcal{I}(G)$) by φ. As a Lie algebra homomorphism, φ can be extended to a homomorphism, which will be denoted also by φ, from $[G]$ onto $[I]$, although we consider the group homomorphism φ only on $\mathcal{G}$.

For the given one-parameter subgroup $\exp Rx=\mathcal{X}$ and the analytic subgroup $\mathcal{L}$ of $\mathcal{G}$, we set $\varphi(x)=x_1$, $\varphi(\mathcal{X})=\mathcal{X}_1$ and $\varphi(\mathcal{L})=\mathcal{L}_1$. By (5.1) we have the following two cases (a) and (b).

(a) *$\mathcal{X}_1$ is a closed straight line, $\mathcal{X}_1\bar{\mathcal{L}}_1$ is locally compact, and $\mathcal{X}_1\bar{\mathcal{L}}_1$ is homeomorphic with $\mathcal{X}_1\times\bar{\mathcal{L}}_1$.*

Let C denote the center of $\mathcal{G}$. $\mathcal{X}$ is a closed straight line and $\varphi^{-1}(\mathcal{X}_1)=\mathcal{X}C$. We set $\varphi^{-1}(\bar{\mathcal{L}}_1)=\mathcal{M}$. Then $\varphi^{-1}(\mathcal{X}_1\bar{\mathcal{L}}_1)=\mathcal{X}C\mathcal{M}=\mathcal{X}\mathcal{M}$ is a locally compact set, and the commutator subgroup of $\mathcal{M}=\bar{\mathcal{L}}C$ is contained in $\bar{\mathcal{L}}$. Hence by (4.2), either $\mathcal{X}\bar{\mathcal{L}}$ is locally compact, or $\overline{\mathcal{X}\bar{\mathcal{L}}}=\overline{\mathcal{X}\mathcal{L}}$ is a torus in $\mathcal{G}/\bar{\mathcal{L}}$. In the first case, if $\mathcal{X}\cap\bar{\mathcal{L}}$ is not the identity, then $\mathcal{X}\bar{\mathcal{L}}/\bar{\mathcal{L}}$ is a circle, and it reduces to the second case.

(b) $$x_1=y_1+z_1\,,\qquad (y_1,\,z_1\in G)$$

$$[x_1]\cap[L_1]=[z_1],\ [x_1]=[y_1]+[z_1],\ [y_1]\cap[z_1]=0\,.$$

Because the representation ξ is minimal, we have $\varphi^{-1}(I)=G$. Since $\varphi([x])=[x_1]\ni z_1$, we can find z in $[x]\cap G$ with $\varphi(z)=z_1$. On the other hand, since $z_1 \in [L_1] = [\varphi(L)]$ there exists $z' \in [L] \cap G$ with $\varphi(z') = z_1$. That $z - z' \in C$ and $[z', L] \subset L$ implies $[z, L] \subset L$. We put $y = x - z$, and we get that $[y, z] = 0$, $\varphi(y) = y_1$ and $x = y + z$.

Next, we set

$$\varphi^{-1}(\bar{\mathcal{Y}}_1) = \mathcal{A},\ \varphi^{-1}(\overline{\mathcal{Z}_1\mathcal{L}_1}) = \mathfrak{B},\ \exp Ry = \mathcal{Y} \text{ and } \exp Rz = \mathcal{Z}\,.$$

$\mathcal{A} = \overline{\mathcal{Y}C}$ is an abelian group, and the commutator subgroup of $\mathfrak{B} = \overline{\mathcal{Z}\mathcal{L}C}$ is contained in $\bar{\mathcal{L}}$. Applying (4.2) to the locally compact set $\mathcal{A}\mathfrak{B} = \varphi^{-1}(\bar{\mathcal{Y}}_1, \overline{\mathcal{Z}_1\mathcal{L}_1})$, we have that either $\mathcal{X}\bar{\mathcal{L}}$ is locally compact, or $\overline{\mathcal{X}\mathcal{L}}/\bar{\mathcal{L}}$ has a toral group structure with $\exp \lambda x\bar{\mathcal{L}}$ as an everywhere dense one-parameter subgroup. When $\mathcal{X}\bar{\mathcal{L}}$ is locally compact, if $\mathcal{X}$ is not a closed straight line or if $\mathcal{X}$ is a closed straight line and $\mathcal{X} \cap \bar{\mathcal{L}}$ is not the identity, it reduces to the second case.

University of Pennsylvania

Bibliography

[1] C. Chevalley, Théorie des groupes de Lie, Vol. III, Paris, Hermann, 1955.

[2] M. Goto, On algebraic Lie algebras, J. Math. Soc. Japan, 1 (1948), 29–45.

[3] M. Goto, Faithful representations of Lie groups I, Math. Japon., 1 (1948), 107–119.

[4] M. Goto, Faithful representations of Lie groups II, Nagoya Math. J., 1 (1950), 91–107.

[5] M. Goto, Orbits of one-parameter groups II (Linear group case), J. Math. Soc. Japan, 22 (1970), 123–133.

[6] S. Helgason, Differential geometry and symmetric spaces, Columbia University, 1962.

[7] N. Jacobson, Lie algebras, Tracts in Math. 10, Interscience, 1962.

[8] Y. Matsushima, On algebraic Lie groups and algebras, J. Math. Soc. Japan, 1 (1948), 46–57.