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The logarithmic derivative and equations of evolution in a Banach space

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1. Introduction.

In [4, Theorems 1 and 2] T. Kato uses the notion of m-monotonicity to establish the existence of solutions to the evolution system

$$u'(t)+A(t)u(t)=0$$

where $A(t)$ is an (possibly nonlinear) operator on a Banach space E whose dual space E^* is uniformly convex. In Theorem 4.1 of this paper we use the logarithmic derivative (which is similar to a Lyapunov function) to extend this result to a general Banach space. In section 2 the logarithmic derivative is defined and certain basic properties are derived. In certain cases we establish a connection between operators which have a logarithmic derivative and those which are monotonic or accretive. In section 3 several existence theorems to ordinary differential equations are given and in section 4 we give the extension of the result of Kato mentioned above. In section 5 sufficient conditions for an operator A to generate a semigroup of operators on E are given.

2. Operators with logarithmic derivative.

Let E be a Banach space over the real or complex field with norm denoted by $|\cdot|$, and let E^* be the dual space of E with the norm on E^* also denoted by $|\cdot|$. We will let $\rightarrow$ denote norm convergence on E and $\xrightarrow{w}$ denote weak convergence on E. For each subset D of E let $H(D, E)$ denote the class of all functions from D into E. In [4], Kato defines a member A of $H(D, E)$ to be monotonic if $|x-y+\rho[Ax-Ay]|\geqq|x-y|$ for all x and y in D and all $\rho>0$. If, in addition, the image of $1+\rho A$ (where $1+\rho A$ is the member B of $H(D, E)$ defined by $Bx=x+\rho Ax$ for all x in D) is E for each $\rho>0$, then A is said to be m-monotonic.

For each x in E define $F(x)=\{f\in E^*: (x, f)=|x|^2=|f|^2\}$ and $G(x)=\{f \in E^*: |f|=1 \text{ and } (x, f)=|x|\}$. It is immediate that if $x\neq 0$, then f is in

$G(x)$ if and only if $|x|f$ is in $F(x)$. Kato [4, Lemma 1.1] shows that a member A of $H(D, E)$ is monotonic if and only if for each x and y in D there is an f in $F(x-y)$ such that $\mathrm{Re}(Ax-Ay, f) \geqq 0$. Hence, it follows that A is monotonic if and only if there is a g in $G(x-y)$ such that $\mathrm{Re}(Ax-Ay, g) \geqq 0$.

DEFINITION 2.1. For each subset D of E the class $LN(D, E)$ will consist of all members A of $H(D, E)$ with the property that there is a constant K such that for each bounded subset Q of D for which the image of Q under A is bounded, and for each pair of positive numbers β and ε, there is a positive number δ such that whenever $0<h \leqq \delta$, x and y are in Q with $|x-y| \geqq \beta$, then

$$(|x-y+h[Ax-Ay]|-|x-y|)/h \leqq K|x-y|+\varepsilon. \tag{2a}$$

If A is in $LN(D, E)$, denote by $L'[A]$ the smallest number K such that the inequality in (2a) holds.

REMARK. If A is in $LN(D, E)$, x and y are in D, and $0<k<h$, then $-|Ax-Ay| \leqq (|x-y+k[Ax-Ay]|-|x-y|)/k \leqq (|x-y+h[Ax-Ay]|-|x-y|)/h \leqq |Ax-Ay|$. Thus, if $x \neq y$, by taking $Q=\{x, y\}$ and $\beta=|x-y|$ in the definition above, we have

$$\lim_{h \to +0} (|x-y+h[Ax-Ay]|-|x-y|)/h \leqq L'[A]|x-y|.$$

PROPOSITION 2.1. *Suppose that A and B are in $LN(D, E)$. Then*

i) *if $\rho>0$, ρA is in $LN(D, E)$ with $L'[\rho A]=\rho L'[A]$,*

ii) *if for each bounded subset Q of D such that $A+B$ is bounded on Q it follows that A and B are bounded on Q, then $A+B$ is in $LN(D, E)$ with $L'[A+B] \leqq L'[A]+L'[B]$, and*

iii) *if a is in the field over E, $L'[A+a1]=L'[A]+\mathrm{Re}(a)$.*

INDICATION OF PROOF. Part i) follows from the equality $(|x-y+h[\rho Ax-\rho Ay]|-|x-y|)h=\rho(|x-y+\rho h[Ax-Ay]|-|x-y|)/(\rho h)$ and part ii) follows from the inequality $(|x-y+h[Ax+Bx-Ay-By]|-|x-y|)/h \leqq (|x-y+2h[Ax-Ay]|-|x-y|)/(2h)+(|x-y+2h[Bx-By]|-|x-y|)/(2h)$. Since $(|x-y+h[ax-ay]|-|x-y|)/h=|x-y|(|1+ha|-1)/h$ and $(|1+ha|-1)/h \to \mathrm{Re}(a)$ as $h \to +0$, we have $L'[a1]=\mathrm{Re}(a)$. Thus, from ii), $L'[A+a1] \leqq L'[A]+\mathrm{Re}(a)$ and $L'[A]=L'[A+a1-a1] \leqq L'[A+a1]+L'[-a1]=L'[A+a1]-\mathrm{Re}(a)$ and iii) follows.

DEFINITION 2.2. A member A of $H(D, E)$ will be called uniformly monotonic if $-A$ is in $LN(D, E)$ and $L'[-A] \leqq 0$. If, in addition, the image of $1+\rho A$ is E for all $\rho>0$, then A will be called uniformly m-monotonic.

PROPOSITION 2.2. *If A is a uniformly monotonic (resp. uniformly m-monotonic) member of $H(D, E)$, then A is monotonic (resp. m-monotonic).*

INDICATION OF PROOF. Let x and y be in D, $h>0$, and g in $G(x-y)$. Then $-\mathrm{Re}(Ax-Ay, g)=[\mathrm{Re}(x-y-h[Ax-Ay], g)-|x-y|]/h \leqq (|x-y-h[Ax$

$-Ay]|-|x-y|)/h$. Since $L'[-A]\leqq 0$, we have, by letting $h\to+0$, that $-\mathrm{Re}(Ax-Ay, g)\leqq 0$ and the proposition follows.

LEMMA 2.1. *If A is a monotonic member of $H(D, E)$ and the image of $1+\rho_0 A$ is E for some $\rho_0>0$, then A is m-monotonic.*

A proof of this lemma can be found in [7, Lemma 4].

For each subset D of E let $LIP(D, E)$ denote the class of all members A of $H(D, E)$ for which there is a constant K such that $|Ax-Ay|\leqq K|x-y|$ for all x and y in D. Denote by $N'[A]$ the smallest constant K for which this inequality holds. If A is in $LIP(D, E)$, x and y are in D, and $h>0$, then the inequality $|(|x-y+h[Ax-Ay]|-|x-y|)/h|\leqq|Ax-Ay|\leqq N'[A]|x-y|$ shows that A is in $LN(D, E)$ and $|L'[A]|\leqq N'[A]$. For each A in $LIP(D, E)$ let $M'[A]=\lim_{h\to+0}(N'[1+hA]-1)/h$. If x and y are in E and $h>0$, then $(|x-y+h[Ax-Ay]|-|x-y|)/h\leqq|x-y|(N'[1+hA]-1)/h\to|x-y|M'[A]$ as $h\to+0$ so that $L'[A]\leqq M'[A]$. If A is a linear member of $LIP(E, E)$, it can be shown that $L'[A]=M'[A]$.

LEMMA 2.2. *If A is in $LIP(E, E)$ and $\rho>0$ is such that $\rho N'[A]<1$, then*

i) *$(1+\rho A)^{-1}$ is in $LIP(E, E)$ and*

ii) *if $0<\delta<1$ and Q is a bounded subset of E, then there is a constant K such that if $0\leqq\rho\leqq\delta$ and x is in Q, then $|(1+\rho A)^{-1}x-(1-\rho A)x|\leqq K\rho^2$.*

INDICATION OF PROOF. The proof is contained in a proof of J. W. Neuberger [6, Lemma 1] and we outline it here. Let $B_0=1$ and for $n\geqq 1$ take $B_n=1-\rho AB_{n-1}$. Let $M>0$ be such that $|Ax|\leqq M$ for all x in Q and let $\beta=\rho N'[A]<1$. If $n\geqq 1$ we have $|B_nx-B_{n-1}x|\leqq\beta|B_{n-2}x-B_{n-1}x|\leqq\cdots\leqq\beta^{n-1}|\rho Ax|\leqq\beta^n K_1$ where $K_1=M/N'[A]$. Consequently, if $m>n\geqq 1$, then $|B_mx-B_nx|\leqq\sum_{i=n+1}^{m}|B_ix-B_{i-1}x|\leqq\beta^{n+1}K_1/(1-\beta)$. It follows that $B_nx\to(1+\rho A)^{-1}x$ and that $(1+\rho A)^{-1}$ is in $LIP(E, E)$ so that i) is true. Since $|(1+\rho A)^{-1}x-(1-\rho A)x|=\lim_{m\to\infty}|B_mx-B_1x|\leqq\beta^2K_1/(1-\beta)$ we have ii).

PROPOSITION 2.3. *If A is in $LIP(E, E)$ then A is monotonic if and only if A is uniformly m-monotonic.*

INDICATION OF PROOF. The "if" part follows from Proposition 2.2. Suppose that A is monotonic. By Lemmas 2.2 and 2.1 we have that A is m-monotonic. Let Q be a bounded subset of E. By ii) of Lemma 2.2 there are constants K and δ such that $|(1+hA)^{-1}x-(1-hA)x|\leqq Kh^2$ for all x in Q and $0<h\leqq\delta$. Thus, since $|(1+hA)^{-1}x-(1+hA)^{-1}y|\leqq|x-y|$, we have $(|x-y-h[Ax-Ay]|-|x-y|)/h=(|(1-hA)x-(1-hA)y|-|x-y|)/h\leqq(|(1+hA)^{-1}x-(1+hA)^{-1}y|+2Kh^2-|x-y|)/h\leqq 2Kh$ and the proposition follows.

LEMMA 2.3. *Suppose that E^* is uniformly convex, A is in $H(D, E)$, and Q is a bounded subset of D for which there is a constant M such that $|Ax|\leqq M$*

for all x *in* Q. *Then for each pair of positive numbers* β *and* ε *there is a* $\delta>0$ *such that if* x *and* y *are in* Q, $|x-y|\geqq\beta$, $0<h\leqq\delta$, *and* g *is the member of* $G(x-y)$, *we have* $\mathrm{Re}\,(Ax-Ay, g)\leqq(|x-y+h[Ax-Ay]|-|x-y|)/h\leqq\mathrm{Re}\,(Ax-Ay, g)+\varepsilon$.

INDICATION OF PROOF. Since E^* is uniformly convex, let ε' be such that if f_1 and f_2 are in E^* with $|f_1|=|f_2|=1$ and $|f_1+f_2|\geqq 2-\varepsilon'$, then $|f_1-f_2|\leqq\varepsilon/(2M)$. Choose $\delta=\varepsilon'\beta/(4M)$ and let g be in $G(x-y)$, $0<h\leqq\delta$, and f be in $G(x-y+h[Ax-Ay])$. Then $\mathrm{Re}\,(Ax-Ay, g)=[\mathrm{Re}\,(x-y+h[Ax-Ay], g)-|x-y|]/h\leqq(|x-y+h[Ax-Ay]|-|x-y|)/h$ which gives the left side of the inequality. By the choice of f,

$$(|x-y+h[Ax-Ay]|-|x-y|)/h=[\mathrm{Re}\,(x-y+h[Ax-Ay], f)-|x-y|]/h$$
$$\leqq\mathrm{Re}\,(x-y, f)/h+|Ax-Ay|-|x-y|/h\,.$$

Transposing terms and multiplying by h we have $|x-y|-h|Ax-Ay|+|x-y+h[Ax-Ay]|-|x-y|\leqq\mathrm{Re}\,(x-y, f)$ and hence, $|x-y|-4hM\leqq\mathrm{Re}\,(x-y, f)$. Thus, $|f+g|\geqq[\mathrm{Re}\,(x-y, f+g)]/|x-y|\geqq 2-4hM/|x-y|\geqq 2-\varepsilon'$. By the choice of ε', $|f-g|\leqq\varepsilon/(2M)$ and since $\mathrm{Re}\,(x-y, f)\leqq|x-y|$ and $\mathrm{Re}\,(Ax-Ay, f-g)\leqq|Ax-Ay|\,|f-g|\leqq\varepsilon$, we have

$$(|x-y+h[Ax-Ay]|-|x-y|)/h=\mathrm{Re}\,(Ax-Ay, f)+[\mathrm{Re}\,(x-y, f)-|x-y|]/h$$
$$\leqq\mathrm{Re}\,(Ax-Ay, g)+\mathrm{Re}\,(Ax-Ay, f-g)$$
$$\leqq\mathrm{Re}\,(Ax-Ay, g)+\varepsilon$$

and the lemma is true.

As an immediate consequence of Lemma 2.3 and the definition of F and G we have

THEOREM 2.1. *If* E^* *is uniformly convex and* A *is in* $H(D, E)$, *these are equivalent:*

i) A *is in* $LN(D, E)$.

ii) *There is a constant* K *such that* $\mathrm{Re}\,(Ax-Ay, g)\leqq K|x-y|$ *for all* x *and* y *in* D *and* g *in* $G(x-y)$.

iii) *There is a constant* K *such that* $\mathrm{Re}\,(Ax-Ay, f)\leqq K|x-y|^2$ *for all* x *and* y *in* D *and* f *in* $F(x-y)$.

Furthermore, if i) *holds, then* $L'[A]$ *is the smallest constant* K *such that the inequality in* ii)*—or* iii)*—holds.*

From Theorem 2.1 and Proposition 2.2 we have

COROLLARY 2.1. *If* E^* *is uniformly convex, then* A *is monotonic* (*resp. m-monotonic*) *if and only if* A *is uniformly monotonic* (*resp. uniformly m-monotonic*).

NOTATION. Suppose that A is in $LN(D, E)$ and $c\leqq -L'[A]$. Then $L'[A+c1]=L'[A]+c\leqq 0$ so that $-A-c1$ is uniformly monotonic. Assume that $-A-c1$ is uniformly m-monotonic and for each positive integer n define

(2b)
1) $J_n^c=[1-n^{-1}(A+c1)]^{-1}$.
2) $A_n^c=-(A+c1)J_n^c=n(1-J_n^c)$.
3) $B_n^c=AJ_n^c=-A_n^c-cJ_n^c=-[n1-(n-c)J_n^c]$.

PROPOSITION 2.4. *If A is in $LN(D, E)$ and there is a $c_0\leqq -L'[A]$ such that $-A-c_0 1$ is uniformly m-monotonic, then $-A-c1$ is uniformly m-monotonic for all $c\leqq -L'[A]$.*

INDICATION OF PROOF. Let $c\leqq -L'[A]$ and choose $\rho>0$ sufficiently small so that $\rho|c-c_0|<1$. Then $1+\rho(-A-c1)=1+\rho(-A-c_0 1)+\rho(c_0-c)1=[1+\rho(c_0-c)]\{1+\rho[1+\rho(c_0-c)]^{-1}[-A-c_0 1]\}$. Since $\rho[1+\rho(c_0-c)]^{-1}>0$, we have that the image of $1+\rho[1+\rho(c_0-c)]^{-1}[-A-c_0 1]$ is E and so the image of $1+\rho(-A-c1)$ is E. The assertion of the proposition now follows from Lemma 2.1.

LEMMA 2.4. *Using the notation above we have*

i) *J_n^c is in $LIP(E, E)$ with $N'[J_n^c]\leqq 1$ for all $n\geqq 1$.*

ii) *A_n^c is in $LIP(E, E)$ with $N'[A_n^c]\leqq 2n$ and $L'[-A_n^c]\leqq 0$ for all $n\geqq 1$.*

iii) *B_n^c is in $LIP(E, E)$ with $N'[B_n^c]\leqq 2n+|c|$ and $L'[B_n^c]\leqq |c|$ for all $n\geqq 1$.*

iv) *If x is in D then $|A_n^c x|\leqq |(A+c1)x|$ and $|B_n^c x|\leqq(1+|c|n^{-1})|(A+c1)x| +|cx|$ for all $n\geqq 1$.*

v) *If x is in the closure of D then $J_n^c x\to x$ as $n\to\infty$.*

INDICATION OF PROOF. i) is immediate since $-A-c1$ is m-monotonic and ii) follows from [**4**, Lemma 2.3] and Proposition 2.3. Since $B_n^c=-A_n^c-cJ_n^c$, iii) follows from i) and ii) and from part ii) of Proposition 2.1. iv) follows from [**4**, Lemma 2.3] and the identity $B_n^c=-A_n^c-cJ_n^c=-A_n^c-c(1-n^{-1}A_n^c)$. v) is Lemma 2.4 of [**4**].

LEMMA 2.5. *Let A be in $LN(D, E)$ and suppose that A has the property that for each sequence (x_n) in D such that $x_n\to x$ and the $|Ax_n|$ are bounded, it follows that $Ax_n\xrightarrow{w}Ax$. Using the notation above we have the following:*

i) *If (y_n) is a sequence in E such that $y_n\to y$ and the $|A_n^c y_n|$ are bounded, then y is in D, $A_n^c y_n\xrightarrow{w}-(A+c1)y$, and $B_n^c y_n\xrightarrow{w}Ay$.*

ii) *If z is in D then $A_n^c z\xrightarrow{w}-(A+c1)z$ and $B_n^c z\xrightarrow{w}Az$.*

INDICATION OF PROOF. It is immediate that $-Ax_n-cx_n\xrightarrow{w}-Ax-cx$. Letting $x_n=J_n^c y_n$ we have $y_n-x_n=n^{-1}A_n^c y_n\to 0$ so that $x_n\to y$. Hence, $A_n^c y_n =-Ax_n-cx_n\xrightarrow{w}-(A+c1)y$ and since $B_n^c=-A_n^c-cJ_n^c$, we have $B_n^c y_n\xrightarrow{w}Ay$. Thus i) is true and part ii) follows from i) with $y_n=z$ and part iv) of Lemma 2.4.

In [**2**] Browder defines a member A of $H(D, E)$ to be accretive if $\mathrm{Re}(Ax-Ay, f)\geqq 0$ for all x and y in D and all f in $F(x-y)$. Thus, A is accretive if and only if $\mathrm{Re}(Ax-Ay, g)\geqq 0$ for all x and y in D and all g in $G(x-y)$, and if A is accretive, then A is monotonic.

PROPOSITION 2.5. *Let A be in $H(D, E)$. Then $-A$ is accretive if and only if* $\lim_{h\to 0}(|x-y+h[Ax-Ay]|-|x-y|)/h\leqq 0$ *for all x and y in D.*

INDICATION OF PROOF. If g is in $G(x-y)$ then $\mathrm{Re}(Ax-Ay, g)=[\mathrm{Re}(x-y+h[Ax-Ay], g)-|x-y|]/h\leqq(|x-y+h[Ax-Ay]|-|x-y|)/h$ for all $h>0$. Thus, if $\lim_{h\to+0}(|x-y+h[Ax-Ay]|-|x-y|)/h\leqq 0$, then $\mathrm{Re}(Ax-Ay, g)\leqq 0$ for all g in $G(x-y)$ so that $-A$ is accretive. Now suppose that $-A$ is accretive. For each $h>0$ let g_h be in $G(x-y+h[Ax-Ay])$. From the above, if g is in $G(x-y)$, then $\mathrm{Re}(Ax-Ay, g)\leqq(|x-y+h[Ax-Ay]|-|x-y|)/h=[\mathrm{Re}(x-y+h[Ax-Ay], g_h)-|x-y|]/h=\mathrm{Re}(x-y, g_h)/h+\mathrm{Re}(Ax-Ay, g_h)-|x-y|/h$. Transposing terms and multiplying by h, we have $|x-y|+h[\mathrm{Re}(Ax-Ay, g)-\mathrm{Re}(Ax-Ay, g_h)]\leqq\mathrm{Re}(x-y, g_h)$. Since $|(x-y, g_h)|\leqq|x-y|$, it follows that $\lim_{h\to+0}(x-y, g_h)=|x-y|$. Since the unit ball in E^* is w^* compact, there is an f in E^* with $|f|\leqq 1$ and a sequence of positive numbers (h_n) such that $\lim_{n\to\infty}h_n=0$ and if $f_n=g_{h_n}$ for each $n\geqq 1$, then $\lim_{n\to\infty}(z, f_n)=(z, f)$ for each z in E. Since $(x-y, f)=\lim_{n\to\infty}(x-y, f_n)=|x-y|, f$ is in $G(x-y)$ and hence, $\mathrm{Re}(Ax-Ay, f)\leqq 0$. Consequently, $\lim_{h\to+0}(|x-y+h[Ax-Ay]|-|x-y|)/h=\lim_{n\to\infty}(|x-y+h_n[Ax-Ay]|-|x-y|)/h_n=\lim_{n\to\infty}[\mathrm{Re}(x-y+h_n[Ax-Ay], f_n)-|x-y|]/h_n\leqq\lim_{n\to\infty}\mathrm{Re}(Ax-Ay, f_n)=\mathrm{Re}(Ax-Ay, f)\leqq 0$ and the proposition is true.

COROLLARY 2.2. *If A is in $H(D, E)$ and K is a constant, then these are equivalent:*

i) $\mathrm{Re}(Ax-Ay, f)\leqq K|x-y|^2$ *for all x and y in D and f in $F(x-y)$.*

ii) $\mathrm{Re}(Ax-Ay, g)\leqq K|x-y|$ *for all x and y in D and g in $G(x-y)$.*

iii) $\lim_{h\to+0}(|x-y+h[Ax-Ay]|-|x-y|)/h\leqq K|x-y|$ *for all x and y in D.*

INDICATION OF PROOF. The proof that i) is equivalent to ii) is immediate. It follows that ii) and iii) are equivalent from Proposition 2.5 and the proof of Proposition 2.1.

3. Ordinary differential equations in $LN(D, E)$.

Let I be an interval in the real line and let $\{A(t): t\in I\}$ be a family of members of $LN(D, E)$. In this section we will be concerned with solving the initial value problem

$$u'(t)=A(t)u(t), \qquad u(a)=z \tag{3a}$$

where a is in I, z is in D, and the function $(t, x)\to A(t)x$ of $I\times D$ into E is continuous and maps bounded subsets of $I\times D$ into bounded subsets of E.

DEFINITION 3.1. If Q is a bounded subset of D, the family $\{A(t): t\in I\}$

is said to have uniform logarithmic derivative on $I\times Q$ if there are constants M and K such that $|A(t)x|\leqq M$ for all (t, x) in $I\times Q$ and for each pair of positive numbers β and ε, there is a positive number δ such that if t is in I, x and y are in Q with $|x-y|\geqq\beta$, and $0<h\leqq\delta$, then

$$(|x-y+h[A(t)x-A(t)y]|-|x-y|)/h\leqq K|x-y|+\varepsilon.$$

LEMMA 3.1. *Suppose that I is a compact interval, Q is a bounded subset of D, and the function $(t, x)\rightarrow A(t)x$ of $I\times D$ into E is continuous and maps bounded subsets of D into bounded subsets of E.*

i) *If $A(t)$ is in $LIP(D, E)$ with $N'[A(t)]\leqq K$ for all t in I then $\{A(t): t\in I\}$ has uniform logarithmic derivative on $I\times Q$.*

ii) *If the family of functions $\{g_x : x\in Q\}$ where $g_x(t)=A(t)x$ is equicontinuous on I and $L'[A(t)]\leqq K$, then $\{A(t): t\in I\}$ has uniform logarithmic derivative on $I\times Q$.*

iii) *If E^* is uniformly convex and $\operatorname{Re}(A(t)x-A(t)y, f)\leqq K|x-y|^2$ for all x and y in Q, t in I, and f in $F(x-y)$, then $\{A(t): t\in I\}$ has uniform logarithmic derivative on $I\times Q$.*

INDICATION OF PROOF. Part i) follows from the inequality $(|x-y+h[A(t)x-A(t)y]|-|x-y|)/h\leqq|A(t)x-A(t)y|\leqq K|x-y|$. Let β and ε be positive numbers and choose $\delta'>0$ such that if $|t-s|\leqq\delta'$, then $|A(t)x-A(s)x|\leqq\varepsilon/3$ for all x in Q. Let $(t_i)_0^n$ be a partition of I such that $|t_i-t_{i-1}|\leqq\delta'$ and choose δ_i so that $(|x-y+h[A(t_i)x-A(t_i)y]|-|x-y|)/h\leqq L'[A(t_i)]|x-y|+\varepsilon/3$ for x and y in Q with $|x-y|\geqq\beta$, and $0<h<\delta_i$. Let $\delta=\min\{\delta_i : 1\leqq i\leqq n\}$. If t is in I, there is a t_i such that $|t-t_i|\leqq\delta'$ so that if x and y are in Q with $|x-y|\geqq\beta$ and $0<h\leqq\delta$, we have $(|x-y+h[A(t)x-A(t)y]|-|x-y|)/h\leqq(|x-y+h[A(t_i)x-A(t_i)y]|-|x-y|)h+|A(t)x-A(t_i)x|+|A(t)y-A(t_i)y|\leqq L'[A(t_i)]|x-y|+\varepsilon/3+\varepsilon/3+\varepsilon/3\leqq K|x-y|+\varepsilon$ and part ii) follows. The proof of part iii) is similar to that of Lemma 2.3 and is omitted.

LEMMA 3.2. *Let I be an open interval and q a continuous function from I into E such that $q'_+(t)$ exists for all t in I. If $p(t)=|q(t)|$ for all t in I, then $p'_+(t)$ exists and*

$$p'_+(t)=\lim_{h\rightarrow+0}(|q(t)+hq'_+(t)|-|q(t)|)/h.$$

Furthermore, if $\delta>0$, $p'_+(t)\leqq(|q(t)+\delta q'_+(t)|-|q(t)|)/\delta$ in as much as the expression in the limit is nonincreasing as $h\rightarrow+0$.

For a proof of this lemma see [**3**, p. 3].

THEOREM 3.1. *Let a be a real number, $T>0$, and $I=[a, a+T]$. Also let z be in E, D a bounded neighborhood of z, and $\{A(t): t\in I\}$ a family of members of $LN(D, E)$ such that*

1) *The function $(t, x)\rightarrow A(t)x$ of $I\times D$ into E is continuous.*

2) *The family $\{A(t): t\in I\}$ has uniform logarithmic derivative on $I\times D$.*

Then there is a $\rho>0$ and a unique continuously differentiable function u from $[a, a+\rho]$ into D such that $u(a)=z$ and $u'(t)=A(t)u(t)$ for all t in $[a, a+\rho]$.

INDICATION OF PROOF. Let M and K be as in Definition 3.1 and assume, without loss, that K is positive. Choose $0<\rho\leqq T$ so that if $|x-z|\leqq\rho M$, then x is in D. For each positive integer n let (t_i^n) be a partition of $[a, a+\rho]$ such that $|t_{i+1}^n-t_i^n|\leqq n^{-1}$. For each $n\geqq 1$ let u_n be the function from $[a, a+\rho]$ into E defined by $u_n(a)=z$, and if $t_i^n\leqq t\leqq t_{i+1}^n$, then $u_n(t)=u_n(t_i^n)+\int_{t_i^n}^t A(s)\,u_n(t_i^n)ds$. It follows that u_n maps $[a, a+\rho]$ into D, $|u_n(t)-u_n(s)|\leqq M|t-s|$, and if $t_i^n\leqq t<t_{i+1}^n$, then $(u_n)'_+(t)=A(t)u_n(t_i^n)$. Suppose that ε is a positive number and for the pair $\beta'=\varepsilon\exp(-K\rho)/6$ and $\varepsilon'=\varepsilon K\exp(-K\rho)/3$, choose $\delta>0$ such that $(|x-y+h[A(t)x-A(t)y]|-|x-y|)/h\leqq K|x-y|+\varepsilon'$ whenever $0<h\leqq\delta$ and x and y are in D with $|x-y|\geqq\beta'$. Choose $n_0\geqq 1$ so that $n_0^{-1}\leqq\min\{\beta'/(2M), \varepsilon\exp(-K\rho)/[12KM(K+\delta^{-1})]\}$. The claim is that whenever $m>n\geqq n_0$, then $|u_n(t)-u_m(t)|\leqq\varepsilon$ for all t in $[a, a+\rho]$. Assume, for contradiction, that there is a t_1 in $[a, a+\rho]$ and integers n and m such that $m>n\geqq n_0$, and that $|u_n(t_1)-u_m(t_1)|>\varepsilon$. Let $p(t)=|u_n(t)-u_m(t)|$ for all t in $[a, a+\rho]$. Then p is continuous, $p(a)=0$, and $p(t_1)>\varepsilon$, so there is a t_0 in (a, t_1) such that $p(t_0)=2\beta'$ and $p(t)\geqq 2\beta'$ for all t in $[t_0, t_1]$. Thus, if t is in $[t_0, t_1]$ there is a pair of integers i and j such that $t_i^n\leqq t<t_{i+1}^n$, $t_j^m\leqq t<t_{j+1}^m$, $(u_n)'_+(t)=A(t)u_n(t_i^n)$, and $(u_m)'_+(t)=A(t)u_m(t_j^m)$. By Lemma 3.2 we have

$$\begin{aligned}
p'_+(t) &\leqq (|u_n(t)-u_m(t)+\delta[A(t)u_n(t_i^n)-A(t)u_m(t_j^m)]|-|u_n(t)-u_m(t)|)/\delta\\
&\leqq (|u_n(t_i^n)-u_m(t_j^m)+\delta[A(t)u_n(t_i^n)-A(t)u_m(t_j^m)]|-|u_n(t_i^n)-u_m(t_j^m)|)/\delta\\
&\quad +2|u_n(t)-u_n(t_i^n)|/\delta+2|u_m(t)-u_m(t_j^m)|/\delta\\
&\leqq K|u_n(t_i^n)-u_m(t_j^m)|+\varepsilon'+2M\delta^{-1}(n^{-1}+m^{-1})\\
&\leqq Kp(t)+2MK(n^{-1}+m^{-1})+\varepsilon'+2M\delta^{-1}(n^{-1}+m^{-1})
\end{aligned}$$

where we used that $|u_n(t_i^n)-u_m(t_j^m)|\geqq|u_n(t)-u_m(t)|-|u_n(t)-u_n(t_i^n)|-|u_m(t_j^m)-u_m(t)|\geqq 2\beta'-2n_0^{-1}M\geqq\beta'$. Thus, $p'_+(t)\leqq Kp(t)+\varepsilon'+4Mn_0^{-1}(K+\delta^{-1})\leqq Kp(t)+2\varepsilon K\exp(-K\rho)/3$ for all t in $[t_0, t_1]$. Solving this differential inequality gives

$$p(t)\leqq p(t_0)\exp(K(t-t_0))+2\varepsilon\exp(-K\rho)[\exp(K(t-t_0))-1]/3\,.$$

Since $p(t_0)=|u_n(t_0)-u_m(t_0)|=\varepsilon\exp(-K\rho)/3$ and $t_1-t_0\leqq\rho$, we have $|u_n(t_1)-u_m(t_1)|=p(t_1)\leqq\varepsilon/3+2\varepsilon/3=\varepsilon$ which is a contradiction to the assumption that $|u_n(t_1)-u_m(t_1)|>\varepsilon$. Consequently, the sequence (u_n) is uniformly Cauchy on $[a, a+\rho]$ and hence, converges to a continuous limit u uniformly on $[a, a+\rho]$. For each integer $n\geqq 1$ define the function g_n from $[a, a+\rho]$ into D by $g_n(t)=A(t)u_n(t_i^n)$ whenever $t_i^n\leqq t<t_{i+1}^n$. By the construction of u_n we have that $|g_n(t)|\leqq M$ and that $u_n(t)=z+\int_a^t g_n(s)ds$ for all t in $[a, a+\rho]$. If $t_i^n\leqq t<t_{i+1}^n$

we have $|u_n(t_i^n)-u(t)|\leqq|u_n(t_i^n)-u_n(t)|+|u_n(t)-u(t)|\leqq n^{-1}M+|u_n(t)-u(t)|$ so that if $g(t)=A(t)u(t)$, then $g_n(t)\to g(t)$ by the continuity of $A(t)$. Furthermore, since the sequence (g_n) is uniformly bounded, it follows by bounded convergence that $u(t)=\lim_{n\to\infty}u_n(t)=\lim_{n\to\infty}z+\int_a^t g_n(s)ds=z+\int_a^t A(s)u(s)ds$. Thus, u is continuously differentiable and satisfies (3a) on $[a, a+\rho]$. Suppose that v is a continuously differentiable function on $[a, a+\rho]$ which satisfies (3a). If $p(t)=|u(t)-v(t)|$ for all t in $[a, a+\rho]$, then $p'_+(t)=\lim_{h\to+0}(|u(t)-v(t)+h[A(t)u(t)-A(t)v(t)]|-|u(t)-v(t)|)/h\leqq Kp(t)$. As $p(a)=0$ we have $p(t)=|u(t)-v(t)|=0$ for all t in $[a, a+\rho]$ so that $v=u$. This completes the proof of the theorem.

THEOREM 3.2. *Let S denote the set of nonnegative real numbers and suppose that $\{A(t): t\in S\}$ is a family of members of $LN(E, E)$ with the following properties:*

1) *The function $(t, x)\to A(t)x$ is continuous.*
2) *The family $\{A(t): t\in S\}$ has uniform logarithmic derivative on bounded subsets of $S\times E$.*
3) *There is a continuous function c from S into the real numbers such that $L'[A(t)]\leqq c(t)$ for all t in S.*

Then for each a in S and z in E, there is a unique continuously differentiable function u from $[a, \infty)$ into E such that

$$u'(t)=A(t)u(t)\,,\qquad u(a)=z \tag{3b}$$

for all t in $[a, \infty)$. Furthermore, $|u(t)-z|\leqq\int_a^t|A(s)z|\exp\left(\int_s^t c(r)dr\right)ds$ for all t in $[a, \infty)$, and if $U(a, t)z$ denotes $u(t)$ for all t in $[a, \infty)$ and z in E, then $U(a, t)$ is in $LIP(E, E)$ with $N'[U(a, t)]\leqq\exp\left(\int_a^t c(s)ds\right)$.

INDICATION OF PROOF. It follows from Theorem 3.1 that there is a solution u to (3b) on some interval $[a, a+\rho)$ where $\rho>0$. Also, u can be extended so long as its image remains in a bounded subset of E. However, so long as u exists, we have that if $p(t)=|u(t)-z|$, then

$$\begin{aligned}p'_+(t)&=\lim_{h\to+0}(|u(t)-z+hA(t)u(t)|-|u(t)-z|)/h\\&\leqq\lim_{h\to+0}(|u(t)-z+h[A(t)u(t)-A(t)z]|-|u(t)-z|)/h+|A(t)z|\\&\leqq L'[A(t)]|u(t)-z|+|A(t)z|\\&\leqq c(t)p(t)+|A(t)z|.\end{aligned}$$

Solving this differential inequality gives $|u(t)-z|\leqq\int_a^t|A(s)z|\exp\left(\int_s^t c(r)dr\right)ds$. It follows that u is bounded on bounded subintervals of $[a, \infty)$ and hence, can be extended to all of $[a, \infty)$. If w is in E and v is a solution to (3b) such that $v(a)=w$, then letting $q(t)=|u(t)-v(t)|$ we have

$$q'_+(t)=\lim_{h\to+0}(|u(t)-v(t)+h[A(t)u(t)-A(t)v(t)]|-|u(t)-v(t)|)/h$$

$$\leqq c(t)q(t).$$

Thus, $|u(t)-v(t)|\leqq|u(a)-v(a)|\exp\left(\int_a^t c(s)ds\right)$ and the assertions of the theorem follow.

COROLLARY 3.1. *Suppose that* $\{A(t): t\in S\}$ *is a family in* $LIP(E, E)$ *for which there is a continuous function* d *from* S *into* S *such that* $N'[A(t)]\leqq d(t)$ *for all* t *in* S. *Furthermore, suppose that for each bounded subset* $I\times Q$ *of* $S\times E$ *there are constants* $M>0$ *and* $\delta>0$ *such that if* (t, s) *is in* $I\times I$ *with* $|t-s|\leqq\delta$ *and* x *is in* Q, *then* $|A(t)x-A(s)x|\leqq|t-s|M(1+|A(s)x|)$. *Then the conclusions of Theorem* 3.2 *are valid.*

INDICATION OF PROOF. Since $L'[A(t)]\leqq N'[A(t)]$ there is a continuous function c on S satisfying condition 3) of Theorem 3.2. By using part i) of Lemma 3.1 and Theorem 3.2 we need only show that the function $(t, x)\to A(t)x$ is continuous and maps bounded subsets of $S\times E$ into bounded subsets of E. This is routine and the proof is omitted.

THEOREM 3.3. *Let* a *be a real number,* $T>0$, *and* $I=[a, a+T]$. *Also let* z *be in* E, D *a bounded neighborhood of* z, *and* $\{A(t): t\in I\}$ *a family of members of* $H(D, E)$ *such that*

1) *The function* $(t, x)\to A(t)x$ *of* $I\times D$ *into* E *is continuous and bounded.*
2) *The family* $\{A(t): t\in I\}$ *is uniformly equicontinuous on* D.
3) *There is a constant* K *such that* $\mathrm{Re}\,(A(t)x-A(t)y, f)\leqq K|x-y|^2$ *for all* x *and* y *in* D, t *in* I, *and* f *in* $F(x-y)$.

Then there is a $\rho>0$ *and a unique continuously differentiable function* u *from* $[a, a+\rho]$ *into* D *such that* $u(a)=z$ *and* $u'(t)=A(t)u(t)$ *for all* t *in* $[a, a+\rho]$.

REMARK. Note that 2) holds if the function $(t, x)\to A(t)x$ is uniformly continuous on $I\times D$. Furthermore, from Corollary 2.2 we have $\lim_{h\to+0}(|x-y+h[A(t)x-A(t)y]|-|x-y|)/h\leqq K|x-y|$ for all x and y in D and t in I.

INDICATION OF PROOF. Assume that $K>0$ and let M be such that $|A(t)x|\leqq M$ for all (t, x) in $I\times D$. Let ρ, (t_i^n), and (u_n) be as in the proof of Theorem 3.1 and suppose that ε is a positive number. Choose $\delta>0$ such that if t is in I and x and y are in D with $|x-y|\leqq\delta$, then $|A(t)x-A(t)y|\leqq\varepsilon K\exp(-K\rho)/2$. Let n_0 be a positive integer such that $n_0^{-1}M\leqq\delta$. Thus, if $k\geqq n_0$ and $t_i^k\leqq t<t_{i+1}^k$, then $|u_k(t)-u_k(t_i^k)|\leqq M|t-t_i^k|\leqq Mk^{-1}\leqq\delta$. Now let $n>m\geqq n_0$ and let $p(t)=|u_n(t)-u_m(t)|$ for all t in $[a, a+\rho]$. If t is in $[a, a+\rho]$ and i and j are integers such that $t_i^n\leqq t<t_{i+1}^n$ and $t_j^m\leqq t<t_{j+1}^m$, then

$$p'_+(t)=\lim_{h\to+0}(|u_n(t)-u_m(t)+h[A(t)u_n(t_i^n)-A(t)u_m(t_j^m)]|-|u_n(t)-u_m(t)|)/h$$

$$\leqq \lim_{h\to+0}(|u_n(t)-u_m(t)+h[A(t)u_n(t)-A(t)u_m(t)]|-|u_n(t)-u_m(t)|)/h$$
$$+|A(t)u_n(t_i^n)-A(t)u_n(t)|+|A(t)u_m(t_j^m)-A(t)u_m(t)|.$$

But $|u_n(t_i^n)-u_n(t)|\leqq\delta$ and $|u_m(t_j^m)-u_m(t)|\leqq\delta$ so that $p'_+(t)\leqq Kp(t)+\varepsilon K\exp(-k\rho)$. Consequently, $p(t)\leqq p(a)\exp(K(t-a))+\varepsilon K\exp(-K\rho)[\exp(K(t-a))-1]/K$. Since $p(a)=0$ and $t-a\leqq\rho$ we have that $p(t)=|u_n(t)-u_m(t)|\leqq\varepsilon$ for all t in $[a, a+\rho]$. Thus, the sequence (u_n) is uniformly Cauchy on $[a, a+\rho]$ and the completion of the proof is essentially the same as in the proof of Theorem 3.1.

THEOREM 3.4. *Let S denote the set of nonnegative real numbers and suppose that $\{A(t): t\in S\}$ is a family of members of $H(E, E)$ with the following properties:*

1) *The function $(t, x)\to A(t)x$ is continuous and maps bounded subsets of $S\times E$ into bounded subsets of E.*
2) *Each point (t, x) in $S\times E$ has a neighborhood $I\times Q$ such that the family $\{A(t): t\in I\}$ is uniformly equicontinuous on Q.*
3) *There is a continuous function c from S into the real numbers such that $\mathrm{Re}(A(t)x-A(t)y, f)\leqq c(t)|x-y|^2$ for all x and y in E, t in S, and f in $F(x-y)$.*

Then the conclusions of Theorem 3.2 hold.

The proof of this theorem is analogous to that of Theorem 3.2 and is omitted.

REMARK. In [5, Theorem 3] Murakami constructs the functions u_n defined in the proofs of Theorems 3.1 and 3.3 and, with the assumption of the existence of a continuously differentiable Lyapunov function, proves that they converge to the solution u. Here we are essentially using the norm as a Lyapunov function but it is not necessarily differentiable. The difference in the suppositions of Theorems 3.1 and 3.3 is that in 3.1 the $A(t)$ may only be continuous but the limits defining the Gateaux differential are uniform in x and y so long as they remain a positive distance apart while in 3.3 we relax the uniform limit of the Gateaux differential and require that the $A(t)$ be uniformly continuous.

4. Evolution equations in $LN(D, E)$.

Let S denote the set of nonnegative real numbers and suppose that $\{A(t): t\in S\}$ is a family of members of $LN(D, E)$ with the following properties:

1) There is a continuously differentiable function c from S into the real numbers such that $-A(t)-c(t)1$ is uniformly m-monotonic for all t in S.
2) There is a continuous function d from $S\times S\times S$ into S such that $|A(t)x$

(4a) $-A(s)x|\leqq|t-s|d(t,s,|x|)(1+|A(t)x|+|A(s)x|)$ for all (t,s) in $S\times S$ and all x in D.

3) If t is in S and (x_n) is a sequence in D such that $x_n\to x$ and $|A(t)x_n|$ are bounded for $n\geqq 1$, then x is in D and $A(t)x_n\xrightarrow{w} A(t)x$.

REMARK. We have from [**4**, Lemma 2.5] that if E^* is uniformly convex, then 1) implies 3). Condition 2) is that of Browder in [**1**]. Note that 3 is satisfied if D is closed and $A(t)$ is demicontinuous for all t in S.

We will be concerned with finding solutions to the evolution system

$$u'(t)=A(t)u(t), \qquad u(a)=z \tag{4b}$$

where a is in S, z is in D, and t is in $[a,\infty)$.

THEOREM 4.1. *Suppose that the family* $\{A(t): t\in S\}$ *satisfies the conditions of* (4a) *and that a is in S and z is in D. Then there is a unique function u from $[a,\infty)$ into D which is Lipschitz continuous on bounded subintervals of $[a,\infty)$ and satisfies* (4b) *in the following sense:*

i) *$u(a)=z$, the weak derivative u'_w of u exists, is weakly continuous, and satisfies $u'_w(t)=A(t)u(t)$ for all t in $[a,\infty)$.*

ii) *The function $t\to A(t)u(t)$ of $[a,\infty)$ into E is Bochner integrable on bounded subintervals of $[a,\infty)$ and $u(t)=z+(B)\int_a^t A(s)u(s)ds$ for all t in $[a,\infty)$. In particular, the derivative u' of u exists almost everywhere on $[a,\infty)$ and $u'(t)=A(t)u(t)$ for almost all t in $[a,\infty)$.*

Furthermore, if for each (a,t) in $S\times S$ with $a\leqq t$ and each z in D, $U(a,t)z$ denotes $u(t)$, then $U(a,t)$ is in $LIP(D,E)$ with $N'[U(a,t)]\leqq\exp\left(-\int_a^t c(s)ds\right)$.

REMARK. If E^* is uniformly convex, then this theorem is essentially Theorems 1 and 2 of Kato in [**4**]. We will prove this theorem with a sequence of lemmas which parallels those of Kato.

NOTATION. For each positive integer n and each t in S let $J_n^c(t)=[1-n^{-1}(A(t)+c(t)1)]^{-1}$, $A_n^c(t)=-[A(t)+c(t)1]J_n^c(t)$, and $B_n^c(t)=A(t)J_n^c(t)$. Note that $J_n^c(t)$, $A_n^c(t)$ and $B_n^c(t)$ satisfy the conclusions of Lemma 2.4. Furthermore, with the assumption of part 3) in (4a), the conclusions of Lemma 2.5 are valid.

In what follows we assume that T is a positive number and I is the interval $[a,a+T]$.

LEMMA 4.1. *For each bounded subset Q of D there is a $\delta>0$ and an $M>0$ such that if x is in Q, (t,s) is in $I\times I$ with $|t-s|\leqq\delta$, then $|A(t)x-A(s)x|\leqq|t-s|M(1+2|A(s)x|)$.*

INDICATION OF PROOF. Take $M=2\sup\{d(t,s,|x|): x\in Q, (t,s)\in I\times I\}$ and let $\delta=1/M$. If x is in Q and $|t-s|\leqq\delta$, then $|A(t)x-A(s)x|\leqq|t-s|M(1+|A(t)x-A(s)x|+2|A(s)x|)/2\leqq\delta M|A(t)x-A(s)x|/2+|t-s|M(1+2|A(s)x|)/2$ and the assertion of the lemma follows.

LEMMA 4.2. *Suppose that Q is a bounded subset of D and K is a positive constant. Then there is a constant K' such that if for some s in I, $|A(s)x| \leqq K$ for all x in Q, then $|A(t)x| \leqq K'$ for all (t, x) in $I \times Q$.*

INDICATION OF PROOF. Let δ and M be as in Lemma 4.1 and let n_0 be an integer such that if (t, s) is in $I \times I$, then $|t-s| \leqq n_0\delta$. Take $K' = 1+3^{n_0}K + \sum_{i=1}^{n_0-1} 3^i$. Suppose that s is in I and $|A(s)x| \leqq K$ for all x in Q. If t is in I and $|t-s| \leqq \delta$, we have $|A(t)x| \leqq |A(t)x - A(s)x| + |A(s)x| \leqq 1+3K$ by Lemma 4.1. Assume that for some $1 \leqq k < n_0$ we have that if $|t-s| \leqq k\delta$, then $|A(t)x| \leqq 1+\sum_{i=1}^{k-1} 3^i + 3^k K$. A simple induction argument shows that this inequality holds with $k = n_0$ and hence, if t is in I, then $|t-s| \leqq n_0\delta$ so that $|A(t)x| \leqq K'$ and the lemma is true.

LEMMA 4.3. *If Q is a bounded subset of E, then there is a constant K such that $|J_n^c(t)x| \leqq K$ for all (t, x) in $I \times Q$ and all $n \geqq 1$.*

INDICATION OF PROOF. Let M be such that $|x| \leqq M$ for all x in Q, let z be in D, and take $K = M + \sup\{|A(t)z + c(t)z| : t \in I\} + 2|z|$. If x is in Q, t is in I, and $n \geqq 1$, then by part i) of Lemma 2.4, $|J_n^c(t)x| \leqq |J_n^c(t)x - J_n^c(t)z| + |J_n^c(t)z| \leqq |x-z| + |[1-n^{-1}A_n^c(t)]z| \leqq |x| + 2|z| + n^{-1}|A_n^c(t)z|$. The lemma now follows from iv) of Lemma 2.4.

LEMMA 4.4. *If Q is a bounded subset of E, there is a $\delta > 0$ and an $M > 0$ such that $|B_n^c(t)x - B_n^c(s)x| \leqq |t-s| M(1+2|B_n^c(s)x|)$ for all $n \geqq 1$, x in Q, and (t, s) in $I \times I$ with $|t-s| \leqq \delta$.*

INDICATION OF PROOF. It follows from part 3) of (2b) that

$$\begin{aligned} B_n^c(t)x - B_n^c(s)x &= [n-c(t)]J_n^c(t)x - [n-c(s)]J_n^c(s)x \\ &= [n-c(t)][J_n^c(t)x - J_n^c(s)x] + [c(s)-c(t)]J_n^c(s)x. \end{aligned}$$

From i) of Lemma 2.4 we have

$$\begin{aligned} |J_n^c(t)x - J_n^c(s)x| &= |J_n^c(t)[1-n^{-1}(A(s)+c(s)1)]J_n^c(s) \\ &\quad - J_n^c(t)[1-n^{-1}(A(t)+c(t)1)]J_n^c(s)x| \\ &\leqq n^{-1}|A(t)J_n^c(s)x - A(s)J_n^c(s)x| \\ &\quad + n^{-1}|c(t)-c(s)|\,|J_n^c(s)x|. \end{aligned}$$

Thus,

$$\begin{aligned} |B_n^c(t)x - B_n^c(s)x| &\leqq |1+n^{-1}c(t)|\,|A(t)J_n^c(s) - A(s)J_n^c(s)x| \\ &\quad + (1+n^{-1})|c(t)-c(s)|\,|J_n^c(s)x| \end{aligned}$$

and from Lemmas 4.1 and 4.3 there is a $\delta > 0$ and constants M' and K such that if $|t-s| \leqq \delta$, then $|B_n^c(t)x - B_n^c(s)x| \leqq |1-n^{-1}c(t)|\,|t-s| M'[1+2|A(s)J_n^c(s)x|] + (1-n^{-1})|c(t)-c(s)|K$. The assertion of the lemma now follows since c is continuously differentiable on I.

Since $B_n^c(t)$ is in $LIP(E, E)$ with $N'[B_n^c(t)] \leqq 2n + |c(t)|$ (see iii) of Lemma 2.4) we have by Lemma 4.4 and Corollary 3.1 that for each $n \geqq 1$, there is a continuously differentiable function u_n from $[a, \infty)$ into E such that

(4c) $$u_n'(t) = B_n^c(t) u_n(t), \qquad u(a) = z$$

for all t in $[a, \infty)$.

LEMMA 4.5. *There is a constant K such that $|u_n(t)| \leqq K$ and $|u_n'(t)| = |B_n^c(t) u_n(t)| \leqq K$ for all $n \geqq 1$ and all t in I.*

INDICATION OF PROOF. Since $L'[B_n^c(t)] \leqq |c(t)|$ for all t in S and all $n \geqq 1$, we have by Corollary 3.1 that the $|u_n(t)|$ are bounded on I. Now let Q be a bounded subset of E which contains $u_n(t)$ for all t in I and $n \geqq 1$. Choose δ and M as in Lemma 4.4 and for each t in I, $0 < h \leqq \delta$, and $n \geqq 1$, let $P_{n,h}(t) = |u_n(t+h) - u_n(t)|$. Then

$$\begin{aligned}(P_{n,h})_+'(t) &= \lim_{k \to +0} (|u_n(t+h) - u_n(t) + k[B_n^c(t+h) u_n(t+h) - B_n^c(t) u_n(t)]| \\ &\quad - |u_n(t+h) - u_n(t)|)/k \\ &\leqq \lim_{k \to +0} (|u_n(t+h) - u_n(t) + k[B_n^c(t+h) u_n(t+h) - B_n^c(t+h) u_n(t)]| \\ &\quad - |u_n(t+h) - u_n(t)|)/k + |B_n^c(t+h) u_n(t) - B_n^c(t) u_n(t)| \\ &\leqq |c(t)| P_{n,h}(t) + hM(1 + 2|B_n^c(t) u_n(t)|).\end{aligned}$$

Consequently, $|u_n(t+h) - u_n(t)| \leqq |u_n(a+h) - u_n(a)| \exp\left(\int_a^t |c(s)| ds\right) + hM \int_a^t (1 + 2|B_n^c(s) u_n(s)|) \exp\left(\int_s^t |c(r)| dr\right) ds$ for all $0 < h \leqq \delta$, $n \geqq 1$, and t in I. Dividing by h, letting $h \to +0$, and noting that $B_n^c(s) u_n(s) = u_n'(s)$, we have $|u_n'(t)| \leqq |u_n'(a)| \exp\left(\int_a^t |c(s)| ds\right) + 2M \int_a^t (1 + 2|u_n'(s)|) \exp\left(\int_s^t |c(r)| dr\right) ds$. Since $|u_n'(a)| = |B_n^c(a) z|$ is bounded by part iv) of Lemma 2.4, it follows from Gronwall's inequality (see e.g. [**3**, p. 19]) that $|u_n'(t)|$ is bounded for all t in I and $n \geqq 1$.

LEMMA 4.6. *If $Q = \{x \in E : x = J_n^c(t) u_n(t)$ for $n \geqq 1$ and t in $I\}$, then Q is bounded and the family $\{A(t) : t \in I\}$ has uniform logarithmic derivative on $I \times Q$ (see Definition* 3.1).

INDICATION OF PROOF. Since $|u_n(t)| \leqq K$, Q is bounded by Lemma 4.3. Since $|A(t) J_n^c(t) u_n(t)| = |B_n^c(t) u_n(t)| \leqq K$, we have by Lemma 4.2 that there is a constant K' such that $|A(s)x| \leqq K'$ for all s in I and x in Q. Let β and ε be positive numbers. From Lemma 4.1 there is a $\delta' > 0$ and an $M' > 0$ such that if $|t-s| \leqq \delta'$ and x is in Q, then $|A(t)x - A(s)x| \leqq |t-s| K_1$ where $K_1 = M'(1+2K')$. Let $(r_i)_0^m$ be a partition of I such that $|r_i - r_{i-1}| \leqq \min\{\delta', \varepsilon/(4K_1)\}$ and choose $\delta_i > 0$ such that if x and y are in Q with $|x-y| \geqq \beta$, and $0 < h \leqq \delta_i$, then $(|x - y + h[A(r_i)x - A(r_i)y]| - |x-y|)/h \leqq L'[A(r_i)]|x-y| + \varepsilon/2$. Now take

$\delta=\min\{\delta_i : 1 \leqq i \leqq m\}$ and let $K_2=\sup\{|c(s)| : s \in I\} \geqq \sup\{L'[A(s)] : s \in I\}$. If t is in I there is an integer i such that $|t-r_i| \leqq \delta'$. Thus, if x and y are in Q with $|x-y| \geqq \beta$, and $0<h \leqq \delta$, then $(|x-y+h[A(t)x-A(t)y]|-|x-y|)/h \leqq (|x-y+h[A(r_i)x-A(r_i)y]|-|x-y|)/h+|A(t)x-A(r_i)x|+|A(t)y-A(r_i)y| \leqq K_2|x-y|+\varepsilon/2+2|t-r_i|K_1$ and the assertion of the lemma follows since $2|t-r_i| \leqq \varepsilon/(2K_1)$.

LEMMA 4.7. *There is a Lipschitz continuous function u from I into E such that $u_n(t) \rightarrow u(t)$ uniformly on I.*

INDICATION OF PROOF. Let Q be as in Lemma 4.6 and suppose that ε is a positive number. Since the family $\{A(t) : t \in I\}$ has uniform logarithmic derivative on $I \times Q$ (Lemma 4.6) let K be as in Definition 3.1 and assume that K is positive. For the pair $\beta'=\varepsilon \exp(-KT)/6$ and $\varepsilon'=\varepsilon K \exp(-KT)/3$, choose $\delta>0$ such that $(|x-y+h[A(t)x-A(t)y]|-|x-y|)/h \leqq K|x-y|+\varepsilon'$ whenever x and y are in Q with $|x-y| \geqq \beta'$ and $0<h \leqq \delta$. Since $|u_n(s)-J_n^c(s)u_n(s)| = n^{-1}|A_n^c(s)u_n(s)| \leqq n^{-1}|B_n^c(s)u_n(s)|+n^{-1}|c(s)J_n^c(s)u_n(s)| \leqq n^{-1}K_1$ for some constant K_1, there is an integer n_0 such that $2n_0^{-1}K_1 \leqq \varepsilon \exp(-KT)/6$ and $n_0^{-1}(2KK_1+4K_1/\delta) \leqq \varepsilon K \exp(-KT)/3$. Suppose, for contradiction, that there are integers $n>m \geqq n_0$ and a t_1 in I such that $|u_n(t_1)-u_m(t_1)|>\varepsilon$. Let $p(t)=|u_n(t)-u_m(t)|$ for all t in I. Since $p(a)=0$ and $p(t_1)>\varepsilon$, there is a t_0 in $[a, t_1)$ such that $p(t_0)=2\beta'$ and $p(t) \geqq 2\beta'$ for all t in $[t_0, t_1]$. We have from Lemma 3.2 that

$$\begin{aligned} p'_+(t) &\leqq (|u_n(t)-u_m(t)+\delta[B_n^c(t)u_n(t)-B_m^c(t)u_m(t)]|-|u_n(t)-u_m(t)|)/\delta \\ &\leqq (|J_n^c(t)u_n(t)-J_m^c(t)u_m(t)+\delta[A(t)J_n^c(t)u_n(t) \\ &\quad -A(t)J_m^c(t)u_m(t)]|-|J_n^c(t)u_n(t)-J_m^c(t)u_m(t)|)/\delta \\ &\quad +2|J_n^c(t)u_n(t)-u_n(t)|/\delta+2|J_m^c(t)u_m(t)-u_m(t)|/\delta\,. \end{aligned}$$

Since $|J_n^c(t)u_n(t)-J_m^c(t)u_m(t)|=|J_n^c(t)u_n(t)-u_n(t)+u_n(t)-u_m(t)+u_m(t)-J_m^c(t)u_m(t)| \geqq |u_n(t)-u_m(t)|-|J_n^c(t)u_n(t)-u_n(t)|-|J_m^c(t)u_m(t)-u_m(t)| \geqq \varepsilon \exp(-KT)/3-2n_0^{-1}K_1 \geqq \varepsilon \exp(-KT)/6=\beta'$ for all t in $[t_0, t_1]$, we have by the choice of β' that

$$\begin{aligned} p'_+(t) &\leqq K|J_n^c(t)u_n(t)-J_m^c(t)u_m(t)|+4n_0^{-1}K_1/\delta+\varepsilon' \\ &\leqq Kp(t)+n_0^{-1}(2KK_1+4K_1/\delta)+\varepsilon' \\ &\leqq Kp(t)+2\varepsilon K \exp(-KT)/3\,. \end{aligned}$$

Thus, for each t in $[t_0, t_1]$ we have $p(t) \leqq p(t_0)\exp(K(t-t_0))+2\varepsilon K \exp(-KT)[\exp(K(t-t_0))-1]/(3K)$ and since $p(t_0)=\varepsilon \exp(-KT)/3$ and $t_1-t_0 \leqq T$, it follows that $p(t_1) \leqq \varepsilon/3+2\varepsilon/3=\varepsilon$. This contradicts the assumption that $p(t_1)>\varepsilon$. Consequently, the sequence (u_n) is uniformly Cauchy and since E is complete, there is a continuous function u from I into E such that $u_n(t) \rightarrow u(t)$ uniformly on I. As $|u'_n(t)|$ are bounded for t in I and $n \geqq 1$, it follows that u is Lipschitz continuous on I so that the lemma is true.

LEMMA 4.8. *The function u in Lemma 4.7 maps I into D, the function $t \to A(t)u(t)$ of I into E is weakly continuous, and for each f in E^* the function $t \to (u(t), f)$ of I into the field over E is continuously differentiable with $d(u(t), f)/dt = (A(t)u(t), f)$ for all t in I.*

INDICATION OF PROOF. Since $u_n(t) \to u(t)$ and $|B_n^c(t)u_n(t)| \leqq K$, we have $|A_n^c(t)u_n(t)|$ are bounded and hence, $u(t)$ is in D, $B_n^c(t)u_n(t) \xrightarrow{w} A(t)u(t)$, and $|A(t)u(t)| \leqq K$ (this follows from the conclusions of Lemma 2.5 which are valid due to the assumption of condition 3) of (4a)). Let δ and M be as in Lemma 4.1 with $Q = \{x \in E : x = u(t) \text{ for } t \text{ in } I\}$. Then if s is in I and $|t-s| \leqq \delta$, $|A(t)u(t) - A(s)u(t)| \leqq |t-s| M(1+2|A(t)u(t)|) \leqq |t-s| M(1+2K)$. Furthermore, since $u(t) \to u(s)$ as $t \to s$, we have by condition 3) of (4a) that $A(s)u(t) \xrightarrow{w} A(s)u(s)$. Hence, $A(t)u(t) - A(s)u(s) = A(t)u(t) - A(s)u(t) + A(s)u(t) - A(s)u(s) \xrightarrow{w} 0$ and it follows that $t \to A(t)u(t)$ is weakly continuous on I. If f is in E^*, then $(u_n(t), f) = (z, f) + \int_a^t (B_n^c(s)u_n(s), f)ds$ for all $n \geqq 1$ and t in I. Since $u_n(t) \to u(t)$, $B_n^c(t)u_n(t) \xrightarrow{w} A(t)u(t)$, and $|(B_n^c(s)u_n(s), f)| \leqq K|f|$, we have $(u(t), f) = (z, f) + \int_a^t (A(s)u(s), f)ds$ and the assertion of the lemma follows.

LEMMA 4.9. *The function $t \to A(t)u(t)$ of I into E is Bochner integrable and for each t in I, $u(t) = z + (B)\int_a^t A(s)u(s)ds$.*

The proof of this lemma is the same as [**4**, Lemma 4.6] and is omitted.

We have now established the existence of a function u from $[a, \infty)$ into D which is Lipschitz continuous on bounded subintervals of $[a, \infty)$ and satisfies parts i) and ii) of Theorem 4.1. Suppose that w is in D and v is a function from $[a, \infty)$ into D which is Lipschitz continuous on bounded subintervals of $[a, \infty)$ and satisfies each of the conditions i) and ii) of u in Theorem 4.1 except that $v(a) = w$. For each t in $[a, \infty)$ let $p(t) = |u(t) - v(t)|$. By Lemma 3.2 $p'_+(t)$ exists for almost all t in $[a, \infty)$ and for all such t,

$$\begin{aligned} p'_+(t) &= \lim_{h \to +0} (|u(t) - v(t) + h[A(t)u(t) - A(t)v(t)]| - |u(t) - v(t)|)/h \\ &\leqq L'[A(t)]\,|u(t) - v(t)|. \end{aligned}$$

By part 1) of (4a) we have that $L'[A(t) + c(t)1] \leqq 0$ so by part iii) of Proposition 2.1, $L'[A(t)] \leqq -c(t)$. Hence, $p'_+(t) \leqq -c(t)p(t)$ for almost all t in $[a, \infty)$ and since p is absolutely continuous on bounded subintervals of $[a, \infty)$, it follows that

$$|u(t) - v(t)| \leqq |z - w| \exp\left(-\int_a^t c(s)ds\right)$$

for each t in $[a, \infty)$. The uniqueness of u and the last assertion of Theorem 4.1 follow easily from this inequality and the proof of Theorem 4.1 is complete.

5. Semi-groups of operators.

In this section we will give sufficient conditions for a member A of $H(D, E)$ to generate a semi-group U of operators in $LIP(E, E)$.

DEFINITION 5.1. A function U from S into $LIP(E, E)$ will be called a semi-group of operators in $LIP(E, E)$ if the following holds:

1) $U(0)=1$ and $U(t)U(s)=U(t+s)$ for all t and s in S.

(5a) 2) There is a constant K such that $N'[U(t)]\leqq \exp(Kt)$ for all t in S.

3) If z is in E and $u_z(t)=U(t)z$ for all t in S then u_z is continuous on S.

If D is a dense subset of E and A is a member of $H(D, E)$, then A is said to be a generator (resp. weak generator) of U if for each z in D, $[U(h)z-z]/h\rightarrow Az$ (resp. $[U(h)z-z]/h \overset{w}{\rightarrow} Az$) as $h\rightarrow +0$.

THEOREM 5.1. *Suppose A is in $H(E, E)$, A is continuous,* $\operatorname{Re}(Ax-Ay, f)\leqq K|x-y|^2$ *for all x and y in E and f in $F(x-y)$, and either*

1) *each z in E has a neighborhood V_z such that the restriction of A to V_z is in $LN(V_z, E)$, or*
2) *A is locally uniformly continuous on E.*

Then A generates a semi-group of operators U satisfying (5a). *Furthermore, u_z is differentiable on S for each z in E and $u_z'(t)=Au_z(t)$ for all t in S.*

INDICATION OF PROOF. The local existence of solutions to $u'(t)=Au(t)$ where A satisfies either 1) or 2) follows from Theorems 3.1 or 3.3. To complete the proof we need only show that u can be extended to S. Let $T>0$ and suppose that u is defined on $[0, T)$. Let $0<t_1<t_2<T$ and for each t in $[0, t_1]$ define $p(t)=|u(t+t_2-t_1)-u(t)|$. Then $p_+'(t)=\lim_{h\rightarrow +0}(|u(t+t_2-t_1)-u(t)+h[Au(t+t_2-t_1)-Au(t)]|-|u(t+t_2-t_1)-u(t)|)/h\leqq Kp(t)$ and hence, $|u(t_2)-u(t_1)|\leqq \exp(KT)|u(t_2-t_1)-u(0)|$. Thus, $\lim_{t\rightarrow T^-} u(t)$ exists and the theorem follows.

THEOREM 5.2. *Suppose that A is in $H(D, E)$ and either of the following is satisfied:*

1) *D is dense in E, $-(A-K1)$ is uniformly m-monotonic, and if (x_n) is a sequence in D such that $x_n\rightarrow x$ and $|Ax_n|$ are bounded, then x is in D and $Ax_n \overset{w}{\rightarrow} Ax$.*
2) *$D=E$, A is demicontinuous on E,* $\operatorname{Re}(Ax-Ay, f)\leqq K|x-y|^2$ *for all x and y in E and f in $F(x-y)$, and each z in E has a neighborhood V_z such that A is bounded on V_z and the restriction of A to V_z is in $LN(V_z, E)$.*

Then A is a weak generator of a semi-group of operators U satisfying (5a). *Also, for each z in D the weak derivative $(u_z)_w'$ of u_z exists on S and $(u_z)_w'(t)=Au_z(t)$ for all t in S. Furthermore, for almost all t in S, $u_z'(t)$ exists and equals $Au_z(t)$.*

INDICATION OF PROOF. If A satisfies 1) then the conclusions are an immediate consequence of Theorem 4.1. In a manner similar to the proof of Theorem 3.1, for each z in E and some $T>0$ we can find a locally Lipschitz continuous function u from $[0, T)$ into E which is weakly differentiable and satisfies $u(0)=z$ and $u_w'(t)=Au(t)$ for all t in $[0, T)$. Thus, for each t in $[0, T)$ we have $u(t)=z+(B)\int_0^t Au(s)ds$ (where (B) denotes the Bochner integral) and hence, $u'(t)$ exists for almost all t in $[0, T)$ and equals $Au(t)$. The proof now follows in a manner similar to the proof of Theorem 5.1 by using the Lebesgue integral in solving the differential inequalities.

REMARK. If A is a continuous member of $H(E, E)$ and A generates a semi-group U satisfying (5a) with $K=0$ and with the functions u_z being differentiable and satisfying $u_z'(t)=Au_z(t)$ for all t in S and z in E, then $-A$ is necessarily accretive. This can easily be seen for if x and y are in E and $p(t)=|u_x(t)-u_y(t)|$, then p is nonincreasing on S and hence, $p_+'(t)\leqq 0$. Consequently, $\lim_{h\to+0}(|x-y+h[Ax-Ay]|-|x-y|)/h=p_+'(0)\leqq 0$ so $-A$ is accretive by Proposition 2.5. If Q is a bounded subset of E and for each $\varepsilon>0$ there is a $\delta>0$ such that if x is in Q and $0<h\leqq\delta$, we have $|[u_x(h)-x]/h-Ax|\leqq\varepsilon$, then the restriction of A to Q is in $LN(Q, E)$ and $-A$ is uniformly monotonic on Q. This can easily be seen for if x and y are in Q and $0<h\leqq\delta$, then

$$\begin{aligned}(|x-y+h[Ax-Ay]|-|x-y|)/h &\leqq (|x-y+[u_x(h)-x-u_y(h)+y]|-|x-y|)/h+2\varepsilon \\ &=(|U(h)x-U(h)y|-|x-y|)/h+2\varepsilon \\ &\leqq 2\varepsilon\end{aligned}$$

since $|U(h)x-U(h)y|\leqq|x-y|$. In particular, if A is locally uniformly continuous on E, then $-A$ is accretive if and only if $-A$ is locally uniformly monotonic (i.e. for each z in E there is a neighborhood V_z of z such that the restriction of A to V_z is in $LN(V_z, E)$ and $L'[A|V_z]\leqq 0$).

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