

On conformal diffeomorphisms of complete Riemannian manifolds with parallel Ricci tensor

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We shall assume, throughout this paper, that Riemannian manifolds are connected and of dimension > 2 , their metrics are positive-definite, and manifolds and diffeomorphisms are of differentiability class C^∞ .

Let M and M^* be Riemannian manifolds with metric tensor g and g^* respectively, and f a diffeomorphism of M onto M^* . If the induced metric f^*g^* of g^* by f is related to g by

$$f^*g^* = \rho^{-2}g,$$

then f is called a *conformal diffeomorphism* of M onto M^* , where ρ is a positively valued scalar field on M . If the scalar field ρ satisfies the equation

$$\nabla\nabla\rho = \phi g,$$

∇ indicating covariant differentiation and ϕ being a scalar field, then the diffeomorphism f is called a *concircular* one, which carries Riemannian circles of M to those of M^* . If ρ is a constant or equal to 1, then f is said to be *homothetic* or *isometric*, respectively.

THEOREM 1. *Let M and M^* be complete Riemannian manifolds with parallel Ricci tensor. If there is a non-isometric conformal diffeomorphism f of M onto M^* , then f is homothetic or both the manifolds M and M^* are isometric to the sphere.*

THEOREM 2. *Let M be a complete Riemannian manifold with parallel Ricci tensor. If M admits a non-isometric conformal transformation f , then occurs one of the following two cases:*

- (1) f is homothetic and M is a Euclidean space, or
- (2) M is isometric to the sphere.

The first and principal part of Theorem 1 was proved by N. Tanaka [6] by a group-theoretic method under some additional conditions on Ricci tensors of M and M^* . Then T. Nagano [5] dealt with the cases excepted by Tanaka, showed that f is properly conformal only in the case where both M and M^* are Einstein manifolds of positive curvature, and completed Theorem

1 as stated above by use of a famous theorem due to K. Yano and T. Nagano [11]:

THEOREM A. *If a complete Einstein manifold M admits a global one-parameter group of non-homothetic conformal transformations, then M is isometric to the sphere.*

A generalization of Theorem A to a conformal diffeomorphism is stated as follows:

THEOREM B. *Let M and M^* be complete Einstein manifolds. If there is a non-homothetic conformal diffeomorphism of M onto M^* , then both M and M^* are isometric to the sphere.*

While T. Nagano stated this fact on the way of the reduction of Theorem 1 to Theorem A, an immediate proof was given by S. Ishihara and Y. Tashiro [2] and Y. Tashiro [7] since a conformal diffeomorphism between Einstein manifolds is concircular, cf. [8, 9].

As a generalization of Theorem 1 or A, the present authors [10] have recently proved the following.

THEOREM C. *If a complete and reducible Riemannian manifold admits a global one-parameter group of non-isometric conformal transformations, then the group is homothetic and the manifold is a Euclidean space.*

It has remained open, however, to give an alternative proof of Theorem 1 by tensor calculus. The purpose of the present paper is to prove Theorem 1 by an elementary method similar to that employed in [10].

§1. Let f be a conformal diffeomorphism of M into M^* , and denote the metric tensors g and f^*g^* by components $g_{\mu\lambda}$ and $g_{\mu\lambda}^*$. Then they are related by the equation

$$(1.1) \quad g_{\mu\lambda}^* = \rho^{-2} g_{\mu\lambda}.$$

We denote the Christoffel symbol, the curvature tensor, the Ricci tensor and the scalar curvature of g by $\left\{ \begin{smallmatrix} \kappa \\ \mu\lambda \end{smallmatrix} \right\}$, $K_{\nu\mu\lambda}{}^\kappa$, $K_{\mu\lambda}$ and κ respectively, where the scalar curvature κ is defined by

$$(1.2) \quad \kappa = \frac{1}{n(n-1)} K_{\mu\lambda} g^{\mu\lambda},$$

and indicate the corresponding quantities of g^* by asterisking. Then we obtain the transformation formulas

$$(1.3) \quad \left\{ \begin{smallmatrix} \kappa \\ \mu\lambda \end{smallmatrix} \right\}^* = \left\{ \begin{smallmatrix} \kappa \\ \mu\lambda \end{smallmatrix} \right\} - \frac{1}{\rho} (\delta_\mu^\kappa \rho_\lambda + \delta_\lambda^\kappa \rho_\mu - g_{\mu\lambda} \rho^\kappa),$$

$$(1.4) \quad K_{\nu\mu\lambda}^* \kappa = K_{\nu\mu\lambda} \kappa + \frac{1}{\rho} (\delta_\nu^* \nabla_\mu \rho_\lambda - \delta_\mu^* \nabla_\nu \rho_\lambda + g_{\mu\lambda} \nabla_\nu \rho^\kappa - g_{\nu\lambda} \nabla_\mu \rho^\kappa) \\ - \frac{1}{\rho^2} \rho_\omega \rho^\omega (\delta_\nu^* g_{\mu\lambda} - \delta_\mu^* g_{\nu\lambda}),$$

$$(1.5) \quad K_{\mu\lambda}^* = K_{\mu\lambda} + \frac{1}{\rho} (n-2) \nabla_\mu \rho_\lambda + \frac{1}{\rho} g_{\mu\lambda} \nabla_\kappa \rho^\kappa - \frac{1}{\rho^2} (n-1) \rho_\kappa \rho^\kappa g_{\mu\lambda},$$

$$(1.6) \quad \kappa^* = \rho^2 \kappa + \frac{2}{n} \rho \nabla_\kappa \rho^\kappa - \rho_\kappa \rho^\kappa,$$

where we have put $\rho_\lambda = \nabla_\lambda \rho$.

We have from (1.6)

$$\nabla_\kappa \rho^\kappa = \frac{n}{2\rho} (\kappa^* - \rho^2 \kappa + \rho_\kappa \rho^\kappa).$$

Substituting this into (1.5) and putting

$$(1.7) \quad L_{\mu\lambda} = K_{\mu\lambda} - \frac{n}{2} \kappa g_{\mu\lambda} \quad \text{and} \quad L_{\mu\lambda}^* = K_{\mu\lambda}^* - \frac{n}{2} \kappa^* g_{\mu\lambda},$$

we obtain the equation

$$(1.8) \quad L_{\mu\lambda}^* = L_{\mu\lambda} + \frac{1}{\rho} (n-2) \nabla_\mu \rho_\lambda - \frac{1}{2\rho^2} (n-2) g_{\mu\lambda} \rho_\kappa \rho^\kappa.$$

Differentiating covariantly this equation with respect to $\left\{ \begin{smallmatrix} \kappa \\ \mu\lambda \end{smallmatrix} \right\}^*$ and substituting (1.3), we obtain the equation

$$(1.9) \quad \nabla_\nu^* L_{\mu\lambda}^* = \nabla_\nu L_{\mu\lambda} + \frac{1}{2\rho^2} (n-2) [\nabla_\nu \nabla_\mu \nabla_\lambda \rho^2 - g_{\mu\lambda} \nabla_\nu (\rho_\kappa \rho^\kappa) \\ - g_{\nu\lambda} \nabla_\mu (\rho_\kappa \rho^\kappa) - g_{\nu\mu} \nabla_\lambda (\rho_\kappa \rho^\kappa)] + \frac{1}{2\rho^2} [2L_{\mu\lambda} \nabla_\nu \rho^2 \\ + L_{\nu\mu} \nabla_\lambda \rho^2 + L_{\nu\lambda} \nabla_\mu \rho^2 - (g_{\nu\lambda} L_{\mu\kappa} + g_{\nu\mu} L_{\lambda\kappa}) \nabla^\kappa \rho^2].$$

Now we assume that the Ricci tensors $K_{\mu\lambda}$ and $K_{\mu\lambda}^*$ are parallel, $\nabla_\nu K_{\mu\lambda} = \nabla_\nu^* K_{\mu\lambda}^* = 0$. Then the scalar curvatures κ and κ^* are constants and the tensors $L_{\mu\lambda}$ and $L_{\mu\lambda}^*$ are also parallel. Hence the scalar field ρ satisfies the differential equation

$$(1.10) \quad \nabla_\nu \nabla_\mu \nabla_\lambda \rho^2 = g_{\mu\lambda} \nabla_\nu (\rho_\kappa \rho^\kappa) + g_{\nu\lambda} \nabla_\mu (\rho_\kappa \rho^\kappa) + g_{\nu\mu} \nabla_\lambda (\rho_\kappa \rho^\kappa) \\ - \frac{1}{n-2} [2L_{\mu\lambda} \nabla_\nu \rho^2 + L_{\nu\lambda} \nabla_\mu \rho^2 + L_{\nu\mu} \nabla_\lambda \rho^2 \\ - (g_{\nu\lambda} L_{\mu\kappa} + g_{\nu\mu} L_{\lambda\kappa}) \nabla^\kappa \rho^2].$$

Applying Ricci's formula to (1.10) itself and its covariant derivative, we have the equations

$$(1.11) \quad K_{\nu\mu\lambda}^* \rho_\kappa = \frac{1}{n-2} [L_{\mu\lambda} \rho_\nu - L_{\nu\lambda} \rho_\mu + (g_{\mu\lambda} L_{\nu\kappa} - g_{\nu\lambda} L_{\mu\kappa}) \rho^\kappa],$$

$$\begin{aligned}
(1.12) \quad & K_{\omega\nu\mu}{}^\kappa \nabla_\lambda \nabla_\kappa \rho^2 + K_{\omega\nu\lambda}{}^\kappa \nabla_\mu \nabla_\kappa \rho^2 = g_{\omega\lambda} \nabla_\nu \nabla_\mu (\rho_\kappa \rho^\kappa) - g_{\nu\lambda} \nabla_\omega \nabla_\mu (\rho_\kappa \rho^\kappa) \\
& + g_{\omega\mu} \nabla_\nu \nabla_\lambda (\rho_\kappa \rho^\kappa) - g_{\nu\mu} \nabla_\omega \nabla_\lambda (\rho_\kappa \rho^\kappa) \\
& - \frac{1}{n-2} [L_{\omega\lambda} \nabla_\nu \nabla_\mu \rho^2 - L_{\nu\lambda} \nabla_\omega \nabla_\mu \rho^2 + L_{\omega\mu} \nabla_\nu \nabla_\lambda \rho^2 \\
& - L_{\nu\mu} \nabla_\omega \nabla_\lambda \rho^2 + (g_{\nu\lambda} L_{\mu\kappa} + g_{\nu\mu} L_{\lambda\kappa}) \nabla_\omega \nabla^\kappa \rho^2 \\
& - (g_{\omega\lambda} L_{\mu\kappa} + g_{\omega\mu} L_{\lambda\kappa}) \nabla_\nu \nabla^\kappa \rho^2].
\end{aligned}$$

We first prove the following.

LEMMA 1. *Let M and M^* be Riemannian manifolds with parallel Ricci tensor and f be a non-homothetic conformal diffeomorphism of M to M^* . If, in addition, M is simply connected, complete and reducible, then the manifold M is the Pythagorean product $M_1 \times M_2$ of two Einstein manifolds M_1 and M_2 , the scalar curvatures k_1 and k_2 of which have one of the following properties:*

- 1) $k_1 + k_2 = 0$ and $k_1 \neq 0$, and consequently $k_2 \neq 0$,
- 2) one of the manifolds, say M_2 , is one-dimensional and $k_1 \neq 0$,
- 3) $k_1 = k_2 = 0$, and M itself is an Einstein manifold of zero curvature.

PROOF. By virtue of de Rham's well known decomposition theorem, the manifold M is a Pythagorean product

$$M = M_1 \times M_2 \times \cdots \times M_r,$$

where M_1, M_2, \dots, M_r are irreducible complete Riemannian manifolds and called *parts* of M . Suppose that the dimension of each part M_s is n_s ($s=1, 2, \dots, r$). Let $(x^{i_1}, x^{i_2}, \dots, x^{i_r})$ be a local coordinate system in a neighborhood of a point such that (x^{i_s}) is a local coordinate system of each part M_s , and $g_{j_s i_s}$ components of the metric tensor of M_s . In such a separate coordinate system, the matrix $(g_{\mu\lambda})$ of components of the metric tensor g is expressed as the direct sum of the matrices $(g_{j_s i_s})$ of components of the metrics of the parts, and the Christoffel symbol, the curvature tensor and the Ricci tensor have pure components only.

Since the Ricci tensor is parallel, we have

$$(1.13) \quad \nabla_{k_s} K_{j_s i_s} = 0 \quad (s=1, 2, \dots, r)$$

on each part M_s . The irreducibility of each part implies that each part M_s is an Einstein manifold, that is,

$$(1.14) \quad K_{j_s i_s} = (n_s - 1) k_s g_{j_s i_s} \quad (s=1, 2, \dots, r),$$

where k_s is the constant scalar curvature of M_s . By our definition, the scalar curvatures κ and k_s satisfy the relation

$$(1.15) \quad n_1(n_1 - 1)k_1 + \cdots + n_r(n_r - 1)k_r = n(n - 1)\kappa.$$

The components of $L_{\mu\lambda}$ on M_s are equal to

$$(1.16) \quad L_{j_s i_s} = \left[(n_s - 1)k_s - \frac{n}{2}\kappa \right] g_{j_s i_s}.$$

Putting $\lambda = i_s$, $\mu = j_s$ and $\nu = h_t$ ($s \neq t$) in (1.11), we have

$$(1.17) \quad L_{j_s i_s} \rho_{h_t} + g_{j_s i_s} L_{h_t k_t} \rho^{k_t} = 0$$

and, substituting (1.16),

$$(1.18) \quad [(n_s - 1)k_s + (n_t - 1)k_t - n\kappa] g_{j_s i_s} \rho_{h_t} = 0.$$

Since ρ is supposed not to be a constant, there exists a point P where the gradient vector ρ_λ does not vanish and so does not the restriction of ρ_λ on some part, say ρ_{h_1} on $M_1(P)$, through P . Therefore we have the relations

$$(1.19) \quad (n_1 - 1)k_1 + (n_s - 1)k_s = n\kappa \quad (s = 2, \dots, r),$$

from which it follows that

$$(1.20) \quad (n_2 - 1)k_2 = \dots = (n_r - 1)k_r.$$

Hence the product $M_2 \times \dots \times M_r$ is an Einstein manifold.

Putting together the product $M_2 \times \dots \times M_r$ into a manifold, we may assume from the beginning that M is the Pythagorean product $M_1 \times M_2$ of two Einstein manifolds M_1 and M_2 . Then, similarly to (1.15) and (1.19), the scalar curvatures k_1 and k_2 satisfy the relations

$$(1.21) \quad n_1(n_1 - 1)k_1 + n_2(n_2 - 1)k_2 = n(n - 1)\kappa$$

and

$$(1.22) \quad (n_1 - 1)k_1 + (n_2 - 1)k_2 = n\kappa.$$

It follows from these equations that

$$(1.23) \quad (n_1 - 1)(n_2 - 1)(k_1 + k_2) = 0.$$

Therefore, if neither M_1 nor M_2 are one-dimensional, we have

$$(1.24) \quad k_1 + k_2 = 0.$$

If $k_1 = 0$ and consequently $k_2 = 0$, or if $k_1 = 0$ and M_2 is one-dimensional, then the manifold M itself is an Einstein one of zero scalar curvature. Q.E.D.

§ 2. We shall derive the differential equations which the scalar field ρ satisfies in the cases 1) and 2) of Lemma 1. For brevity, let (x^i, x^p) be a separate local coordinate system of $M = M_1 \times M_2$ ($i, j, k = 1, \dots, n_1$; $p, q = n_1 + 1, \dots, n$) and write k for the non-zero constant scalar curvature k_1 of M_1 .

Eliminating k_2 from (1.21) and (1.22), we have the relation

$$(2.1) \quad n\kappa = (2n_1 - n)k.$$

The components L_{ji} of $L_{\mu\lambda}$ belonging to M_1 given by (1.16) are then equal to

$$(2.2) \quad L_{ji} = \frac{n-2}{2} kg_{ji}.$$

Since

$$\begin{aligned} (n_2 - 1)k_2 &= n\kappa - (n_1 - 1)k \\ &= [(2n_1 - n) - (n_1 - 1)]k \\ &= (n_1 - n + 1)k \end{aligned}$$

by (1.22) and (2.1), the components L_{qp} belonging to M_2 are equal to

$$(2.3) \quad L_{qp} = -\frac{n-2}{2} kg_{qp}$$

including the case $n_2 = 1$.

Putting the indices $\lambda = i$, $\mu = j$, $\nu = k$ and $\omega = p$ in the equation (1.12), we have

$$(2.4) \quad \nabla_p \nabla_i (\rho_\kappa \rho^\kappa) = 0,$$

and, putting $\lambda = i$, $\mu = p$, $\nu = j$ and $\omega = q$,

$$(2.5) \quad g_{ji} \nabla_q \nabla_p (\rho_\kappa \rho^\kappa - k\rho^2) = g_{qp} \nabla_j \nabla_i (\rho_\kappa \rho^\kappa + k\rho^2).$$

From the equation (2.4), we may put

$$(2.6) \quad \rho_\kappa \rho^\kappa = \phi_1 + \phi_2,$$

where the functions ϕ_1 and ϕ_2 respectively depend on (x^i) and (x^p) only. From the equation (2.5), we may put

$$(2.7) \quad \begin{aligned} \nabla_j \nabla_i (\rho_\kappa \rho^\kappa + k\rho^2) &= \phi g_{ji}, \\ \nabla_q \nabla_p (\rho_\kappa \rho^\kappa - k\rho^2) &= \phi g_{qp}, \end{aligned}$$

where ϕ is a function determined as follows. Differentiating covariantly the first equation of (2.7) with respect to x^p and taking account of (2.1), we have

$$k \nabla_p \nabla_j \nabla_i \rho^2 = (\nabla_p \phi) g_{ji}.$$

Comparing this equation with (1.10) for $\lambda = i$, $\mu = j$ and $\nu = p$, we obtain

$$\nabla_p \phi = k \nabla_p (\phi_2 - k\rho^2),$$

and similarly from (1.10) and the second of (2.7)

$$\nabla_i \phi = -k \nabla_i (\phi_1 + k\rho^2).$$

Hence ϕ is given in the form

$$\phi = -k(\phi_1 - \phi_2 + k\rho^2 + b),$$

b being a constant.

Therefore the equations (2.7) turn to

$$(2.8) \quad \begin{aligned} \nabla_j \nabla_i (\phi_1 + k\rho^2) &= -k(\phi_1 - \phi_2 + k\rho^2 + b)g_{ji}, \\ \nabla_q \nabla_p (\phi_2 - k\rho^2) &= -k(\phi_1 - \phi_2 + k\rho^2 + b)g_{qp}. \end{aligned}$$

These equations show that the function $\phi_1 - \phi_2 + k\rho^2$ is a special concircular scalar field on each part. Subtracting the equation (1.10) referred to M_1 and M_2 from the covariant derivatives of the equations (2.8), we see that the functions ϕ_1 and ϕ_2 satisfy the equations

$$(2.9) \quad \begin{aligned} \nabla_k \nabla_j \nabla_i \phi_1 &= -k(2g_{ji} \nabla_k \phi_1 + g_{kj} \nabla_i \phi_1 + g_{ki} \nabla_j \phi_1), \\ \nabla_r \nabla_q \nabla_p \phi_2 &= k(2g_{qp} \nabla_r \phi_2 + g_{rq} \nabla_p \phi_2 + g_{rp} \nabla_q \phi_2). \end{aligned}$$

§ 3. We shall prove the following simple lemmas, the proof of which we have not seen.

LEMMA 2. *Let g and g^* be complete Riemannian metrics on a manifold M , and C a differentiable curve defined on the interval $[0, 1)$ in M . If the length of the curve C is bounded with respect to the metric g , then so is it with respect to the metric g^* .*

PROOF. Denote by $|X|$ and $|X|^*$ the lengths of a vector X with respect to g and g^* , respectively. Since the metric g is complete and the curve C is bounded with respect to g , the curve C lies in a compact subset of M . If we make X vary over unit vectors at points in the subset, the length $|X|^*$ is bounded. Hence, for any vector X at any point in the subset, we have the inequality

$$|X|^* \leq K|X|,$$

K being a constant.

On the other hand, the curve C tends to a point $C(1)$ as t tends to 1. Adding the point $C(1)$ to C , we get the curve \bar{C} which is continuous on the closed interval $[0, 1]$ and differentiable on $[0, 1)$. If the tangent vector field of C is denoted by $\dot{C}(t)$, then the inequality

$$|\dot{C}(t)|^* \leq K|\dot{C}(t)|$$

is valid in $[0, 1)$. The integral of the right hand side from 0 to 1 is convergent, and so is the integral of the left hand side. Hence the length of \bar{C} is finite with respect to g^* , and the length of the curve C is bounded with respect to g^* .

LEMMA 3. *Let g be a complete metric on a manifold M and f be a diffeomorphism of M . Then the induced metric f^*g is also complete.*

PROOF. Let C be a geodesic with respect to the Riemannian connection

of the induced metric f^*g . Then the curve $f(C)$ is a geodesic with respect to the Riemannian connection of the metric g , and has the same arc length as C . Since g is complete, $f(C)$ is extendable for any large value of the arc length and so is C . Hence the induced metric f^*g is complete. Q. E. D.

§ 4. We are now going to give a

PROOF OF THEOREM 1. If, in addition to the assumptions of the theorem, both M and M^* are irreducible, then they are Einstein manifolds. If there is a non-homothetic conformal diffeomorphism of M onto M^* , then M and M^* are isometric to the sphere by virtue of Theorem B. Hence it suffices for us to show that a conformal diffeomorphism f is homothetic if M or M^* is reducible.

We may assume that the manifolds M and M^* are simply connected, by taking the universal covering manifolds if necessary, and M is reducible. If f is supposed not to be homothetic, then it follows from Lemma 1 that M is the Pythagorean product $M_1 \times M_2$ of Einstein manifolds M_1 and M_2 and occurs one of the following cases:

- (I) $k_1 = k = -c^2$ and $k_2 = -k = c^2$,
- (II) $k_1 = k = -c^2$ and $n_2 = 1$,
- (III) $n_1 = 1$ and $k_2 = -k = c^2$,
- (IV) M itself is an Einstein manifold of zero curvature,

where c is a positive constant.

CASE (I). We first suppose that the restriction of the gradient vector field ρ_λ on $M_1(P)$ through a point P does not identically vanish. Then there is a geodesic curve C in $M_1(P)$, along which ρ is not constant. Denote by s the arc length of the curve C . Since $M_1(P)$ is complete, C is infinitely extendable.

The first equations of (2.8) and (2.9) reduce along C to the ordinary differential equations

$$(4.1) \quad -\frac{d^2}{ds^2}(\phi_1 - c^2 \rho^2) = c^2(\phi_1 - c^2 \rho^2 + A)$$

$$(4.2) \quad \frac{d^3 \phi_1}{ds^3} = 4c^2 \frac{d\phi_1}{ds},$$

where we have put $-\phi_2(P) + b = A$. The general solution of (4.2) is written in the form

$$(4.3) \quad \phi_1 = A_1 e^{2cs} + B_1 e^{-2cs} + C_1$$

and consequently the general solution of (4.1) is in the form

$$(4.4) \quad \rho^2 = \frac{1}{c^2} A_1 e^{2cs} + \frac{1}{c^2} B_1 e^{-2cs} + A_2 e^{cs} + B_2 e^{-cs} + C_2,$$

where the coefficients A_1, B_1 and so on are integral constants.

By our assumption, at least one of A_1, B_1, A_2 and B_2 is different from zero. If, for example, $A_1 \neq 0$, then A_1 should be positive because ρ is positively valued. Put $A_1 = a^2$ and take a value s_0 so large that the inequality

$$\rho > \frac{a}{2c} e^{cs}$$

is valid for $s > s_0$. Let C^* be the image of C under the conformal diffeomorphism, and s^* the arc length of C^* . Then s^* is related to s by the differential equation

$$\frac{ds^*}{ds} = \frac{1}{\rho},$$

from which we have the inequality

$$s^* - s_0^* = \int_{s_0}^s \frac{1}{\rho} ds < \frac{2}{a} e^{-cs_0},$$

s_0^* being the value corresponding to s_0 . Hence the length of C^* is bounded. This contradicts to Lemma 2 because the length of C is unbounded.

We next assume that the components ρ_i identically vanish on $M_1(P)$ through any point P and consequently all over M . Then ρ is independent of points of M_1 , and $\phi_1 = 0$. We see from the first equation of (2.8) that

$$(4.5) \quad \phi_2 + c^2 \rho^2 = b.$$

Since M_2 is a complete Einstein manifold of positive scalar curvature, it is compact by Myers' theorem [4]. Hence ρ takes the minimal value m at a point of M_2 , where $\phi_2 = \rho_p \rho^p$ vanishes. Thus we see $b = c^2 m^2$ and

$$(4.6) \quad \phi_2 + c^2 \rho^2 = c^2 m^2.$$

Since $\phi_2 \geq 0$, it follows from (4.6) that ϕ_2 identically vanishes. Thus ρ should be a constant, and the diffeomorphism f is homothetic.

CASE (II). In the same way as that in Case (I), ρ is independent of points of M_1 . We take the arc length s as coordinate of M_2 . Then we have

$$\phi_2 = \rho_p \rho^p = \left(\frac{d\rho}{ds} \right)^2,$$

and the equation (4.5) reduces to

$$\left(\frac{d\rho}{ds} \right)^2 + c^2 \rho^2 = b.$$

The general solution of this equation is written in the form

$$\rho = A \cos(cs + B),$$

A and B being constants. Since ρ is negative for some value of s , this is a

contradiction.

CASE (III). As is seen from the discussion in § 2, the equation (2.2) is valid with $k = -k_2 = c^2$ and so are the other equations in this case. Therefore, by taking M_1 itself for the geodesic curve C , the argument in Case (I) is also applicable to this case. Q. E. D.

CASE (IV). Applying the above discussions to the inverse diffeomorphism f^{-1} of M^* onto M , we may consider only the case where M and M^* are Einstein manifolds and M is of zero curvature. Then the diffeomorphism f is homothetic by virtue of Theorem B. Q. E. D.

PROOF OF THEOREM 2. By means of Lemma 3, the theorem follows immediately from Theorem 1 and the fact that a complete Riemannian manifold admitting a non-isometric homothetic transformation is isometric to the Euclidean space [12], see also [1, 3]. Q. E. D.

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