J. Math. Soc. Japan
Vol. 21, No. 1, 1969

On the inertia groups of homology tori

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(Received Feb. 15, 1968)

§ 1. Introduction.

The inertia group $I(M)$ of an oriented closed smooth manifold M is defined to be the subgroup of Θ_n consisting of those homotopy spheres $\tilde{S}$ which satisfy the condition $M\sharp\tilde{S}=M$, where Θ_n is the group of homotopy n-spheres. This group $I(M)$ is one of the diffeomorphy invariants of M.

The inertia groups of manifolds have been studied by I. Tamura [**14**], C. T. C. Wall [**18**], S. P. Novikov [**10**], W. Browder [**1**] and A. Kosinski [**6**]. The following problems have been proposed by them as important ones:

(I) Is it combinatorially (or topologically) invariant?

(II) Does it depend on more than the tangential homotopy equivalence class at the manifold? (W. Browder cf. [**7**])

(III) Is it contained in $\Theta(\partial\pi)$, if we restrict the manifold within π-manifolds? (S. P. Novikov [**10**])

In this paper the following facts will be proved which answer the problems above.

The inertia group of $S^3\times S^{14}$ is not combinatorially (therefore not topologically) invariant and depends on more than the tangential homotopy equivalence class of $S^3\times S^{14}$.

For $\tilde{S}^{14}\neq S^{14}$, $I(S^3\times\tilde{S}^{14})$ contains a homotopy sphere $\tilde{S}^{17}$ which does not belong to $\Theta_{17}(\partial\pi)$.

The author wishes to express his hearty thanks to Professor I. Tamura for suggesting the viewpoint of this research and for showing the attitude of mind in mathematics and also to Mr. H. Sato, Mr. M. Kato and Mr. T. Akiba for many invaluable discussions.

§ 2. Notations and results.

In this paper all the manifolds are compact connected smooth oriented manifolds and the diffeomorphisms are orientation preserving. We write $M_1=M_2$ for manifolds M_1, M_2, if there is an orientation preserving diffeomorphism $f\colon M_1\to M_2$.

Let Θ_q be the group of homotopy q-spheres and Γ_q the pseudo-isotopy group of diffeomorphism of S^{q-1}. It is well known that Θ_q and Γ_q are equivalent ($q \neq 3$) (Smale [**12**]). A subgroup of q-dimensional homotopy spheres which bound parallelizable manifolds is denoted by $\Theta_q(\partial\pi)$.

The inertia group of a closed differentiable manifold M^n is defined to be the group $\{\tilde{S} \in \Theta_n \mid M^n \# \tilde{S} = M^n\}$ which is denoted by $I(M)$.

Now we shall define pairings K_1, K_2: for $0 < p < q$:

$$K_1 : \pi_p(SO_{q+1}) \times \Gamma_{q+1} \longrightarrow \Gamma_{p+q+1},$$
$$K_2 : \pi_p(SO_{q+1}) \times \pi_q(SO_{p+1}) \longrightarrow \Gamma_{p+q+1}.$$

Let $h \in \pi_p(SO_{q+1})$, $r \in \Gamma_{q+1} = \pi_0(\text{diff } S^q)/\pi_0(\text{diff } D^{q+1})$. (In the following we will use the same symbol for an element of a group as its representative.) We define the diffeomorphism

$$F: S^p \times S^q \longrightarrow S^p \times S^q \quad \text{by} \quad F(x, y) = \big(x, rh(x)r^{-1}(y)\big).$$

Attaching two manifolds, $W_1 = D^{p+1} \times S^q$ and $W_2 = S^p \times D^{q+1}$, by the diffeomorphism $F: S^p \times S^q \to S^p \times S^q$, we have a manifold

$$\Sigma = D^{p+1} \times S^q \bigcup_F S^p \times D^{q+1}.$$

Making use of the Mayer-Vietoris exact sequence, it is easy to see that the manifold Σ is a homotopy sphere for $p < q$. We assume that the orientation of the manifold $A \bigcup_f B$ compatible with the first part A, is given. The pairing K_1 is defined by $K_1(h, r) = \Sigma$.

We shall now prove that this does not depend on the choice of representatives.

Let $h' \in \pi_p(SO_{q+1})$ be other representative and $H: S^p \times I \to SO_{q+1}$ be a homotopy between h and h'. We write $H(x, t)$ as $h_t(x)$. Let $r' \in \Gamma_{q+1}$ be other representative and $R: S^q \times I \to S^q \times I$ be a pseudo isotopy between r and r'. Let $p_1: S^q \times I \to S^q$ and $p: S^q \times I \to I$ be the projections to the first and to the second respectively.

We now define the diffeomorphism $G: S^p \times S^q \times I \to S^p \times S^q \times I$ by $G(x, y, t) = \Big(x, R\big(h_t(x)p_1R^{-1}(y, t),\ p_2R^{-1}(y, t)\big)\Big)$.

Attaching two manifold $D^{p+1} \times S^q \times I$ and $S^p \times D^{q+1} \times I$ by the diffeomorphism $G: S^p \times S^q \times I \to S^p \times S^q \times I$, we have a manifold $X = D^{p+1} \times S^q \times I \bigcup_G S^p \times D^{q+1} \times I$. The boundary ∂X is composed of the disjoint sum Σ and $-\Sigma'$ where Σ' is a homotopy sphere made from h' and r'.

Making use of the Mayer-Vietoris exact sequence, it is easy to see that inclusions $i: \Sigma \to X$ and $i': \Sigma' \to X$ give the homotopy equivalences. Therefore Σ is diffeomorphic to Σ' by the h-cobordism theorem. Since $\Theta_n = \Gamma_n$ ($n \neq 3$), Σ and Σ' are the same element of Γ_{p+q+1}.

Next we define the pairing K_2. Let $h_1 \in \pi_p(SO_{q+1})$, $h_2 \in \pi_q(SO_{p+1})$. We consider two bundles $(B_1, S^{p+1}, D^{q+1}, p_1)$ and $(B_2, S^{q+1}, D^{p+1}, p_2)$ with characteristic maps h_1 and h_2 respectively. We can write $B_1 = D_+^{p+1} \times D^{q+1} \underset{h_1}{\cup} D_-^{p+1} \times D^{q+1}$ and $B_2 = D_+^{q+1} \times D^{p+1} \underset{h_2}{\cup} D_-^{q+1} \times D^{p+1}$ where $D_\pm^{p+1} \times D^{q+1} = p_1^{-1}(D_\pm^{p+1})$ and $D_\pm^{q+1} \times D^{p+1} = p_2^{-1}(D_\pm^{q+1})$. The plumbing manifold of B_1 and B_2 is defined to be the oriented differentiable $(p+q+2)$-manifold obtained as a quotient space of $B_1 \cup B_2$ by identifying $p_1^{-1}(D_-^{p+1}) = D_-^{p+1} \times D^{q+1}$ and $p_2^{-1}(D_+^{q+1}) = D_+^{q+1} \times D^{p+1}$ by the relation $(x, y) = (y, x)$ $(x \in D_-^{p+1} = D^{p+1},\ y \in D^{q+1} = D_+^{q+1})$ and is denoted by $B_1 \vee\!\!\!\vee B_2$.

The boundary $\partial(B_1 \vee\!\!\!\vee B_2)$ can be seen as follows. Let $f\colon S_+^p \times S^q \to S_-^q \times S^p$ be the diffeomorphism defined by $f(x, y) = \big(h_1(x)y,\ h_2(h_1(x)y)x\big)$. Attaching two manifold $D_+^{p+1} \times S^q$ and $D_-^{q+1} \times S^p$ by the diffeomorphism $f\colon S_+^p \times S^q \to S_-^q \times S^p$, we have $D_+^{p+1} \times S^q \underset{f}{\cup} D_-^{q+1} \times S^p = \partial(B_1 \vee\!\!\!\vee B_2)$. Therefore $\partial(B_1 \vee\!\!\!\vee B_2)$ is a homotopy sphere by the same argument of the pairing K_1. The pairing K_2 is defined by $K_2(h_1, h_2) = \partial(B_1 \vee\!\!\!\vee B_2)$ and one can easily prove that it is well-defined like K_1. $\Gamma'_{p,q}$ denotes the subgroup of Γ_{p+q-1} generated by the image of the pairing $K_2\colon \pi_{p-1}(SO_q) \times \pi_{q-1}(SO_p) \to \Gamma_{p+q-1}$. $\Gamma''_{p,q}$ denotes the subgroup of $\Gamma'_{p,q}$ generated by the image of the restricted pairing $K'_2\colon \pi_{p-1}(SO_q) \times s\pi_{q-1}(SO_{p-1}) \to \Gamma_{p+q-1}$ where s is a natural map $s\colon \pi_{q-1}(SO_{p-1}) \to \pi_{q-1}(SO_p)$.

Then the following theorems will be proved.

THEOREM A. *Let M^m be the simply connected π-manifold with $H_i(M^m) = 0$ for $i \neq 0, p, q, p+q = m$. Then $I(M^m) \subset \Gamma'_{p+1,q}$ for $p < q-1$, $q < 2p$.*

THEOREM B. *There exists a manifold M^m such that $\Gamma''_{p+1,q} = I(M^m)$ for $p < q$, $p+q = m$.*

THEOREM C. $K_1\big(\pi_p(SO_{q+1}), \tilde{S}^{q+1}\big) = I(S^p \times \tilde{S}^{q+1})$, *for $p \neq q$, $p+q \geqq 4$.*

COROLLARY 1. *Let $\tilde{S}^{14} \neq S^{14}$. Then $I(S^3 \times \tilde{S}^{14})$ contains $\tilde{S}^{17}$ which does not belong to $\Theta_{17}(\partial\pi)$.*

COROLLARY 2. *Let $\tilde{S}^{10}$ be the generator of the 3-component of $\Theta_{10} \cong Z_2 \oplus Z_3$. Then $I(S^3 \times \tilde{S}^{10})$ is equal to Θ_{13}.*

COROLLARY 3. *$I(S^p \times S^q) = 0$ for $p+q \geqq 5$ therefore $I(S^3 \times S^{14}) \neq I(S^3 \times \tilde{S}^{14})$ and $I(S^3 \times S^{10}) \neq I(S^3 \times \tilde{S}^{10})$. These show that the inertia group is neither PL homeomorphism invariant nor tangential homotopy equivalence invariant and that the conjecture of Novikov is negative.*

COROLLARY 4. *If $\tilde{S}^q$ is embeddable in M^{p+q} with trivial normal bundle, then $I(M)$ contains $K_1\big(\pi_p(SO_q), \tilde{S}^q\big)$. (Cf. Theorem of Munkres in* [7].)

REMARK. Smooth structures on $S^p \times S^q$ are completely classified by combining Theorem C and the Novikov's work [**10**].

§ 3. A lemma.

In this section we assume $p<q$. Let c_1 be the zero cross section of the bundle $(B_2, S^{q+1}, D^{p+1}, p_2)$. If the characteristic map h_2 of B_2 is contained in Image s where $s: \pi_q(SO_p) \to \pi_q(SO_{p+1})$, then we can write $B_2 = B_2' \oplus 0_1$ where 0_1 denotes the trivial 1-disk bundle and B_2' is D^p bundle over S^{q+1} with the characteristic map $h_2' \in \pi_p(SO_q)$ such that $sh_2' = h_2$. The trivial 1-disk bundle 0_1 can be written as $0_1 = S^{q+1} \times I[-1, 1]$. Let c_1' be the zero cross section of B_2' and c_2 the cross section of B_2 which is written as $c_1' \oplus \frac{1}{2}$ using the above expression.

In B_2, we take two tubular neighborhoods T_1 and T_2 of c_1 and c_2 respectively such that $T_1 \cap T_2 = \phi$, $T_1 \subset \operatorname{Int} B_2$, $T_2 \subset \operatorname{Int} B_2$ and $T_1 \cap D_+^{q+1} \times D^{p+1} = D_+^{q+1} \times U_\varepsilon$, $T_2 \cap D_+^{q+1} \times D^{p+1} = D_+^{q+1} \times U_\varepsilon'$ where $D_+^{q+1} \times D^{p+1}$ denotes the first part of B_2 and U_ε and U_ε' are ε neighborhoods of 0×0 and $0 \times \frac{1}{2}$ in $D^p \times I[-1, 1] = D^{p+1}$ respectively.

Since c_2 is diffeotopic to c_1, T_1 and T_2 are diffeomorphic to B_2. Let $X = B_1 \between B_2$. We connect ∂T_1 and ∂X by an imbedding $l: I[0, 1] \to X$ such that $l(\operatorname{Int} I) \cap T_1 = \phi$, $l(I) \cap T_2 = \phi$, $l(\operatorname{Int} I) \cap \partial X = \phi$, $l(I) \cap T_1 = l(0)$, $l(I) \cap \partial X = l(1)$ and $l(I)$ is contained in $(\operatorname{Int} D_-^{q+1}) \times D^{p+1}$. We take a tubular neighborhood T of $l(I)$ in $X - \operatorname{Int} T_1 - T_2$, which is clearly diffeomorphic to $I \times D^{p+q+1}$. Let $T_1' = T_1 \cup T$. Let Y denote $X - \left(\operatorname{Int} T_1' \cup l(1) \times \operatorname{Int} D^{p+q+1}\right)$.

LEMMA. *Y is diffeomorphic to T_2 for* $\dim Y \geqq 6$.

PROOF. In case where $p>1$. According to Smale [**10**], if the natural inclusion $\iota: T_2 \to Y$ induces a homotopy equivalence and $\pi_1(Y - \operatorname{Int} T_2) = \pi_1(\partial T_2) = \pi_1(\partial Y) = \{1\}$, then T_2 is diffeomorphic to Y. It is easy to see that $\pi_1(Y - \operatorname{Int} T_2) = \pi_1(\partial T_2) = \pi_1(\partial Y) = \{1\}$. Firstly we prove that $\iota: T_2 \to Y$ induces an isomorphism of homology groups. We put $Y' = \partial T_1' - \left(l(1) \times \operatorname{Int} D\right)$.

We shall examine the next commutative diagram.

$$\begin{tikzcd}
\longrightarrow H_{i+1}(Y, Y') \arrow[r, "\partial_*"] \arrow[d, "\approx"] & H_i(Y') \arrow[r] \arrow[d] & H_i(Y) \arrow[r] \arrow[d] & H_i(Y, Y') \longrightarrow \arrow[d, "\approx"] \\
\longrightarrow H_{i+1}(X, T_1') \arrow[r] & H_i(T_1') \arrow[r] & H_i(X) \arrow[r] & H_i(X, T_1') \longrightarrow
\end{tikzcd} \tag{*}$$

where $H_i(Y, Y')$ is isomorphic to $H_i(X, T_1')$ by the excision isomorphism.

$1 \leqq i \leqq q$: It is easy to see that $H_i(Y) \approx 0$ from the diagram (*).

$i = q+1$: In the diagram (*), putting $i = q+1$, the isomorphisms $H_{q+1}(Y') \approx H_{q+1}(T_1') \approx Z$ and $H_q(Y') \approx H_q(T_1') \approx 0$ hold. Hence we obtain the isomorphisms $H_{q+1}(Y) \approx H_{q+1}(X) \approx Z$ by the five lemma. The composition $\iota' \circ \iota$ of the natural maps $H_{q+1}(T_2)\xrightarrow{\iota}H_{q+1}(Y)\xrightarrow{\iota'}H_{q+1}(X)$ is an isomorphism and ι' is an isomorphism from the above. Consequently ι is an isomorphism.

$q+2\leqq i$: The isomorphisms $H_i(Y)\approx H_i(X)\approx 0$ are easily deduced from the diagram (*) likewise. On the other hand $H_i(T_2)$ is zero except for $i=0$, $q+1$. Thus we can conclude that the natural inclusion $\iota: T_2\to Y$ induces the isomorphism of homology groups. Hence the natural inclusion $\iota'': \partial T_2\to Y-\mathrm{Int}\,T_2$ induces the isomorphism of homology groups by the excision isomorphism i.e. ι'' gives a homotopy equivalence. Making use of the Poincaré-Lefschetz duality theorem, the natural inclusion $\iota''': \partial Y\to Y-\mathrm{Int}\,T_2$ induces the isomorphism of homology groups i.e. ι''' gives a homotopy equivalence (see J. H. C. Whitehead [19]). Hence $(Y-\mathrm{Int}\,T_2, \partial T_2, \partial Y)$ is an h-cobordism and T_2 is diffeomorphic to Y.

In case where $p=1$:
Since $\pi_q(SO_2)=0$ for $q\geqq 2$, one has $T_2=S^{q+1}\times D^2$. Since $\pi_1(X)=\{1\}$ and $Y=X-\left(\mathrm{Int}\,T'_1\cup l(1)\times\mathrm{Int}\,D^{q+2}\right)$, any element of $\pi_1(Y)$ is homotopic into $\partial T'_1-l(1)\times\mathrm{Int}\,D^{q+2}$ by the Van Kampen's theorem. As the generator of $\pi_1\left(\partial T'_1-l(1)\times\mathrm{Int}\,D^{q+2}\right)\cong\pi_1(\partial T'_1-\mathrm{Int}\,D^{q+2})\cong\pi_1(S^{q+1}\times S^1-\mathrm{Int}\,D^{q+2})\cong Z$, one can take a fibre $*\times S^1$ of $\partial T'_1$. But this is homotopic to $\partial D^2\times 0$ when we write B_1 as $D^2\times D^{q+1}\cup D^2\times D^{q+1}$ and homotopic to zero in Y. On the other hand, one has $\pi_1(\partial T_2)=\pi_1(S^{q+1}\times S^1)\cong Z$ and $\pi_1(\partial Y)=\pi_1(\partial T'_1\,\#\,\partial X)\cong Z$. Since the generator of $\pi_1(\partial Y)$ is homotopic to the generator of $\pi_1(\partial T_2)$ in $(Y-\mathrm{Int}\,T_2)$ and $\pi_1(Y)=\{1\}$ we obtain $\pi_1(Y-\mathrm{Int}\,T_2)\cong Z$. Apparently inclusions of universal coverings $\widetilde{\partial T_2}\to\widetilde{Y-\mathrm{Int}\,T_2}$ and $\widetilde{\partial Y}\to\widetilde{Y-\mathrm{Int}\,T_2}$ induce homology isomorphisms. Thus inclusions $\partial T_2\to Y-\mathrm{Int}\,T_2$ and $\partial Y\to Y-\mathrm{Int}\,T_2$ are homotopy equivalences (see J. H. C. Whitehead [19]). When $\pi_1\cong Z$, Whitehead group is trivial and s-cobordism theorem (M. Kervaire [4]) implies that $Y-\mathrm{Int}\,T_2=\partial T_2\times I$. Consequently Y is diffeomorphic to T_2. This completes the proof of Lemma.

§ 4. Proof of Theorems.

(a) Proof of Theorem A. Firstly we shall prove this theorem when M^m bounds a π-manifold W^{m+1} which is $\left[\frac{m+1}{2}\right]$-connected. If $\tilde{S}\in I(M)$, there exists a diffeomorphism $f: M^m-\mathrm{Int}\,D^m\to M^m-\mathrm{Int}\,D^m$ such that $f|\partial D^m\in\Gamma_m$ represents $\tilde{S}$ (see I. Tamura [13]). (Here we identified Γ_m and Θ_m by the theorem of Smale.) Using this diffeomorphism f, we construct a manifold $W\cup_f W$ which is denoted by X. Clearly the boundary ∂X is diffeomorphic to $\tilde{S}$. One has easily $\pi_1(X)=\{1\}$ by the Van Kampen's theorem. Making use of the Mayer-Vietoris exact sequence $\to H_i(M-\mathrm{Int}\,D)\to H_i(W)\oplus H_i(W)\to H_i(X)\to$

$H_{i-1}(M-\mathrm{Int}\,D)\to$ and the Poincaré-Lefschetz duality theorem, $H_i(W)\cong H^{m+1-i}(W, M)$, it is easy to see that

$$H_i(X)\approx\begin{cases} Z & i=0 \\ Z\oplus\cdots\oplus Z & i=p+1,\ q \\ 0 & \text{otherwise}. \end{cases}$$

Let $a_1', \cdots, a_k'\in H_p(M)$ and $b_1', \cdots, b_k'\in H_q(M)$ (for some k) be bases whose intersection numbers are $a_i'\circ b_j'=\delta_{ij}$. Let $f_i': S^p\to M$, $i=1, \cdots, k$ be the mapping such that $[f_i'(S^p)]=a_i'$ where $[f_i'(S^p)]$ denotes the homology class represented by $f_i'(S^p)$. By Whitney's imbedding Theorem [**20**], we may suppose that f_i' $(i=1, \cdots, k)$ are imbeddings and $f_i'(S^p)\cap f_j'(S^p)=\phi$ $i\neq j$. Let $\boldsymbol{i}: M-\mathrm{Int}\,D\to W$ be a natural inclusion map. Since $\boldsymbol{i}_*[f_i'(S^p)]=0$ and $\boldsymbol{i}_*f_*[f_i'(S^p)]=0$, we can extend f_i' and $f\circ f_i'$ to $f_i^+: D_+^{p+1}\to W$ and $f_i^-: D^{p+1}\to W$. These give an imbedding $f_i:\tilde{S}^{p+1}\to X$ such that $[f_i(\tilde{S}^{p+1})]=a_i$, where a_i is a generator of $H_{p+1}(X)$ such that $\partial_* a_i=a_i'$ where ∂_* is a boundary homomorphism of Mayer-Vietoris exact sequence:

$$\begin{array}{c} \to H_{p+1}(W)\oplus H_{p+1}(W)\to H_{p+1}(X)\xrightarrow[\approx]{\partial_*} H_p(M-\mathrm{Int}\,D)\to H_p(W)\oplus H_p(W)\to \\ \qquad\wr\wr\qquad\qquad\wr\wr\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\wr\wr\qquad\quad\wr\wr \\ \qquad 0\qquad\qquad\ 0\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\ 0\qquad\quad\ 0 \end{array}$$

Here we may assume that $\tilde{S}^{p+1}$ is a natural sphere and $f_i(S^{p+1})\cap f_j(S^{p+1})=\phi$ $i\neq j$. Let $N(f_i)$ be a tubular neighborhood of $f_i(S^{p+1})$ in $\mathrm{Int}\,X$ $(i=1, \cdots, k)$ such that $N(f_i)\cap N(f_j)=\phi$ $i\neq j$. $N(f_i)$ is a D^q-bundle over S^{p+1}; $(N(f_i), S^{p+1}, D^q, \bar{p}_i)$. Let $\hat{X}=N(f_1)\natural\cdots\natural N(f_k)\subset\mathrm{Int}\,X$ be a boundary connected sum of $N(f_1), \cdots, N(f_k)$ in $\mathrm{Int}\,X$. According to Smale [**12**], we have a handlebody decomposition as follows:

$$X=\big(N(f_1)\natural\cdots\natural N(f_k)\big)\cup D_1^q\times D_1^{p+1}\cup\cdots\cup D_k^q\times D_k^{p+1},$$

and that we can suppose that the handle $D_i^q\times D_i^{p+1}$ represents the homology class b_i $(i=1, \cdots, k)$ where b_i denotes the image of b_i' by the natural isomorphism $H_q(M)\xrightarrow[\approx]{} H_q(X)$. Thus the homotopy type of X is given by $X\simeq S_1^{p+1}\vee\cdots\vee S_k^{p+1}\cup e_1^q\cup\cdots\cup e_k^q$. Since each e_i^q attaches to equators of $S_1^{p+1}\vee\cdots\vee S_k^{p+1}$, X has the homotopy type $X\simeq S_1^{p+1}\vee\cdots\vee S_k^{p+1}\vee S_1^q\vee\cdots\vee S_k^q$. According to I. Tamura [**15**], X can be written as $\big(N(f_1)\veebar N(g_1)\big)\natural\cdots\natural\big(N(f_k)\veebar N(g_k)\big)$. (Where g_i is an imbedding of the homology generator b_i.) Since we can write $\partial\big(N(f_i)\veebar N(g_i)\big)=K_2(h_1^i, h_2^i)$ where $h_1^i\in\pi_p(SO_q)$, $h_2^i\in\pi_{q-1}(SO_{p+1})$ are characteristic maps of the bundles $N(f_i)$ and $N(g_i)$ respectively, $I(M^m)\subset\Gamma'_{p+1,q}$. Thus Theorem A is proved when M^m bounds a π-manifold W^{m+1} which is $\left[\frac{m+1}{2}\right]$-connected.

Secondly we shall prove that the general case is reduced to the case above. One has easily that $I(M)=I(M\#\tilde{S})$ and $I(M)+I(M')\subset I(M\# M')$, therefore if one proves that $M\#\tilde{S}$ or $M\# M\#\tilde{S}$ bounds a π-manifold W which is $\left[\frac{m+1}{2}\right]$-connected, the proof of Theorem A is complete. If $m\neq 8k+6$, there exists a homotopy sphere $\tilde{S}$ such that $M\#\tilde{S}$ is a boundary of a π-manifold W. (Cf. E. H. Brown and F. P. Peterson [2].)

If $m=8k+6$ there exists a homotopy sphere such that $M\#\tilde{S}$ or $M\# M\#\tilde{S}$ is a boundary of a π-manifold W.

At first we will kill the fundamental group of W. $H_i(W^{m+1})$ for $i\leqq\left[\frac{m+1}{2}\right]-1$ can be killed by surgeries inductively.

Case $m+1=2n+1$. Since $H_n(\partial W)\approx H_{n+1}(\partial W)\approx 0$, $H_n(W)$ can be killed (see C. T. C. Wall [**17**]).

Case $m+1=4n$, there exists a π-manifold W' such that index $W'=-$index W and $\partial W'$ is a homotopy sphere and that $H_i(W')=0$ for $1\leqq i\leqq 2n-1$. Since $H_{2n}\big(\partial(W\natural W')\big)\approx H_{2n-1}\big(\partial(W\natural W')\big)\approx 0$ and index $W\natural W'=0$, we can kill $H_{2n}(W\natural W')$ completely by surgeries (see J. Milnor [**9**]).

Case $m+1=4n+2$. $H_{2n+1}(W\natural W)$ can be killed, since $H_{2n+1}\big(\partial(W\natural W)\big)$ $=H_{2n+1}(M\# M)\approx 0$, $H_{2n}\big(\partial(W\natural W)\big)=H_{2n}(M\# M)\approx 0$ and Arf invariant of $W\natural W$ is zero. Consequently $M\#\tilde{S}$ or $M\# M\#\tilde{S}$ for some $\tilde{S}^m$ bounds a π-manifold W^{m+1} which is $\left[\frac{m+1}{2}\right]$-connected. Thus Theorem A is completely proved.

(b) PROOF OF THEOREM B. Let $\alpha_i=K_2(h_1^i, sh_2^i)\in\Gamma''_{p+1,q}$ for $h_1^i\in\pi_p(SO_q)$ $h_2^i\in\pi_{q-1}(SO_p)$ $i=1,\cdots,k$ such that $\{\alpha_i\ (i=1,\cdots,k)\}$ generate $\Gamma''_{p+1,q}$. Let (B_i, S^{p+1}, D^q, p_i) and $(B_i', S^q, D^{p+1}, p_i')$ be disk bundles over spheres with characteristic maps h_1^i, sh_2^i respectively. We denote $B_i\vee\!\!\!\vee B_i'$ by X_i. By Lemma in §3 X_i can be written as $T_i\underset{F_i}{\cup}T_i$ where T_i is a disk bundle over sphere with characteristic map sh_2^i and F_i is an orientation reversing diffeomorphism $F_i\colon\partial T_i-\operatorname{Int}D\to\partial T_i-\operatorname{Int}D$. Since there is an orientation reversing diffeomorphism $R\colon T_i\to -T_i$ using a cross section, $\alpha_i=\partial X_i=\partial(T_i\underset{F_i}{\cup}T_i)=\partial(T_i\underset{F_iR}{\cup}-T_i)$. Hence $I(\partial T_i)$ contains α_i. Thus if we take $\partial T_1\#\cdots\#\partial T_k$ as M, $I(M)=I(\partial T_1\#\cdots\#\partial T_k)\supset I(\partial T_1)+\cdots+I(\partial T_k)=\Gamma''_{p+1,q}$. Next we shall prove that reversed inclusion $I(M)\subset\Gamma''_{p+1,q}$ holds. For any element $\alpha\in I(M)$, there is a diffeomorphism $F\colon M-\operatorname{Int}D\to M-\operatorname{Int}D$, such that $F|\partial D\in\Gamma_{p+q}$ represents α. We put $T=T_1\natural\cdots\natural T_k$. Then we construct a manifold X such that $X=T\underset{F}{\cup}T$ where F is the above map. Since $H_i(T)$ is clearly zero for $i\leqq\left[\frac{m+1}{2}\right]$, $X=T\underset{F}{\cup}T$ can be written as $X=(B_1\vee\!\!\!\vee T_1)\natural\cdots\natural(B_k\vee\!\!\!\vee T_k)$ by the analogous method in the proof of Theorem A, where B_i is a disk bundle over sphere with some characteristic map $h_1'^i \in \pi_p(SO_q)$ $i=1, \cdots, k$. Hence $\alpha = \partial X$ is contained in $\Gamma''_{p+1,q}$. Thus $I(M) = \Gamma''_{p+1,q}$.

(c) Proof of Theorem C. When $p > q$, we have $S^p \times \tilde{S}^{q+1} = S^p \times S^{q+1}$. (See W. C. Hsiang and J. Levine and R. H. Szczarba [3].) From the proof of Corollary 3 in the later, we have $I(S^p \times \tilde{S}^{q+1}) = I(S^p \times S^{q+1}) = \{0\}$. On the other hand $K_1\left(\pi_p(SO_{q+1}), \tilde{S}^{q+1}\right)$ is contained in $I(S^p \times \tilde{S}^{q+1})$ by Lemma. Therefore Theorem C trivially holds when $p \geq q$. Now we may assume $p < q$. First we shall prove that $I(S^p \times \tilde{S}^{q+1})$ contains $K_1\left(\pi_p(SO_{q+1}), \tilde{S}^{q+1}\right)$. Let $\alpha = K_1(h, \tilde{S}^{q+1}) \in K_1\left(\pi_p(SO_{q+1}), \tilde{S}^{q+1}\right)$. We now construct two disk bundles, B_1, B_2 as follows. Let $(B_1, S^{p+1}, D^{q+1}, p_1)$ be a disk bundle with a characteristic map h, and $(B_2, \tilde{S}^{q+1}, D^{p+1}, p_2)$ be the trivial bundle over a homotopy sphere $\tilde{S}^{q+1}$. On the other hand the pairing K_1 can be interpreted as follows. Let $r \in \Gamma_{q+1}$ be a corresponding element of $\tilde{S}^{q+1} \in \Theta_{q+1}$. One defines the diffeomorphism $F' : S^p \times S^q \to S^p \times S^q$ by $F'(x, y) = \left(x, rh(x)y\right)$. Attaching two manifolds $W_1 = D^{p+1} \times S^q$ and $W_2 = S^p \times D^{q+1}$ by the diffeomorphism $F' : S^p \times S^q \to S^p \times S^q$, we have a homotopy sphere $\Sigma' = D^{p+1} \times S^q \bigcup_{F'} S^p \times D^{q+1}$ for $p < q$. Then Σ' is diffeomorphic to Σ by the diffeomorphism $f : D^{p+1} \times S^q \bigcup_F S^p \times D^{q+1} \to D^{p+1} \times S^q \bigcup_{F'} S^p \times D^{q+1}$ defined by

$$f = id \times r^{-1} \qquad \text{on } D^{p+1} \times S^q$$

and

$$f = id \times id \qquad \text{on } S^p \times D^{q+1}.$$

It is easy to see that f is a diffeomorphism between Σ and Σ'. Let G_1 be a diffeomorphism $G_1 : S^p \times D^{q+1} \to S^p \times D^{q+1}$ defined by $G_1(x, y) = \left(x, h(x)y\right)$. Let $B_1 = D_+^{p+1} \times D^{q+1} \bigcup_{G_1} D_-^{p+1} \times D^{q+1}$. Let G_2 be a diffeomorphism $G_2 : S^q \times D^{p+1} \to S^q \times D^{p+1}$ defined by $G_2(x, y) = \left(r(x), y\right)$. Let $B_2 = D_+^{q+1} \times D^{p+1} \bigcup_{G_2} D_-^{q+1} \times D^{p+1}$. We define $B_1 \diamondsuit B_2$ to be the oriented differentiable $(p+q+2)$-manifold obtained as a quotient space of $B_1 \cup B_2$ by identifying $D_-^{p+1} \times D^{q+1}$ of B_1 and $D_+^{q+1} \times D^{p+1}$ of B_2 in such a way that $(x, y) = (y, x)$ $(x \in D_-^{p+1} = D^{p+1},\ y \in D^{q+1} = D_+^{q+1})$. Let $G_1' = G_1 | S^p \times S^q$ and $G_2' = G_2 | S^q \times S^p$. Using a diffeomorphism $R : S^p \times S^q \to S^q \times S^p$ defined by $R(x, y) = (y, x)$, we define $G_2'' = R^{-1} G_2' R$. Then we have $F' = G_2'' G_1'$ and $\partial(B_1 \diamondsuit B_2) = D_+^{p+1} \times S^q \bigcup_{G_2'' G_1'} D_-^{q+1} \times S^p = D_+^{p+1} \times S^q \bigcup_{F'} D_-^{q+1} \times S^p = \Sigma' = \Sigma$. Thus α can be written as $\partial(B_1 \diamondsuit B_2)$. Considering a trivial S^p bundle over $\tilde{S}^{q+1}$ in place of S^p bundle over S^{q+1} of Lemma, quite analogously, one has that $B_1 \diamondsuit B_2$ is diffeomorphic to $B_2 \bigcup_H B_2$ where H is a diffeomorphism $H : \partial B_2 - \mathrm{Int} D \to \partial B_2 - \mathrm{Int} D$.

Therefore $\alpha=\partial(B_1 \natural B_2)=\partial(B_2 \underset{H}{\cup} B_2)$ implies that the inertia group of $\partial B_2= S^p\times\tilde{S}^{q+1}$ contains α. Conversely for any element $\alpha\in I(S^p\times\tilde{S}^{q+1})$, there is a diffeomorphism $H\colon S^p\times\tilde{S}^{q+1}-\mathrm{Int}\,D\to S^p\times\tilde{S}^{q+1}-\mathrm{Int}\,D$ such that $H|\partial D\in\Gamma_{p+q+1}$ represents α. Using this diffeomorphism we construct a manifold $D^{p+1}\times\tilde{S}^{q+1} \underset{H}{\cup} D^{p+1}\times\tilde{S}^{q+1}$ which is denoted by X. Clearly $\partial X=\alpha$ and like the proof of Theorem A, we can prove that X can be written as $B_1 \natural (\tilde{S}^{q+1}\times D^{p+1})$ where B_1 is a disk bundle over sphere with some characteristic map $h\in\pi_p(SO_{q+1})$. This implies $\alpha=K_1(h, \tilde{S}^{q+1})$ and completes the proof of Theorem C.

§ 5. Proof of Corollaries.

(a) Proof of Corollary 1.

Since the pairing K_1 is equal to that of Novikov (see [11] p. 235), next diagram commutes up to sign.

$$\begin{tikzcd}
\pi_p(SO_{q+1})\times\Gamma_{q+1} \arrow[r, "K_1"] \arrow[d, "J_p\times\omega"'] & \Gamma_{p+q+1} \arrow[d, "\omega"] \\
G_p\times G_{q+1}/\mathrm{Im}\,J_{q+1} \arrow[r, "C"'] & G_{p+q+1}/\mathrm{Im}\,J_{p+q+1}
\end{tikzcd}$$

where G_i is the stable homotopy group $G_i=\pi_{i+k}(S^k)$ and ω is the Kervaire-Milnor map [5] and J_p denotes the Hopf-Whitehead homomorphism and C is the composition.

The Kervaire-Milnor braid

$$\begin{tikzcd}
0\cong\pi_{14}(SO) \arrow[dr] \arrow[rr, bend left] & & \pi_{14}(F) \arrow[dr, "\cong"'] \arrow[rr, bend left] & & P_{14}\cong Z_2 \\
& F\Gamma_{14} \arrow[ur] \arrow[dr, "\cong"'] & & \pi_{14}(F, SO) \arrow[ur, "\phi"'] \arrow[dr] & \\
0\cong P_{15} \arrow[ur] \arrow[rr, bend right] & & \Gamma_{14} \arrow[ur, "\omega"'] \arrow[rr, bend right] & & \pi_{13}(SO)\cong 0
\end{tikzcd}$$

shows that $\omega(\tilde{S}^{14})$ (where $\tilde{S}^{14}\neq S^{14}$) is κ or $\kappa+\sigma^2$ (in Toda's notation) since $\phi(\sigma^2)\neq 0$ (see Levine [8]). But according to Toda's tables [16], $\nu\circ(\kappa+\sigma^2)$

$=\nu\circ\kappa\not\equiv 0$ (mod Im J). Hence $I(S^3\times\tilde{S}^{14})=K_1\left(\pi_3(SO),\tilde{S}^{14}\right)$ is not contained in $\Theta_{17}(\partial\pi)$. This makes the proof complete.

(b) PROOF OF COROLLARY 2. Analogously, for the generator $\tilde{S}^{10}$ of the three component of $\Theta_{10}=Z_2\oplus Z_3$, $K_1\left(\pi_3(SO),\tilde{S}^{10}\right)=\Theta_{13}$ (cf. S. P. Novikov [**11**] and A. Kosinski [**6**]). So by Theorem C, we have that $I(S^3\times\tilde{S}^{10})$ is equal to the whole group Θ_{13}.

(c) PROOF OF COROLLARY 3. Let $\tilde{S}^{p+q}\in I(S^p\times S^q)$. Then there is a diffeomorphism $f: S^p\times S^q-\mathrm{Int}\,D\to S^p\times S^q-\mathrm{Int}\,D$ such that $f|\partial D\in\Gamma_{p+q}$ represents $\tilde{S}^{p+q}$. We now construct a manifold $D^{p+1}\times S^q\bigcup_f S^p\times D^{q+1}$ which is denoted by X. If $p<q$, the homology groups of X are zero except for dimension zero. On the other hand $\pi_1(\partial X)=\pi_1(\tilde{S}^{p+q})=\{1\}$. Hence X is diffeomorphic to a disk and $\tilde{S}^{p+q}=\partial X$ is a natural sphere. This implies that $I(S^p\times S^q)=0$ for $p<q$. Making use of Kosinski's Theorem [**6**], and Wall's [**18**], $I(S^p\times S^p)=0$ is obtained for $2p\geqq 6$. Therefore $I(S^3\times S^{14})\neq I(S^3\times\tilde{S}^{14})$ and $I(S^3\times S^{10})\neq I(S^3\times\tilde{S}^{10})$. These show that the inertia group is neither PL homeomorphism invariant nor tangential homotopy equivalence invariant. Since $I(S^3\times\tilde{S}^{14})$ is not contained in $\Theta_{17}(\partial\pi)$, the conjecture of Novikov is negative.

(d) PROOF OF COROLLARY 4. This is obtained as an easy application of Theorem C.

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