J. Math. Soc. Japan
Vol. 20, No. 3, 1968

On provably recursive functions and ordinal recursive functions*

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(Received Oct. 24, 1966)

A recursive function $\phi(x)$ is defined to be $U(\mu yT(e, x, y))$, if $\forall x\exists yT(e, x, y)$, where U and T are primitive recursive and e is an integer; but nothing is said about the theory in which the predicate $\forall x\exists yT(\mathbf{e}, x, y)$ is provable. The investigation of reasonable theories $\mathfrak{T}$ in which provable recursiveness in $\mathfrak{T}$ is defined by $\vdash_{\mathfrak{T}}\forall x\exists yT(\mathbf{e}, x, y)$ forms an interesting branch of recursive function theory, and the functions provably recursive in such $\mathfrak{T}$ constitute a not unnatural subclass of the class of computable functions. We will give a characterization of provable recursiveness for certain theories.

Let $\mathfrak{T}$ be the theory of natural numbers or a subtheory of analysis. A recursive function $\phi(x)$ is called "provably recursive in $\mathfrak{T}$", if $\vdash_{\mathfrak{T}}\forall x\exists yT(\mathbf{e}, x, y)$, where e is a Gödel number of ϕ. Let $\prec$ be a primitive recursive well-ordering of natural numbers with $\neg n'\prec 0$ for every n. We call $\prec$ a *provable primitive recursive well-ordering in* $\mathfrak{T}$, if the sentence "$\prec$ is a well-ordering" is provable in $\mathfrak{T}$ (cf. § 3). A number-theoretic function ϕ is called "ordinal recursive with respect to $\prec$" ($\prec$-recursive), if it is defined by "defining equations" of primitive recursive form and by transfinite induction with respect to $\prec$. (For the precise definition, cf. [**8a**] and § 2.)

In [**11**], Takeuti defined *GLC*, a Gentzen-style simple type theory containing t-variables of the first order and f-variables with finitely many argument-places and stated his fundamental conjecture (FC) about *GLC*; (that Gentzen's Hauptsatz for *LK*, that is the cut elimination theorem, holds in *GLC* as well.) Takeuti proved that FC holds for many subsystems of *GLC* by using transfinite

* The author wishes to express her heartfelt thanks to Professor G. Takeuti for his invaluable advice during the preparation of this paper. This work was done at Hughes Aircraft Company, Fullerton, California, and was sponsored by Air Force Systems Command, Research and Technology Division, Rome Air Development Center, Griffiss Air Force Base, New York, 13442, under Contract AF 30(602)-3754. An earlier version was read at the RADC-HAC Joint Symposium on Logic. Computability and Automata held at Oriskany, N.Y. in August 1965. The author is indebted to Professor J. Myhill, Drs. F.B. Cannonito and V.H. Dyson, and Mr. G.E. Cash for reading this paper in manuscript and suggesting a number of linguistic improvements.

induction on various systems of ordinal diagrams introduced in [**14**] and [**17**][0]. This provides constructive consistency-proofs of certain subsystems of analysis, for example, arithmetic with the Π_1^1-comprehension axiom:

$$\forall x_1 \cdots \forall x_n \ \forall \varphi_1 \cdots \forall \varphi_m \exists \varphi \forall x(x \in \varphi \Leftrightarrow A(x, x_1, \cdots, x_n, \varphi_1, \cdots, \varphi_m))$$

where $\varphi, \varphi_1, \cdots, \varphi_m$ are variables of the second order and $A(x, x_1, \cdots, x_n, \varphi_1, \cdots, \varphi_m)$ does not contain φ and is in Π_1^1-form; i.e. in the form $\forall \phi B$ with B arithmetic.

Let $\mathfrak{T}$ be such a subsystem of analysis, $O(\mathfrak{T})$ the system of ordinal diagrams used to prove the consistency of $\mathfrak{T}$ and well-ordered by $\prec$, and $|O(\mathfrak{T})|$ the order-type of $O(\mathfrak{T})$. For an element s of $O(\mathfrak{T})$, let $\prec^s$ denote the suborder of $\prec$ up to s, that is, $\prec^s$ is defined by $\forall x \forall y(x \prec^s y \Leftrightarrow x \prec y \wedge y \prec s)$. By the technique of arithmetization the relations "s is an element of $O(\mathfrak{T})$", "$a \prec b$" and "$a \prec^s b$" become primitive recursive predicates, and $\prec$ and $\prec^s$ become primitive recursive well-orderings of natural numbers.

THEOREM 1. *Let ϕ be a provably recursive function in $\mathfrak{T}$. Then we find an element s of $O(\mathfrak{T})$ such that ϕ is $\prec^s$-recursive.*

THEOREM 2. *If $\prec$ is a provable primitive recursive well-ordering in $\mathfrak{T}$, then every $\prec$-recursive function is provably recursive in $\mathfrak{T}$.*

Theorems 1 and 2 apply also to Gentzen's theory NN of natural numbers, yielding results which have been obtained by Kreisel[1].

In [**20**] Takeuti proved that the following condition (†)[2] is satisfied by $\mathfrak{T}$, where $\mathfrak{T}$ is any of the theories NN (classical arithmetic), RNN and f-CNN. (Some intuitive idea of the latter two theories may be gained by observing that in each of them the following form of the comprehension axiom holds.

$$\forall x_1 \cdots \forall x_n \exists \varphi \forall x(x \in \varphi \Leftrightarrow A(x, x_1, \cdots, x_n))$$

where $A(x, x_1, \cdots, x_n)$ does not contain φ or any free second-order variable; but in each of them the power of this axiom is somewhat lessened by restrictions on the inference schema

0) In this paper we simply say "ordinal diagrams of finite order" referring to the system of ordinal diagrams of order n developed in [**14**] or the system $O(\{0, \cdots, n\}, N)$ $(=O(\{0, \cdots, n\}, N, \phi))$ where N denotes the set of natural numbers in their usual order, developed in [**17**]. The system $O(\{0, \cdots, n\}, N)$ which is order-isomorphic to the system of ordinal diagrams of order $n+1$ in [**14**], will sometimes be referred to as the system of ordinal diagrams of order n. By a system of "ordinal diagrams of infinite order" we understand a system $O(I, A)$, where, at least, I is not a finite set.

1) It was suggested to the author that Kreisel be credited for first having proved in [**6**] Theorems 1 and 2 for the case of arithmetic. Professor Kreisel suggested referring to [**9**], particularly 3.3234 and 3.3421.

2) Though the condition (†) in [**20**] is stated incorrectly, the results there remain correct by reading (†) in the present form.

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}.$$

These theories will be exactly defined in § 1.)

(†) *For every ordinal σ less than $|O(\mathfrak{T})|$, there is a provable primitive recursive well-ordering in $\mathfrak{T}$ whose order-type is σ.*

Theorem 2 implies:

COROLLARY. *When (†) holds for $\mathfrak{T}$, for every ordinal σ less than $|O(\mathfrak{T})|$, there is a provable primitive recursive well-ordering $\prec$ in $\mathfrak{T}$ such that the order-type of $\prec$ is σ and every $\prec$-recursive function is provably recursive in $\mathfrak{T}$.*

In [4], Gödel defined "primitive recursive functionals of finite type" (PR functionals) and showed that every ordinal recursive function (or order $<\varepsilon_0$ with respect to the usual standard ordering of order ε_0) is a PR functional. Tait in [10] stated that the converse, that is, "Every PR functional of type (0, 0) is ordinal recursive", was established by a result of Kreisel[3] and established it himself by a method more direct than Kreisel's. From these results together with our theorem we see that provable recursiveness in arithmetic coincides with Gödel's primitive recursiveness, which is essentially proved by Gödel [4].

In [20] and [21], Takeuti remarked that Gentzen's result of [3] can be stated in a more general form, namely: The order-type of any provable recursive well-ordering in $\mathfrak{T}$ is less than $|O(\mathfrak{T})|$; and this applies to most systems $\mathfrak{T}$ whose consistency has been proved by an application of the proof of FC to the corresponding subsystem of *GLC*. We will show that this technique can also be imitated for *SJNN*, if we take a suitable slightly larger system for $O(SJNN)$[3a].

§ 1. Preliminary definitions.

In the following we will restate several Gentzen-style theories of natural numbers formalized in the first or second order predicate calculus, and developed in [2], [15], [18], [19] and [21].

The subsystem G^1LC of *GLC* is the second order predicate calculus, where f-variables are predicate variables with argument-places only of the first order. We recall several notions concerning G^1LC (see [11], [12] or [21] for these notions as well as G^1LC). A *semi-formula* is a formula or is obtained from a formula by substituting bound t-variables for one or more free ones. Note that while each formula is a semi-formula, not all semi-formulas are formulas;

3) Professor Kreisel informed us that the paper that Tait referred to is [8b].

3a) Kreisel has suggested that "recursive" above might plausibly be replaced by "Σ_1^1" following his method in [8c], but we have not verified the details.

the distinction being that a semi-formula is a formula if and only if each occurrence in it of a bound variable is quantified. A *semi-variety* is the form abstracted from a semi-formula and is expressed as $\{x_1, \cdots, x_n\}$ $F(x_1, \cdots, x_n)$, where $F(a_1, \cdots, a_n)$ is a semi-formula, $a_1, \cdots, a_n$ are pairwise distinct free t-variables and $x_1, \cdots, x_n$ are pairwise distinct bound t-variables not contained in $F(a_1, \cdots, a_n)$. A semi-variety is called a *variety*, if it does not contain any free occurrence of bound variables. A *quasi-formula* is a semi-formula or a semi-variety. By $A \vdash B$ we understand $\neg A \vee B$.

Throughout this paper, by a *mathematical beginning sequence* we understand a sequence $\Gamma \rightarrow \Delta$ with the following properties; every formula of Γ or Δ is primitive recursive and contains no logical symbol, and every sequence obtained from $\Gamma \rightarrow \Delta$ by replacing all free t-variables in Γ and Δ by arbitrary natural numbers (i.e. numerals) is true. By a *logical beginning sequence* and a *beginning sequence for equality* we mean a sequence of the form $D \rightarrow D$ and $s=t$, $A(s) \rightarrow A(t)$ respectively (s and t arbitrary terms).

An inference in a proof-figure is called *implicit* if the fibre (Formelbund[4]) through the principal formula (Hauptformel) of this inference ends in a cut-formula (Schnittformel); otherwise it is called *explicit* (cf. [**12**] or [**21**] for the precise definition of "implicit" and "explicit").

Let $\sharp$ be a universal quantifier for predicate variables ($\forall$ on an f-variable) and $\natural$ a variable or a logical symbol. If $\natural$ appears in the scope of $\sharp$, we say "$\sharp$ ties $\natural$". If $\sharp$ appears as the leftmost $\forall$ in the form $\forall \varphi B$ in a quasi-formula A and $\natural$ is an $\forall$ on an f-variable appearing in the scope B of $\sharp$ and $\natural$ ties φ, we say "$\sharp$ affects $\natural$ in A".

1.1 *Gentzen's system NN of the theory of natural numbers* (given in [**2**]): NN is obtained from LK ([**1**]) by the following modifications.

1.1.1 Every beginning sequence of NN is a logical or mathematical beginning sequence or a beginning sequence for equality.

1.1.2 The following inference schema "induction" (VJ-Schlussfigur) is added:

$$\frac{A(a), \Gamma \rightarrow \Delta, A(a')}{A(0), \Gamma \rightarrow \Delta, A(t)}$$

where a is not contained in any of $A(0)$, Γ and Δ, and t is an arbitrary term. $A(a)$ and $A(a')$ are called the *principal-formulas* and a is called the *eigenvariable* of this induction.

The consistency of NN is proved by using transfinite induction up to ε_0 (cf. [**2**]). By a result of [**3**], (†) is true for NN ([**20**]).

1.2 *The theory RNN of natural numbers* (developed in [**15**]): RNN is

4) Here and in the following Gentzen's terminologies in [**1**]—[**3**] are sometimes given in parentheses.

obtained from G^1LC by the following modifications:

1.2.1 Every beginning sequence of *RNN* is a logical or mathematical beginning sequence or a beginning sequence for equality.

1.2.2 The inference schema "induction" is added.

1.2.3 Every implicit inference $\forall$ left on an f-variable in a proof-figure of *RNN* is restricted by the condition that its principal formula be regular. The definition of a regular formula is seen in § 2, Chapter I of [**15**]. Briefly, a formula A is *regular*, if the following condition is satisfied: Let $\sharp$ and $\natural$ be an arbitrary pair of $\forall$'s on f-variables in A occurring in the forms $\sharp\phi B(\phi)$ and $\natural\varphi C(\varphi)$ respectively. If $\natural\varphi C(\varphi)$ appears in $B(\phi)$ and $\natural$ is negative with respect to $\sharp$, then $\natural$ is isolated, where $\natural$ is called *isolated* if the following conditions are satisfied:

(i) $C(\varphi)$ contains no free f-variable.

(ii) No $\forall$ on an f-variable affects $\natural$.

(iii) $\natural$ does not affect any $\forall$ on an f-variable.

The consistency of *RNN* is proved by using transfinite induction on ordinal diagrams of finite order in [**15**]. It is proved in [**16**] that (†) is true for *RNN*.

1.3 *The theory f-CNN of natural numbers* (developed in [**18**]): *f-CNN* is obtained from G^1LC by the following modifications.

1.3.1 Every beginning sequence of *f-CNN* is a logical or mathematical beginning sequence or a beginning sequence for equality.

1.3.2 The inference schema "induction" is added.

1.3.3 Every implicit inference $\forall$ left on an f-variable in a proof-figure of *f-CNN* is restricted by the condition that its principal-formula be f-closed; that is, if

$$\frac{F(V),\ \Gamma \rightarrow \Delta}{\forall\varphi F(\varphi),\ \Gamma \rightarrow \Delta}$$

is such an inference, $\forall\varphi F(\varphi)$ does not contain any free f-variable.

The consistency of *f-CNN* is proved in [**18**] by using transfinite induction on ordinal diagrams of finite order. It is proved in [**16**] that (†) is true for *f-CNN*[5] ([**20**]).

1.4 *The system ID with inductive definition* (developed in [**19**]): This system is obtained from G^1LC by the following modifications.

5) Every principal formula of an inference $\forall$ left on an f-variable used in [**16**] is isolated and f-closed. Moreover, if

$$\frac{F(V),\ \Gamma \rightarrow \Delta}{\forall\varphi F(\varphi),\ \Gamma \rightarrow \Delta}$$

is an inference $\forall$ left on an f-variable used in [**16**], then V is isolated. That is, the formal theory of the ordinal diagrams (of finite order) can be developed within any of *RNN*, *f-CNN*, *SJNN*, and *SMINN*.

Let $I(a)$ and $a<^*b$ be primitive recursive predicates, $<^*$ being a well-ordering of the set $\{a: I(a)\}$, and let $A_0, A_1, A_2, \cdots$ be new symbols for predicates with two argument-places.

1.4.1 Every beginning sequence is a logical or mathematical beginning sequence or a beginning sequence for equality, or a sequence of the following form (referred to as a "beginning sequence for inductive definition"):

$$I(s),\, A_j(s, t) \rightarrow G_j(s, t, \{x, y\}(A_j(x, y) \wedge x <^* s))$$

or

$$I(s),\, G_j(s, t, \{x, y\}(A_j(x, y) \wedge x <^* s)) \rightarrow A_j(s, t)\,,$$

where $j=0, 1, 2, \cdots$ and s, t are arbitrary terms. Here $G_j (j=0, 1, 2, \cdots)$ are arbitrary formulas satisfying the following conditions: (i) $G_j(a, b, \alpha)$ does not contain $A_j, A_{j+1}, \cdots$. (ii) If $G_j(a, b, \alpha)$ contains a figure of the form $\forall\varphi A(\varphi)$, then $A(\beta)$ does not contain any bound f-variable.

1.4.2 The inference schema "induction" is added.

1.4.3 Every implicit inference $\forall$ left on an f-variable of the form

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall\varphi F(\varphi), \Gamma \rightarrow \Delta}$$

is restricted by the condition that $F(\alpha)$ does not contain any bound f-variables. ($F(\alpha)$ may contain $A_0, A_1, A_2, \cdots$ and V may contain bound f-variables and $A_0, A_1, A_2, \cdots$). The consistency of *ID* is proved by using transfinite induction on a system of ordinal diagrams (of a certain infinite order, cf. [**17**] and [**19**]). It is not known whether (†) for *ID* is true or not.

1.5 *SINN* (developed in [**21**]). *SINN* is obtained from G^1LC by the following modifications.

1.5.1 Every beginning sequence of *SINN* is a logical or mathematical beginning sequence or a beginning sequence for equality.

1.5.2 The inference schema "induction" is added.

1.5.3 Every implicit inference $\forall$ left on an f-variable of the form

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall\varphi F(\varphi), \Gamma \rightarrow \Delta}$$

is restricted by the condition that V be semi-isolated, where V is called semi-isolated if every $\forall$ on f-variable $\mathfrak{h}$ in V satisfies the conditions (ii) and (iii) in 1.2.3. The consistency of this system is proved by using transfinite induction on a system of ordinal diagrams (of a certain infinite order. Cf. [**17**] and [**21**]).

1.6 *The system SMINN. SMINN* is obtained from G^1LC by the following modifications.

1.6.1 Every beginning sequence of *SMINN* is a logical or mathematical beginning sequence or a beginning sequence for equality.

1.6.2 The inference schema "induction" is added.

1.6.3 Every implicit inference $\forall$ left on an f-variable in a proof-figure of *SMINN* is restricted by the condition that its principal formula be semi-isolated. The consistency of this system is proved by using transfinite induction on ordinal diagrams of finite order by a slight modification of the consistency-proof of S_2 in [**21**][6]. It is proved in [**16**] that (†) is ture for *SMINN*[5].

1.7 *SJNN* (given in [**21**]). *SJNN* is obtained from *SINN* by the following restrictions.

1.7.1 Every formula in a beginning sequence of *SJNN* is without logical symbols.

1.7.2 Every principal formula of an induction in *SJNN* is semi-isolated.

The consistency of this system is proved in [**21**] by using transfinite induction on ordinal diagrams of finite order. It is proved in [**16**] that (†) is true for *SJNN*[5].

1.8 *The system EID with extended inductive definition* (defined in [**21**]).

Let $I(a)$ and $a <^* b$ be as in 1.4 and $A_0, A_1, A_2, \cdots$ be symbols for predicates, where $A_j(a, b, \alpha)$ is a formula.

Let A be a semi-formula, $A_j(a, b, V)$ a semi-formula in A and $\natural$ an arbitrary variable or logical symbol contained in V. Then we say "$\natural$ is tied by A_j in A." We say "$\sharp$ affects A_j in $\forall\varphi F$", if $\sharp$ is the outermost $\forall$ of $\forall\varphi F$ and φ is tied by A_j in $\forall\varphi F$. The notion "semi-isolated" for this system is extended as follows: A semi-formula A is called *semi-isolated*, if any $\forall$ on an f-variable contained in A does not affect any other $\forall$ on an f-variable or $A_0, A_1, \cdots$ in A.

A semi-variety $\{x_0, \cdots, x_m\}H(x_0, \cdots, x_m)$ is called *semi-isolated*, if $H(a_0, \cdots, a_m)$ is semi-isolated.

The system *EID* is obtained from G^1LC by the following modifications.

1.8.1 Every beginning sequence is a logical or mathematical beginning sequence or a beginning sequence for equality, or a sequence of the following form (called a *beginning sequence for inductive definition*):

$$I(s), A_j(s, t, V) \rightarrow G_j(s, t, V, \{x, y\}(A_j(x, y, V) \wedge x <^* s))$$

or

$$I(s), G_j(s, t, V, \{x, y\}(A_j(x, y, V) \wedge x <^* s)) \rightarrow A_j(s, t, V),$$

where $j = 0, 1, 2, \cdots$, $G_j(a, b, \alpha, \beta)$ is an arbitrary semi-isolated formula containing none of $A_j, A_{j+1}, \cdots$, V is an arbitrary variety and s, t are arbitrary terms.

1.8.2 The inference schema "induction" is added.

1.8.3 Every implicit inference $\forall$ left on an f-variable of the form

6) The consistency-proof of *SMINN* is given in § 4. The system S_2 is obtained from *SMINN* by deleting the inference schema "induction".

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}$$

is restricted by the condition that V be semi-isolated. The consistency of *EID* is proved by using transfinite induction on a system of ordinal diagrams of certain infinite order (cf. [**21**]). It is not known whether (†) for *EID* is true or not.

1.9 *The system* S_1 (defined in [**12**]). S_1 is obtained from *SJNN* by deleting the inference schema "induction".

§ 2. Ordinal recursiveness of provably recursive functions.

In this section, we prove Theorem 1 proposed in the introduction, beginning by defining several notions.

DEFINITION 1a. Let $S(a)$ and $a \prec b$ be primitive recursive predicates such that $\prec$ is a well-ordering of $\{a : S(a)\}$, $\neg n' \prec 0$ for every natural number n and $b \prec a \rightarrow S(a) \wedge S(b)$. A number-theoretic function ϕ is called *$\prec$-recursive* if and only if one of the following holds (cf. [**5**] and [**8a**]):

(I) $\phi(a) = a'$.

(II) $\phi(a_1, \cdots, a_n) = 0$.

(III) $\phi(a_1, \cdots, a_n) = a_i, 1 \leqq i \leqq n$.

(IV) $\phi(a_1, \cdots, a_n) = \psi(\chi_1(a_1, \cdots, a_n), \cdots, \chi_m(a_1, \cdots, a_n))$.

where ψ and χ_i $(1 \leqq i \leqq m)$ are $\prec$-recursive.

$$\text{(V)} \quad \begin{cases} \phi(0, a_2, \cdots, a_n) = \psi(a_2, \cdots, a_n), \\ \phi(a', a_2, \cdots, a_n) = \chi(a, \phi(a, a_2, \cdots, a_n), a_2, \cdots, a_n), \end{cases}$$

where ψ and χ are $\prec$-recursive.

$$\text{(VI)} \quad \begin{cases} \phi(0, a_2, \cdots, a_n) = \psi(a_2, \cdots, a_n), \\ \phi(a', a_2, \cdots, a_n) = \chi(a, \phi(\tau^*(a, a_2, \cdots, a_n), a_2, \cdots, a_n), a_2, \cdots, a_n), \end{cases}$$

where ψ, χ and τ are $\prec$-recursive and

$$\tau^*(a, a_2, \cdots, a_n) = \begin{cases} \tau(a, a_2, \cdots, a_n) & \text{if} \quad \tau(a, a_2, \cdots, a_n) \prec a', \\ 0 & \text{otherwise}. \end{cases}$$

By a *Gentzen-style theory of natural numbers* we mean a theory of natural numbers formalized in *LK* or in G^1LC and containing *NN* as a subsystem. (For example, any of the systems given in 1.1–1.8)

DEFINITION 2. Let τ be a Gentzen-style theory of natural numbers. A recursive function $\phi(a_1, \cdots, a_n)$ is called *provably recursive* in $\mathcal{T}$, if the following sequence is provable in $\mathcal{T}$:

$$\rightarrow \forall x_1 \cdots \forall x_n \exists y \mathrm{T}_n(\mathbf{e}, x_1, \cdots, x_n, y),$$

where T_n expresses Kleene's primitive recursive predicate T_n (cf. [**5**]) and e is a Gödel number of ϕ. In the following T_1 will be abbreviated by T.

THEOREM 1a. *Let* $\mathfrak{T}$ *be one of the systems given in* 1.1–1.6 *and* 1.8. *Let* $\phi(a_1, \cdots, a_n)$ *be a provably recursive function in* $\mathfrak{T}$, *and e a Gödel number of* ϕ *such that the sequence*

$$\rightarrow \forall x_1 \cdots \forall x_n \exists y \mathrm{T}_n(\mathbf{e}, x_1, \cdots, x_n, y)$$

is provable in $\mathfrak{T}$. *Then we can find an element s of* $\mathrm{O}(\mathfrak{T})$ *such that* ϕ *is* $\dot{<}$-*recursive, where* $\dot{<}$ *is the primitive recursive well-ordering of natural numbers obtained by arithmetizing the suborder* $\prec^s$ *of the well-ordering* $\prec$ *of* $\mathrm{O}(\mathfrak{T})$ *up to s.*

PROOF. Without loss of generality, we may assume that $n=1$.

2.0 *Outline of the proof.* We define "degree", "proof-figure with degree", "proof-figure of order n", etc., in a manner analogous to that used to define these concepts in the consistency-proof of $\mathfrak{T}$, and use the same assignment of an element of $\mathrm{O}(\mathfrak{T})$ to every sequence of a proof-figure of $\mathfrak{T}$ as in the consistency-proof of $\mathfrak{T}$[7].

The sequence $\rightarrow \exists y \mathrm{T}(\mathbf{e}, a, y)$, where a is a free t-variable, is provable in $\mathfrak{T}$ according to our assumption. Let $\mathfrak{P}_0$ be a proof-figure ending with the sequence $\rightarrow \exists y \mathrm{T}(\mathbf{e}, a, y)$ in $\mathfrak{T}$, and such that every free t-variable except a in $\mathfrak{P}_0$ is used as an eigenvariable in $\mathfrak{P}_0$ and the eigenvariables in $\mathfrak{P}_0$ are different from each other and a; let s be the element of $\mathrm{O}(\mathfrak{T})$ assigned to $\mathfrak{P}_0$ and $\prec^s$ the suborder of $\prec$ up to s. Let $\dot{<}$ be the primitive recursive well-ordering of natural numbers which is obtained from $\prec^s$ by arithmetization. We will show that $\mathbf{P}$ is $\dot{<}$-recursive, where $\mathbf{P}$ is the process which, for given m, computes n such that $\mathrm{T}(\mathbf{e}, \mathbf{m}, \mathbf{n})$ from the proof-figure $\mathfrak{P}_0$.

In the following we will fix a primitive recursive enumeration of sequences in a proof-figure in $\mathfrak{T}$, and call the number corresponding to a sequence its

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7) See § 1, for the reference to the consistency-proof of $\mathfrak{T}$. Here a proof-figure of $\mathfrak{T}$ may contain the inference schema "substitution" with certain restrictions, if $\mathfrak{T}$ is not *NN*. The inference schema "substitution" is of the form

$$\frac{A_1, \cdots, A_m \rightarrow B_1, \cdots, B_n}{A_1\binom{V}{\alpha}, \cdots, A_m\binom{V}{\alpha} \rightarrow B_1\binom{V}{\alpha}, \cdots, B_n\binom{V}{\alpha}}$$

where α is a free f-variable, V is a variety with the same number of argument-places as α and

$$A_i\binom{V}{\alpha} \quad \text{or} \quad B_j\binom{V}{\alpha}$$

is a formula obtained from A_i or B_j by substituting V for $\alpha(1 \leqq i \leqq m$ and $1 \leqq j \leqq n)$, cf. [**11**], [**15**] and [**21**].

ν-number. By the *ν-number of an inference* we mean the ν-number of the lower sequence of the inference. First, we define the reduction of proof-figures in $\mathcal{T}$.

2.1 *Reduction of proof-figures in $\mathcal{T}$.* Let $o(\mathfrak{P})$ denote the element of $O(\mathcal{T})$ assigned to a proof-figure $\mathfrak{P}$. Let $\mathbf{R}$ be the set of proof-figures $\mathfrak{P}$ (with degree or of order n) in $\mathcal{T}$ having the following properties:

(P1) The end-sequence of $\mathfrak{P}$ is of the form

$$\to \exists y T(\mathbf{e}, \mathbf{m}, y),\ T(\mathbf{e}, \mathbf{m}, \mathbf{n}_1), \cdots, T(\mathbf{e}, \mathbf{m}, \mathbf{n}_k)\ (k \geqq 0),$$

or

$$\to T(\mathbf{e}, \mathbf{m}, \mathbf{n}_1), \cdots, T(\mathbf{e}, \mathbf{m}, \mathbf{n}_k)\ (k \geqq 1),$$

where $\mathbf{m}$ and $\mathbf{n}_i (0 \leqq i \leqq k)$ are numerals.

(P2) Every free t-variable in $\mathfrak{P}$ is used as an eigenvariable.

(P3) The eigenvariables in $\mathfrak{P}$ are pairwise distinct.

(P4) $o(\mathfrak{P})$ is not larger than s with respect to the well-ordering of $O(\mathcal{T})$.

We will define the reduction r of proof-figures $\mathfrak{P}$ in $\mathbf{R}$.

2.1.1 If the end-piece (Endstück) of $\mathfrak{P}$ contains an induction, $r(\mathfrak{P})$ is the proof-figure obtained from $\mathfrak{P}$ by applying the "VJ-Reduktion" (3.3 of [2]) to the bottommost induction with the smallest ν-number. The end-sequence of $r(\mathfrak{P})$ is the same as that of $\mathfrak{P}$.

2.1.2 If the end-piece of $\mathfrak{P}$ does not contain any induction and it contains an explicit logical inference (which is an inference $\exists$-right on a t-variable), $r(\mathfrak{P})$ is the proof-figure obtained from $\mathfrak{P}$ as follows: Let $\mathfrak{J}$ be the explicit logical inference with the smallest ν-number in the end-piece of $\mathfrak{P}$ which appears as the bottommost one of such inferences in the string (Faden) to which it belongs.

$$\begin{array}{c} \vdots \\ \dfrac{\Gamma \to \Delta,\ T(\mathbf{e}, \mathbf{m}, \mathbf{n})}{\Gamma \to \Delta,\ \exists y T(\mathbf{e}, \mathbf{m}, y)}\ \mathfrak{J} \\ \vdots \\ \Gamma_0 \to \Delta_0 \end{array}$$

$r(\mathfrak{P})$ is defined to be the proof-figure obtained from $\mathfrak{P}$ by replacing the above part of the proof-figure by the following:

$$\begin{array}{c} \vdots \\ \Gamma \to \Delta,\ T(\mathbf{e}, \mathbf{m}, \mathbf{n}) \\ \hline \text{Some exchanges and a weakening} \\ \hline \Gamma \to T(\mathbf{e}, \mathbf{m}, \mathbf{n}),\ \exists y T(\mathbf{e}, \mathbf{m}, y) \\ \vdots \\ \Gamma_0 \to T(\mathbf{e}, \mathbf{m}, \mathbf{n}),\ \Delta_0 \end{array}$$

REMARK. *Elimination of beginning sequences for equality* in the end-piece of a proof-figure. If the end-piece of $\mathfrak{P}$ does not contain any induction or explicit logical inference, we can eliminate beginning sequences for equality contained in the end-piece by 8.4, Chapter 2 of [**21**]. By $\mathfrak{P}\#$ we denote the proof-figure thus obtained from $\mathfrak{P}$.

2.1.3 If the end-piece of $\mathfrak{P}$ does not contain any induction, or explicit logical inference, and contains a logical beginning sequence, $\mathrm{r}(\mathfrak{P})$ is the proof-figure obtained from $\mathfrak{P}\#$ by applying the reduction 8.5, Chapter 2 of [**21**] (or 3.3 of [**15**]) to the sequence with the smallest ν-number.

REMARK. *Elimination of weakenings* (Verdünnungen) in the end-piece of a proof-figure. Let $\mathfrak{Q}$ be a proof-figure such that the end-piece of $\mathfrak{Q}$ does not contain any induction, logical inference, or beginning sequence for equality. By $\mathfrak{Q}^*$ we denote the proof-figure obtained from $\mathfrak{Q}$ by applying the reduction 8.6, Chapter 2 of [**21**]; i.e., the end-place of $\mathfrak{Q}^*$ does not contain any induction, logical inference, beginning sequence for equality, or weakening.

2.1.4 For the case where $\mathfrak{T}$ is *ID* or *EID*: Let the end-piece of $\mathfrak{P}$ not contain any induction, logical inference, beginning sequence for equality, or logical beginning sequence, but assume it contains a beginning sequence for inductive definition. Then $\mathrm{r}(\mathfrak{P})$ is the proof-figure obtained from $\mathfrak{P}^*$ by applying the reduction 3.6 of [**19**] or 9.1, Chapter 4 of [**21**], respectively, to the beginning sequence for inductive definition with the smallest ν-number.

2.1.5 Let the end-piece of $\mathfrak{P}$ not contain any induction, logical inference, beginning sequence for equality, logical beginning sequence, or beginning sequence for inductive definition. Then the end-piece of $\mathfrak{P}^*$ does not contain any cut, or contains a suitable cut. In the former case, let $\mathrm{r}(\mathfrak{P})$ be $\mathfrak{P}^*$. (In this case the end-piece of $\mathfrak{P}^*$ is $\mathfrak{P}^*$ itself, and the reduction is completed.) In the latter case, let $\mathrm{r}(\mathfrak{P})$ be the proof-figure obtained from $\mathfrak{P}^*$ by applying the essential reduction (Verknüpfungsreduktion) to the suitable cut with the smallest ν-number[8].

This completes the definition of the reduction.

Let $\mathrm{o}(\mathfrak{P})$ denote the element of $O(\mathfrak{T})$ assigned to a proof-figure $\mathfrak{P}$. The reduction has the following properties:

(Q1) For every $\mathfrak{P}$ in **R**, $\mathrm{r}(\mathfrak{P})$ is also in **R** and the second argument of T in the end-formula of $\mathfrak{P}$ is preserved by r.

---

8) A cut in the end-piece of a proof-figure is called *suitable* if and only if each cut-formula of this cut has a fibre that ends with this cut-formula, and contains the principal formula of a logical inference whose lower sequence is an uppermost sequence of the end-piece. For the existence of a suitable cut, see 6.4 of [**12**], or 9, Chapter 2 of [**21**]. For the essential reduction, see 3.5 of [**2**] for *NN*; 4.2–4.5 of [**15**] for *RNN* and *f-CNN*; 4 of [**19**] for the system with inductive definition; 10, Chapter 2 of [**21**] for the systems given in 1.5–1.8.

(Q2) $o(r(\mathfrak{P}))$ is not larger than $o(\mathfrak{P})$ with respect to the well-ordering of $O(\mathfrak{T})$.

(Q3) If $o(r(\mathfrak{P}))=o(\mathfrak{P})$, then $o(r(r(\mathfrak{P})))=o(\mathfrak{P})$.

(Q4) Let $o(r(\mathfrak{P}))=o(\mathfrak{P})$. Then $\mathfrak{P}$ does not contain any logical inference, free variable, induction, or weakening, and every beginning sequence of $\mathfrak{P}$ is a mathematical beginning sequence.

2.2 *Arithmetization* (conclusion of the proof). Let $P(a)$ be primitive recursive predicate which states that a is (the Gödel number of) a proof-figure in **R**.

Let $o(a)$ be the function expressing (the Gödel number of) the element of $O(\mathfrak{T})$ assigned to a, provided that $P(a)$ holds, $a \prec b$ the arithmetization of the well-ordering $\prec$ of $O(\mathfrak{T})$, and $r(a)$ the function which expresses (the Gödel number of) the proof-figure obtained from a by applying the reduction r in 2.1, provided that $P(a)$ holds. "$r(a)=a$" will mean that the reduction is over. In this case, the end-sequence of a is of the form $\rightarrow \mathrm{T}(\mathbf{e}, \mathbf{m}, \mathbf{n}_1) \cdots, \mathrm{T}(\mathbf{e}, \mathbf{m}, \mathbf{n}_k)$ $(k \geqq 1)$. As is easily seen, $o(a)$ and $r(a)$ can be chosen to be primitive recursive. Defining "$a \dot{\prec} b$" and "$a \equiv b$" by "$o(a) \prec o(b) \wedge P(a) \wedge P(b)$" and "$o(a) = o(b) \wedge P(a) \wedge P(b)$", respectively, we can consider $\{a : P(a)\}$ well-ordered by $\dot{\prec}$, whose order-type is that of $\prec^s$. Let $\chi(a)$ be defined as follows:

$$\chi(a)=\begin{cases} \text{the least } n \text{ such that } \mathrm{T}(\mathbf{e}, \mathbf{m}, \mathbf{n}) \text{ is an end-formula of } a \\ \qquad\qquad \text{and } \mathrm{T}(e, m, n) \text{ is true, if } P(a) \wedge r(a)=a\,, \\ 0 \text{ otherwise}. \end{cases}$$

χ is a primitive recursive function. Let $\psi_1(a)$ be the cocharacteristic function of $P(a) \wedge r(a)=a$. Let ψ be defined as follows:

$$\begin{cases} \psi(0)=0\,, \\ \psi(a')=\psi_1(a')+\psi(\tau^*(a))\,, \end{cases}$$

where $\tau(a)=r(a')$, and $\tau^*(a)=\tau(a)$ if $\tau(a) \dot{\prec} a'$; 0 otherwise. The function ψ is $\dot{\prec}$-recursive. Let p be the Gödel number of the proof-figure obtained from $\mathfrak{P}_0$ (in 2.0) by substituting **m** for the free t-variable a throughout $\mathfrak{P}_0$. Then $\phi(m)$ is given by $U(\psi(p))$. This completes the proof of Theorem 1a.

Definition 1b. Let $S(a)$ and $a \prec b$ be as in Definition 1a, and let $S_1(a)$ and $a \prec_1 b$ be primitive recursive predicates such that $\prec_1$ is a well-ordering of $\{a : S_1(a)\}$, $\neg n' \prec_1 0$ for every natural number n, and $b \prec_1 a \rightarrow S_1(a) \wedge S_1(b)$. A number-theoretic function ϕ is called $(\prec, \prec_1)$-recursive, if and only if one of (I)-(VI) in Definition 1a holds (where ψ, χ, χ_i, $(1 \leqq i \leqq m)$ and τ are regarded as $(\prec, \prec_1)$-recursive), or else the following holds:

$$\text{(VII)} \quad \begin{cases} \phi(0, a_2, \cdots, a_n)=\psi(a_2, \cdots, a_n) \\ \phi(a', a_2, \cdots, a_n)=\chi(a, \phi(\tau^*(a, a_2, \cdots, a_n), a_2, \cdots, a_n), a_2, \cdots, a_n)\,, \end{cases}$$

where ψ, χ and τ are $(\prec, \prec_1)$-recursive and

$$\tau^*(a, a_2, \cdots, a_n) = \begin{cases} \tau(a, a_2, \cdots, a_n) & \text{if } \tau(a, a_2, \cdots, a_n) \prec_1 a', \\ 0 & \text{otherwise}. \end{cases}$$

THEOREM 1b. *Let $\phi(a_1, \cdots, a_n)$ be a provably recursive function in SJNN, and e a Gödel number of ϕ such that the sequence*

$$\rightarrow \forall x_1 \cdots \forall x_n \exists y \mathrm{T}_m(\mathbf{e}, x_1, \cdots, x_n, y)$$

is provable in SJNN. Then we can find an element s of O(*SJNN*) *such that ϕ is $({}^{s}\!\!\prec, {}^{1}\!\!\prec)$-recursive, where ${}^{s}\!\!\prec$ is the primitive recursive well-ordering of natural numbers obtained from $\prec^s$ by arithmetization and ${}^{1}\!\!\prec$ is a primitive recursive well-ordering of natural numbers with the order-type ω^ω.*

PROOF. Let $e(a)$ be the formula

$$\forall \varphi(\varphi[0] \wedge \forall y(\varphi[y] \vdash \varphi[y']) \vdash \varphi[a]).$$

We begin the proof by defining the reduction of proof-figures in *SJNN*. Without loss of generality, we may assume that $n = 1$.

2.3 *Reduction of proof-figures in SJNN.* Let $\mathfrak{P}$ be a proof-figure ending with the sequence $\rightarrow \exists y \mathrm{T}(\mathbf{e}, a, y)$ in *SJNN*.

2.3.1 Let $\mathrm{q}_1(\mathfrak{P})$ be the figure obtained from $\mathfrak{P}$ by replacing every induction

$$\frac{A(a), \Gamma \rightarrow \Delta, A(a')}{A(0), \Gamma \rightarrow \Delta, A(t)}$$

occurring in $\mathfrak{P}$ by the following form:

$$\begin{array}{c}
\begin{array}{cc}
\begin{array}{c} \vdots \\ A(0) \rightarrow A(0) \\ \hline\hline A(0), \Gamma \rightarrow \Delta, A(0) \end{array}
&
\begin{array}{c} \vdots \\ A(a), \Gamma \rightarrow \Delta, A(a') \\ \hline\hline \Gamma \rightarrow \Delta, \forall x(A(x) \vdash A(x')) \end{array}
\end{array} \\ \hline
\begin{array}{cc}
A(0), \Gamma \rightarrow \Delta, A(0) \wedge \forall x(A(x) \vdash A(x')) & \begin{array}{c} \vdots \\ A(t) \rightarrow A(t) \end{array}
\end{array} \\ \hline\hline
A(0) \wedge \forall x(A(x) \vdash A(x')) \vdash A(t), A(0), \Gamma \rightarrow \Delta, A(t) \\ \hline\hline
\forall x e(x), A(0), \Gamma \rightarrow \Delta, A(t)
\end{array}$$

(where the double bars indicate that several inference schemata in S_1 (actually, inference schemata in LK) are applied), and then by making an obvious modification, so that $\mathrm{q}_1(\mathfrak{P})$ is a proof-figure in S_1 whose end-sequence is of the form

$$\forall x e(x) \rightarrow \exists y \mathrm{T}(\mathbf{e}, a, y) \quad \text{or} \quad \rightarrow \exists y \mathrm{T}(\mathbf{e}, a, y),$$

according as $\mathfrak{P}$ does or does not contain an induction.

2.3.2 Let F be a formula. By F^e we denote as in 7.1 of [**11**] the *restriction of F depending on the predicate e*, that is, the relativization of F to the predicate e. Let Γ be a list $A_1, \cdots, A_n$ of formulas. By Γ^e we denote the list $A_1^e, \cdots, A_n^e$. Let $\mathfrak{Q}$ be a proof-figure ending with $\Gamma \to \Delta$ in S_1. As is easily seen, we can construct proof-figures ending respectively with each of the following sequences in S_1:

(1) $\to e(0)$.

(2) $\to \exists x e(x)$.

(3) $\to \forall x(e(x) \leftarrow e(x'))$.

(4) $\to (\forall x e(x))^e$.

(5) $(e(0))^e, (\exists x e(x))^e, (\forall x(e(x) \leftarrow e(x')))^e, (\forall x e(x))^e, \Gamma^{*e} \to \Delta^e$,

where Γ^* is the result of deleting the formulas in the right sides of (1)-(3) and $\forall x e(x)$ from Γ, and where (5) is obtained from $\mathfrak{Q}$ by the process given in 7.6.1 of [**11**]. From the proof-figures ending with (1)-(5) we obtain a proof-figure ending with the sequence $\Gamma^{*e} \to \Delta^e$ in S_1. By $\mathrm{q}_2(\mathfrak{Q})$ we denote the proof-figure thus obtained from $\mathfrak{Q}$.

2.3.3 Let $\mathfrak{Q}$ be a proof-figure in S_1 ending with the sequence

$$\to \exists y(\mathbf{e}(y) \wedge \mathrm{T}(e, a, y)).$$

From this proof-figure we obtain a proof-figure ending with $\to \exists y \mathrm{T}(\mathbf{e}, a, y)$ of S_1. We will denote this operation by q_3.

2.3.4 Let $\mathfrak{Q}$ be a proof-figure in S_1, and $\mathrm{r}_1(\mathfrak{Q})$ be the proof-figure obtained from $\mathfrak{Q}$ by applying the reduction with respect to the grade of a proof-figure given in 9, Chapter 3 of [**21**], and modifying it so that every free t-variable except a is used as an eigenvariable and the eigenvariables are different from each other and a. Starting from $\mathrm{q}_3(\mathrm{q}_2(\mathrm{q}_1(\mathfrak{P})))$ and making a finite number of applications of r_1, we obtain a proof-figure of S_1 ending with the sequence $\to \exists y \mathrm{T}(\mathbf{e}, a, y)$[9].

2.4 *Arithmetization* (conclusion of the proof). We will complete the proof in the same way as in the proof of Theorem 1a. We will use the following abbreviations:

$P_0(b)$ for "b is the Gödel number of a proof-figure ending with $\to \exists y \mathrm{T}(\mathbf{e}, a, y)$ in *SJNN*".

---

9) Here we use the *grade* defined as follows: Let A be a quasi-formula of S_1. The *grade* of A (written $g(A)$) is defined to be

$$\omega^{g_1(A)} + g_2(A),$$

where $g_1(A)$ and $g_2(A)$ are the first and the second grades of A in 8.1 of Chapter 3 of [**21**], i.e.: If A is semi-isolated, then $g_1(A)$ is 0; otherwise $g_1(A)$ is $\max(g_1(B), g_1(C))+1$, or $g_1(B)+1$, or $g_1(B)$, according as A is of the form $B \wedge C$, or one of the forms $\neg B$, $\forall x B$ and $\forall \varphi B$, or $\{x_1, \cdots, x_n\}B$, respectively. $g_2(A)$ is the number of logical symbols contained in A. The *grade of a cut* $\mathfrak{J}(g(\mathfrak{J}))$ is the grade of the cut-formula, and the *grade of a proof-figure* $\mathfrak{P}(g(\mathfrak{P}))$ is $\Sigma_{\mathfrak{J}} g(\mathfrak{J})$, where Σ indicates natural sum and $\mathfrak{J}$ ranges over the cuts in $\mathfrak{P}$ such that the cut-formulas are not semi-isolated, if such exist; otherwise $g(\mathfrak{P})$ is defined to be 0.

$P(b)$ for " b is the Gödel number of a proof-figure of S_1 ".

$q_1(b)$ for the Gödel number of $\mathrm{q}_1(\mathfrak{P})$, provided that b is the Gödel number of a proof-figure $\mathfrak{P}$ and $P_0(b)$ holds.

$q_2(b)$ for the Gödel number of $\mathrm{q}_2(\mathfrak{P})$, provided that b is the Gödel number of a proof-figure $\mathfrak{P}$ and $P(b)$ holds.

$q_3(b)$ for the Gödel number of $\mathrm{q}_3(\mathfrak{P})$, provided that b is the Gödel number of a proof-figure $\mathfrak{P}$ of S_1 whose end-sequence is $\rightarrow \exists y(e(y) \wedge \mathrm{T}(\mathbf{e}, a, y))$.

$r_1(b)$ for the Gödel number of $\mathrm{r}_1(\mathfrak{P})$, provided that b is the Gödel number of a proof-figure $\mathfrak{P}$ and $P(b)$ holds.

Each of P_0, P, q_1, q_2, q_3 and r_1 can be chosen to be primitive recursive.

Let $\overset{1}{\prec}$ express the well-ordering of the set of the Gödel numbers of ordinals smaller than ω^ω, $o_1(b)$ the Gödel number of the grade of b, provided that $P(b)$ holds, and let " $a \mathrel{|\!\!\prec} b$ " be " $o_1(a) \overset{1}{\prec} o_1(b) \wedge P(a) \wedge P(b)$ ". Let $\theta(b)$ be the co-characteristic function of " $P(b) \wedge r_1(b) = b$ ". We define χ as follows:

$$\begin{cases} \chi(0) = 0 \\ \chi(b') = \theta(b') \cdot b' + \chi(\tau_1{}^*(b)), \end{cases}$$

where $\tau_1(a) = r_1(a')$, and $\tau_1{}^*(a) = \tau_1(a)$ if $\tau_1(a) \mathrel{|\!\!\prec} a'$; 0 otherwise. The function χ is a $\mathrel{|\!\!\prec}$-recursive function. Consider $\chi(q_3(q_2(q_1(b))))$, where $P_0(b)$ holds. This can be regarded as the Gödel number of the proof-figure $\mathfrak{P}_0$ in 2.0, where $\mathfrak{T}$ is *SMINN*, so that we can apply the reduction r for *SMINN* to this proof-figure. Thus, by the proof of Theorem 1a, $\phi(m)$ is $(\overset{\circ}{\prec}, \mathrel{|\!\!\prec})$-recursive and is given by $U(\phi(\sigma(m, \mathrm{a}; \chi(q_3(q_2(q_1(b)))))))$, where $P_0(b)$ holds and $\sigma(m, \mathrm{a}; q)$ is the primitive recursive function which gives the Gödel number of the proof-figure obtained from a proof-figure $\mathfrak{Q}$ in S_1 with the Gödel number q by substituting the numeral $\mathbf{m}$ for the free t-variable a throughout $\mathfrak{Q}$.

§ 3. Provable recursiveness of $\prec$-recursive functions.

Let $S(a)$ and $a \prec b$ be primitive recursive predicates such that $\prec$ is a well-ordering of $\{a : S(a)\}$, $\neg n' \prec 0$ for every natural number n and $b \prec a \rightarrow S(a) \wedge S(b)$.

DEFINITION 3a. Extending *NN* by adjoining a free predicate variable $\mathcal{E}$ of one argument-place, we formulate " transfinite induction on $\prec$ " $TI(\prec)$ by

$$\forall x(S(x) \wedge \forall y(y \prec x \vdash \mathcal{E}(y)) \vdash \mathcal{E}(x)) \vdash \forall x(S(x) \vdash \mathcal{E}(x))$$

(cf. [3]). We call $\prec$ a *provable primitive recursive well-ordering in NN*, if $TI(\prec)$ is provable in the system extended by addition of the predicate variable $\mathcal{E}$.

DEFINITION 3b. Let $\mathfrak{T}$ be a Gentzen-style theory of natural numbers formalized in G^1LC. " Transfinite inductionon $\prec$ " $TI(\prec)$ for $\mathfrak{T}$ is formulated by

$$\forall\varphi(\forall x(S(x) \wedge \forall y(y \prec x \vdash \varphi[y]) \vdash \varphi[x]) \vdash \forall x(S(x) \vdash \varphi[x])) .$$

We call $\prec$ a *provable primitive recursive well-ordering* in $\mathfrak{T}$, if $TI(\prec)$ is provable in $\mathfrak{T}$.

DIGRESSION. Let $\mathfrak{T}$ be a Gentzen-style theory of natural numbers. For each formula $Q(a)$ of $\mathfrak{T}$, let $TI(Q, \prec)$ be the formula

$$\forall x(S(x) \wedge \forall y(y \prec x \vdash Q(y)) \vdash Q(x)) \vdash \forall x(S(x) \vdash Q(x)) .$$

If $\mathfrak{T}$ is NN (extended by addition of the predicate variable $\mathcal{E}$), or full analysis (i.e. the Gentzen-style theory of natural numbers formalized in G^1LC without any restriction on $\forall$ left on an f-variable), then $TI(Q, \prec)$ is provable in $\mathfrak{T}$ for every formula $Q(a)$ and for every provable primitive recursive well-ordering well-ordering $\prec$ in $\mathfrak{T}$. This is also true in the case where $\mathfrak{T}$ is one of RNN, f-CNN, ID and $SMINN$, since in such a $\mathfrak{T}$ the inference $\forall$ left on an f-variable

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall\varphi F(\varphi), \Gamma \rightarrow \Delta} \qquad (*)$$

has no restriction on V, and $TI(\prec)$ is of the form $\forall\varphi F(\varphi)$ where $F(\alpha)$ contains no bound or free f-variable other than α (cf. 1.2.3, 1.3.3, 1.4.3 and 1.6.3). However it should be noticed that this is presumably not true for subsystems of analysis in general, e.g. $SINN$ in which there is a restriction on V in (*).

Let $\mathfrak{T}$ be NN (1.1), or a Gentzen-style theory of natural numbers satisfying the following condition: The inference $\forall$ left on an f-variable of the following form is available in this theory:

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall\varphi F(\varphi), \Gamma \rightarrow \Delta}$$

where V is arithmetical (that is, contains no free or bound f-variables) and $F(\alpha)$ is arithmetical in α. For example, $\mathfrak{T}$ can be any of the systems given in 1.2–1.8.

THEOREM 2. *If $\prec$ is a provable primitive well-ordering in $\mathfrak{T}$, than every $\prec$-recursive function is provably recursive in $\mathfrak{T}$.*

PROOF. Let $\prec$ be a provable primitive recursive well-ordering in $\mathfrak{T}$ and ϕ a $\prec$-recursive function. We will prove the theorem by mathematical induction on the number of steps required to construct ϕ. We follow the Gödel numbering of Kleene [**5**, §§ 50–56]. Since there is no danger of confusion, we will use Kleene's notation in our formal theory. We consider here only the case where the last schema used to define ϕ is (VI), since the other cases are easy. For simplicity, let ϕ be defined as follows:

$$(0) \qquad \begin{cases} \phi(0) = 0 \\ \phi(a') = \psi(a, \phi(\tau^*(a))) . \end{cases}$$

By the hypothesis of induction, we can find systems of equations G and H with the respective Gödel numbers g and h such that G and H define recursively ψ and τ respectively, and the following sequences are provable in $\mathfrak{T}$.

$$\begin{aligned}&\rightarrow \forall x \forall y \exists z T_2(\mathbf{g}, x, y, z)\\ &\rightarrow \forall x \exists y T(\mathbf{h}, x, y)\,.\end{aligned} \tag{1}$$

Let E_0 be the system of equations obtained by translating "$\tau^*(a)=0$" and (0) using $\boldsymbol{g}, \boldsymbol{h}, \boldsymbol{f}, \boldsymbol{a}$, for "$\psi$", "$\tau^*$", "$\phi$", "$a$", respectively, where $\boldsymbol{g}$ and $\boldsymbol{h}$ are the principal function letters of G and H, respectively. Let e be the Gödel number of the system G, H, E_0. (We assume that the principal and auxiliary function letters are properly chosen.) Then the following sequences are provable in $\mathfrak{T}$.

$$\rightarrow \exists y T(\mathbf{e}, 0, y)\,. \tag{2}$$

$$T_2(\mathbf{g}, a, 0, b),\ T(\mathbf{h}, a, c),\ \neg U(c) \prec a' \rightarrow \exists y T(\mathbf{e}, a', y)\,. \tag{3}$$

$$\forall x \forall y \exists z T_2(\mathbf{g}, x, y, z),\ T(\mathbf{h}, a, c),\ \neg U(c) \prec a' \rightarrow \exists y T(\mathbf{e}, a', y) \quad \text{(from (3))}. \tag{4}$$

$$\forall x \forall y \exists z T_2(\mathbf{g}, x, y, z),\ \forall x \exists y T(\mathbf{h}, x, y),\ \neg S(a) \rightarrow \exists y T(\mathbf{e}, a, y) \tag{5}$$

(by (2), (4), and a mathematical beginning sequence $b \prec a \rightarrow S(a)$ (cf. the beginning of this section)).

$$T(\mathbf{e}, U(c), d),\ T_2(\mathbf{g}, a, U(d), b),\ T(\mathbf{h}, a, c),\ U(c) \prec a' \rightarrow \exists y T(\mathbf{e}, a', y)\,. \tag{6}$$

$$\begin{aligned}&\forall x(x \prec a' \vdash \exists y T(\mathbf{e}, x, y)),\ \forall x \forall y \exists z T_2(\mathbf{g}, x, y, z)\,,\\ &\qquad T(\mathbf{h}, a, c),\ U(c) \prec a' \rightarrow \exists y T(\mathbf{e}, a', y)\end{aligned} \tag{7}$$

(from (6)).

$$\begin{aligned}&\forall x(x \prec a' \vdash \exists y T(\mathbf{e}, x, y)),\ \forall x \forall y \exists z T_2(\mathbf{g}, x, y, z)\,.\\ &\qquad \forall x \exists y T(\mathbf{h}, x, y) \rightarrow \exists y T(\mathbf{e}, a', y)\end{aligned} \tag{8}$$

(by (4) and (7)).

$$\forall x \forall y \exists z T_2(\mathbf{g}, x, y, z),\ \forall x \exists y T(\mathbf{h}, x, y),\ S(a) \rightarrow \exists y T(\mathbf{e}, a, y) \tag{9}$$

(from (2), (8), and the hypothesis that "$\prec$" is a provable primitive recursive well-ordering in $\mathfrak{T}$). Then from (1), (5), and (9), the sequence $\rightarrow \forall x \exists y T(\mathbf{e}, x, y)$ is provable in $\mathfrak{T}$, which completes the proof.

§ 4. Alternative consistency-proof of *SJNN*.

The consistency of *SJNN* was proved in [**21**] by using transfinite induction on ordinal diagrams of finite order with the help of the restriction theory developed in § 7, [**11**]. In this section we will give an alternative proof of the consistency of *SJNN* along the line of the consistency-proof of S_1 given in

[**21**]. We will sketch a proof of the theorem of [**20**] for *SJNN*, which will be seen to hold in a somewhat weakened form owing to the use of a larger system of ordinals.

LEMMA 1. *The system SMINN is consistent.*

PROOF. The consistency of *SMINN* easily follows from the cut-elimination theorem for semi-isolated proof-figures (Theorem 1 of [**21**], Chapter 3]), following the consistency-proof of *SINN* (Theorem 1 of [**21**], Chapter 2]) by the following addition to 5.1-5.7 in Chapter 3 of [**21**]:

(1) If S_1 and S_2 are the upper sequence and the lower sequence of an induction, then the o.d. of S_2 is $(n+2;\ a+2, \sigma)$, where σ is the o.d. of S_1 and a is the number of logical symbols in the principal formula of the induction.

(2) If S_1 and S_2 are the upper sequence and the lower sequence of an inference "term-replacement"[10], then the o.d. of S_2 is equal to that of S_1.

THEOREM 3. *The system SJNN is consistent.* (Theorem 4 of Chapter 3 of [**21**]).

PROOF. (Alternative) Regarding *SMINN* and *SJNN* as S_2 and S_1 in the consistency-proof of S_1 (Theorem 3 of [**21**, Chapter 3]), respectively, and adjoining the following statement to 9.2[11] there, we have the consistency-proof of *SJNN*: If $\mathfrak{J}_1$ is an "induction" of the form

$$\frac{A(a),\ \Pi_2 \rightarrow \Lambda_2,\ A(a')}{A(0),\ \Pi_2 \rightarrow \Lambda_2,\ A(t)},$$

then since the principal formula of an induction of *SJNN* is semi-isolated, neither $A(a)$ nor $A(0)$ is equivalent to the formula B which is not semi-isolated. By our assumption, the proof-figure $\mathfrak{Q}_1$ ending with the sequence

$$A(a),\ \Pi_2^*,\ \Gamma \rightarrow \Delta,\ \Lambda_2,\ A(a')$$

is defined. Then we construct the proof-figure:

10) A term-replacement is the inference schema "Termeinsetzung" in [**3**]:

$$\frac{\Gamma_1,\ A(s),\ \Gamma_2 \rightarrow \Delta}{\Gamma_1,\ A(t),\ \Gamma_2 \rightarrow \Delta} \quad \text{or} \quad \frac{\Gamma \rightarrow \Delta_1,\ A(s),\ \Delta_2}{\Gamma \rightarrow \Delta_1,\ A(t),\ \Delta_2}$$

where s and t are terms which do not contain any free t-variables and stand for the same numeral; cf. 8 of Chapter 2 of [**21**].

11) The case where the uppermost cut $\mathfrak{J}$ with the maximal grade of the cuts whose cut-formulas are not semi-isolated is of the following form:

$$\frac{\Gamma \rightarrow \Delta,\ \forall\varphi F(\varphi) \quad \forall\varphi F(\varphi),\ \Pi \rightarrow \Lambda}{\Gamma,\ \Pi \rightarrow \Delta,\ \Lambda}.$$

Let B be the right cut-formula of $\mathfrak{J}$, and $\Pi_1 \rightarrow \Lambda_1$ the right upper sequence of $\mathfrak{J}$ or an arbitrary sequence above. We construct a proof-figure ending with $\Pi^*_1, \Gamma \rightarrow \Delta, \Lambda_1$, where Π^*_1 is obtained from Π_1 by deleting the formulas equivalent to B. Let $\mathfrak{J}_1$ be the inference whose lower sequence is $\Pi_1 \rightarrow \Lambda_1$, and assume that the proof-figure $\mathfrak{Q}_1$ corresponding to the upper sequence of $\mathfrak{J}_1$ has been defined.

$$\begin{array}{c} \mathfrak{Q}_1 \\ \vdots \\ \dfrac{A(a), \Pi_2^*, \Gamma \to \Delta, \Lambda_2, A(a')}{A(0), \Pi_2^*, \Gamma \to \Delta, \Lambda_2, A(t)} \end{array};$$

Thus we complete the proof.

Takeuti remarked in [**20**] that Gentzen's results of [**3**] can be stated in a more general form, and in [**20**] and in [**21**] he proved the following theorem: *The order-type of any provable recursive well-ordering in* $\mathfrak{T}$ *is less than* $|O(\mathfrak{T})|$, where $\mathfrak{T}$ is any of the systems 1.1–1.5, and 1.8 (and naturally 1.6). We will prove this theorem for *SJNN* in the following form:

THEOREM 4. *The order-type of any provable recursive well-ordering in SJNN is less than the order-type of* $(0, \omega, (0, 0, 0))$ *in* $O(n, \omega+1)$, *where* n *is a positive integer.*

PROOF. We begin the proof of Theorem 4 by recalling the definition of a "*TJ*-proof-figure" from [**20**].

4.1 *TJ-proof-figure. A TJ-proof-figure with respect to SJNN* (or simply, a *TJ*-proof-figure) is defined to be a figure which is obtained from a proof-figure of *SJNN* by modifying it as follows:

4.1.1 The beginninng sequences of *SJNN* and the sequences of the following form called *TJ*-upper sequences (*TJ*-Obersequenzen), are allowed as beginning sequences:

$$S(t), \forall x(x \prec t \vdash \mathcal{E}[x]) \to \mathcal{E}[t],$$

where $S(a)$ and $a \prec b$ are (primitive) recursive predicates such that $\prec$ is a well-ordering of $\{a : S(a)\}$, $\neg n' \prec 0$ for every natural number n and $b \prec a \to S(a) \wedge S(b)$, t is an arbitrary term, and $\mathcal{E}$ is a free f-variable.

4.1.2 The inference schema "term-replacement" is added[10].

4.1.3 The end-sequence is of the form

$$\to \mathcal{E}[\mathbf{s}_1], \cdots, \mathcal{E}[\mathbf{s}_n],$$

where $\mathbf{s}_1, \cdots, \mathbf{s}_n$ are numerals.

4.2 *TJ-proof-figure of order n.* We define a *TJ*-proof-figure with degree and *TJ*-proof-figure of order n in the same way as in the consistency-proof of *SMINN*, where a *TJ*-proof-figure with degree satisfies the conditions 2.1–2.2, Chapter 3 of [**21**] and also

4.2.1 Every implicit $\forall$ left on an f-variable is restricted by the condition that the principal formula is semi-isolated. We will assign an ordinal diagram of order $n+2$ (o.d.) to every sequence in a *TJ*-proof-figure of order n in the same way as in the consistency-proof of *SMINN* with the following additional statement:

4.2.2 The o.d of a *TJ*-upper sequence is

$$(n+2, 0, (n+2, 0, (n+2, 0, (n+2, 0, 0\#(n+2, 0, 0))))),$$

where the o.d. of a beginning sequence of *SJNN* is 0, (cf. § 3 of [3]).

4.3 *Grade of a TJ-proof-figure.* Let A be a quasi-formula of *SJNN*. The *grade* of A (written as $g'(A)$) is defined to be $(0, \omega, 0^{(g_1(A))}\#0^{(g_2(A))}$ where $0^{(i)}$ is defined by $0^{(0)}=0$ and $0^{(i+1)}=0^{(i)}\#0$, (cf. Footnote 9 for $g_1(A)$ and $g_2(A)$). Let the *grade of a cut* $\mathfrak{J}$ in a *TJ*-proof-figure ($g'(\mathfrak{J})$) be the grade of the cut-formula. The *grade of a TJ-proof-figure* $\mathfrak{P}(g'(\mathfrak{P}))$ is taken to be $g'(\mathfrak{J}_1)\#\cdots\#g'(\mathfrak{J}_m)$, where $\mathfrak{J}_1, \cdots, \mathfrak{J}_m$ are all the cuts in $\mathfrak{P}$ with cut-formulas that are not semi-isolated, if such exist; otherwise $g'(\mathfrak{P})$ is defined to be 0.

4.4 *Ordinal diagram of a TJ-proof-figure.* (Conclusion of the proof.) Let $\mathfrak{P}$ be a *TJ*-proof-figure. The ordinal diagram of $\mathfrak{P}$ is defined to be $g'(\mathfrak{P})$ if there exists a cut in $\mathfrak{P}$ whose cut-formula is not semi-isolated; otherwise the ordinal diagram of $\mathfrak{P}$ is defined to be the ordinal diagram of $\mathfrak{P}$ regarded as a *TJ*-proof-figure of order n. Modifying the proof of Theorem 3 along the line of the proof of the theorem in [20] for the system $\mathfrak{S}_1$, we can complete the proof of Theorem 4.

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