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On the exponential decay of solutions for some partial differential equations

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In [1] C. Morawetz has shown, using the energy integrals (the a-b-c method initiated by K. Friedrichs), that a solution of the wave equation

$$u_{tt}-\Delta u+\alpha u=0\,, \qquad \alpha>0\,,$$

which vanishes in the forward light cone $|x|<t$ $(t>0)$ and has finite energy

$$\int_{t=0}(|\nabla u|^{2}+u_{t}^{2}+\alpha u^{2})dv<\infty\,,$$

vanishes identically.

The purpose of the present note is to show that the above result can be extended in the following sharpened form to the case of partial differential equations of more general type: Let u be a solution for the evolution equations of certain types, e. g. the Klein-Gordon wave equation or the Schrödinger wave equation. If this solution u decays exponentially with time on any compact set, then the solution u must vanish identically. The author wishes to express his hearty thanks to Professor K. Yosida who kindly pointed out mistakes in the manuscript and gave him many valuable advices, and also to Professor H. Fujita for his many valuable suggestions.

§ 1. Notations and results.

We denote by G a whole space E^{n} or the exteriors (or interiors) of bounded $(n-1)$-dimensional hypersurfaces which are locally of class C^{4}. Let $x=(x_{1}, \cdots, x_{n})$ be the generic point of E^{n} whose length $\sqrt{x_{1}^{2}+\cdots+x_{n}^{2}}$ is denoted by $|x|$, $D^{\alpha}=D_{1}^{\alpha_{1}}\cdots D_{n}^{\alpha_{n}}$ be a general derivative where $D_{j}=\dfrac{\partial}{\partial x_{j}}$ and α stands for the multi-index $\alpha=(\alpha_{1}, \cdots, \alpha_{n})$ whose length $\alpha_{1}+\cdots+\alpha_{n}$ is also denoted by $|\alpha|$. Let $\|\cdot\|$ be the norm of $L^{2}(G)$, $(\cdot, \cdot)$ be the scalar product in $L^{2}(G)$ and $\|\cdot\|_{L^{2}(K)}$ be the norm of $L^{2}(K)$. Let H_{j} be the totality of those complex-valued functions v in $L^{2}(G)$ for which the distribution derivatives $D^{\alpha}v$ also lie in $L^{2}(G)$ for $|\alpha|\leqq j$. Then H_{j} is a Hilbert space under the norm

$$\|v\|_{j}=\{\sum_{|\alpha|\leqq j}\|D^{\alpha}v\|^{2}\}^{\frac{1}{2}}\,.$$

Let $\overset{\circ}{H}_j$ be the completion of $C_0^\infty(G)$ under the norm of H_j, $B^j(\bar{G})$ be the totality of those functions in $C^j(\bar{G})$ for which the derivatives $D^\alpha v$ are bounded continuous on $\bar{G}$ for $|\alpha|\leqq j$, where $\bar{G}$ means the closure of G.

Let us consider the equation

(1) $$P\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u = a_{00}(x)\frac{\partial^2 u}{\partial t^2}+2\sum_{j=1}^n a_{0j}(x)\frac{\partial^2 u}{\partial x_j\partial t}+a_0(x)\frac{\partial u}{\partial t}$$
$$-\sum_{i,j=1}^n a_{ij}(x)\frac{\partial^2 u}{\partial x_i\partial x_j}+\sum_{j=1}^n a_j(x)\frac{\partial u}{\partial x_j}+a(x)u=0$$

satisfying conditions:

(2) $a_{ij}=a_{ji}\in B^2(\bar{G})$, $a_j\in B^1(\bar{G})$, $a\in B^0(\bar{G})$, (for $i, j=0, 1, \cdots, n$).

There exists a positive constant $\delta_0>0$ such that

(3) $$|\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j|\geqq \delta_0|\xi|^2$$

for any $x\in\bar{G}$ and any $\xi=(\xi_1, \cdots, \xi_n)\in E^n$.

(4) If $n=2$, $a_{ij}(x)$ is real-valued for $i, j=1, 2$.

Let us also consider operators $A(z)$ and $A'(z)$ in $L^2(G)$, defined as follows:

$\mathscr{D}(A(z))=\{v\,;\ v\in H_2,\ v=0$ on the boundary of $G\}$; and

$$A(z)v=P\Big(\frac{\partial}{\partial x}, -iz\Big)v=-\sum_{i,j=1}^n a_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}+\sum_{j=1}^n a_j\frac{\partial v}{\partial x_j}+av$$
$$+(iz)^2a_{00}v-2iz\sum_{j=1}^n a_{0j}\frac{\partial v}{\partial x_j}-iza_0v\,,$$

$\mathscr{D}(A'(z))=\{v\,;\ v\in H_2,\ v=0$ on the boundary of $G\}$; and

$$A'(z)v=-\sum_{i,j=1}^n\frac{\partial^2}{\partial x_i\partial x_j}(\bar{a}_{ij}v)-\sum_{j=1}^n\frac{\partial}{\partial x_j}(\bar{a}_jv)+\bar{a}v$$
$$+(\overline{iz})^2a_{00}v+2\overline{iz}\sum_{j=1}^n\frac{\partial}{\partial x_j}(a_{0j}v)-\overline{iz}\bar{a}v\,.$$

Here $\mathscr{D}(B)$ denotes the domain of an operator B, and $\bar{b}$ the complex conjugate of b. We shall summarize some properties of $A(z)$.

(5) $$\mathscr{D}(A(z))=\mathscr{D}(A'(z))=H_2\cap\overset{\circ}{H}_1\,.$$

(6) $A'(z)^*=A(z)$, where $A'(z)^*$ is the adjoint operator of $A'(z)$ in $L^2(G)$.

There exists a positive constant c_1, depending on z, such that

(7) $$\|v\|_2\leqq c_1\{\|A(z)v\|+\|v\|\}\qquad\text{for}\quad v\in\mathscr{D}(A(z))\,.$$

The reader will find the proof in e.g. F. Browder [2], [3].

We shall prove the following

THEOREM. *Assume that there exist a complex number* z_0 *with* $\operatorname{Im} z_0\geqq 0$ *and a curve* γ *connecting* z_0 *and* $-z_0$ *such that for any* z *on* γ *the resolvent set of*

$A(z)$ contains the origin. Let u be a solution of (1) *with the zero Dirichlet boundary condition, such that for some positive constant c_2 and for any t in* $(-\infty, \infty)$,

(8) $$\|u(\cdot, t)\| \leqq c_2 e^{\mathrm{Im}\, z_0 \cdot |t|}\,.$$

Let U be a non-empty (not necessarily bounded) open subset of G. If u satisfies the following condition, then u must vanish in $U\times(-\infty, \infty)$: For any compact set K in U, there exist positive constants $c_3 = c_3(K)$ and $\varepsilon = \varepsilon(K)$ such that

(9) $$\| u(\cdot, t)\|_{L^2(K)} \leqq c_3 e^{-(\mathrm{Im}\, z_0 + \varepsilon)t}\,, \qquad t \geqq 0\,.$$

Here and in the following, $\eta = \eta(K)$ means that η depends on K.

By a solution of (1) with the zero Dirichlet boundary condition, we mean a function u with the following properties.

(i) $u(\cdot, t)$ takes values in $L^2(G)$ and is continuously differentiable for $-\infty < t < \infty$ in the norm of $L^2(G)$.

(ii) $u(\cdot, t)$ takes values in $\mathring{H}_1$ and is continuous for $-\infty < t < \infty$ in the norm of H_1.

(iii) $u(\cdot, t)$ satisfies the variational equation:

$$\int_{t_1}^{t_2}\int_G \Big[-a_{00}\frac{\partial u}{\partial t}\frac{\partial \Phi}{\partial t} - 2\sum_{j=1}^{n}\frac{\partial u}{\partial t}\frac{\partial(a_{0j}\Phi)}{\partial x_j} + a_0\frac{\partial u}{\partial t}\Phi$$
$$+\sum_{j=1}^{n}\frac{\partial u}{\partial x_j}\Big(\frac{\partial}{\partial x_i}(a_{ij}\Phi)\Big) + \sum_{j=1}^{n} a_j\frac{\partial u}{\partial x_j}\Phi + a\Phi\Big]dxdt$$
$$+\int_G a_{00}\frac{\partial u(x, t_2)}{\partial t}\Phi(x, t_2)dx - \int_G a_{00}\frac{\partial u(x, t_1)}{\partial t}\Phi(x, t_1)dx = 0$$

for any t_1, t_2 with $-\infty < t_1 < t_2 < \infty$ and any $\Phi \in \mathring{H}_1(G\times(-\infty, \infty))$.

Of special interest are the following equations:

(10) $$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + a(x)u = 0, \text{ where } \inf a(x) > 0\,.$$

(11) $$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + a(x)u = 0\,.$$

(12) $$\frac{1}{i}\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u}{\partial x_j}\Big) + a(x)u = 0.$$

Let us assume that the coefficients $a_{ij}(x)$ and $a(x)$ in (10), (11) and (12) are real-valued, then we have a series of corollaries.

COROLLARY 1. *Let u be a solution of the Klein-Gordon wave equation* (10) *with the zero Dirichlet boundary condition, which has finite energy*

(13) $$\Big[\int_G\Big(\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j} + \Big(\frac{\partial u}{\partial t}\Big)^2 + au^2\Big)dx\Big]_{t=0} < \infty\,.$$

Let U be a non-empty (not necessarily bounded) open subset of G. If u satisfies the following condition, then u must vanish in $U\times(-\infty, \infty)$: For any compact set K in U, there exist positive constants $c_4=c_4(K)$ and $\varepsilon=\varepsilon(K)$ such that

$$\|u(\cdot, t)\|_{L^2(K)} \leqq c_4 e^{-\varepsilon t}, \qquad t \geqq 0.$$

PROOF. Let us take $z_0=0$ and a curve γ shrinking to the origin. By the definition and (5), we have $\mathcal{D}(A(0))=H_2 \cap \overset{\circ}{H}_1$,

$$A(0)v=-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial v}{\partial x_j}\right)+a(x)v \qquad \text{for} \quad v\in \mathcal{D}(A(0))$$

and $\inf a(x)>0$, so that $A(0)$ is a positive self-adjoint operator in $L^2(G)$, (see (6) and (7)). Hence the resolvent set of $A(0)$ contains the origin. By (13), it follows from the conservation of energy that

$$\sup_{-\infty<t<\infty}\int_G \Big[\sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}+\Big(\frac{\partial u}{\partial t}\Big)^2+au^2\Big]dx<\infty,$$

so that $\sup\limits_{-\infty<t<\infty}\|u(\cdot, t)\|<\infty$ in view of $\inf a(x)>0$. Hence u satisfies the estimate (8) with $z_0=0$. Moreover, by the assumption, u satisfies the estimate (9) with $z_0=0$. Thus, applying the theorem to the present case, we obtain Corollary 1.

COROLLARY 2. *Let u be a solution of* (10) *with the zero Dirichlet boundary condition which has finite energy* (13). *If there exists a monotone increasing function $\psi(s)$ on $[0, \infty)$ such that u vanishes in the interior of the forward cone $\{(x, t);\ \psi(|x|)<t\ (t>0)\}$, then u vanishes identically.*

PROOF. Corollary 2 is an immediate consequence of Corollary 1.

REMARK 1. The result obtained by C. Morawetz [1], is a special case of Corollary 2.

COROLLARY 3. *Let u be a solution of the heat equation* (11) *with the zero Dirichlet boundary condition such that for some positive constants c_5 and β and for any t in $(-\infty, \infty)$*

$$\|u(\cdot, t)\|\leqq c_5 e^{\beta|t|}.$$

Let U be a non-empty (not necessarily bounded) open subset of G. If u satisfies the following condition, then u must vanish in $U\times(-\infty, \infty)$: For any compact set K in U, there exist positive constants $c_6=c_6(K)$ and $\varepsilon=\varepsilon(K)$ such that

$$\|u(\cdot, t)\|_{L^2(K)}\leqq c_6 e^{-(\beta+\varepsilon)t}, \qquad t\geqq 0.$$

PROOF. In order to apply the theorem to the present case, we have only to show that there exist a complex number z_0 with $\operatorname{Im} z_0=\beta$ (>0) and a curve γ connecting z_0 and $-z_0$, such that for z on γ the resolvent set of $A(z)$ contains the origin. The resolvent set of $A(z)$ contains the origin if and only if the resolvent set of $A(0)$ contains iz. Moreover, the spectral set of $A(0)$

is contained in $[r, \infty)$ for some real r. Hence it is clear that there exist a z_0 and a curve γ with required properties.

COROLLARY 4. *Let u be a solution of the Schrödinger wave equation* (12) *with the zero Dirichlet boundary condition. Let U be a non-empty (not necessarily bounded) open subset of G. If u satisfies the following condition, then u must vanish in $U\times(-\infty, \infty)$: For any compact set K in U, there exist positive constants $c_7=c_7(K)$ and $\varepsilon=\varepsilon(K)$ such that*

$$\|u(\cdot, t)\|_{L^2(K)} \leqq c_7 e^{-\varepsilon t}, \qquad t \geqq 0. \tag{14}$$

PROOF. Let $u_\lambda = e^{-i\lambda t}u$ for a real number λ, then u_λ is a solution of the equation

$$P\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\Big)u_\lambda = \frac{1}{i}\frac{\partial u_\lambda}{\partial t} - \sum_{j=1}^{n}\frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial u_\lambda}{\partial x_j}\Big) + au_\lambda + \lambda u_\lambda = 0.$$

Hence $\|u_\lambda(\cdot, t)\|$ is uniformly bounded in t. Moreover, u_λ decays in the same manner as the solution u (see (14)). Let us define $A_\lambda(z)$, corresponding to the above equation, in the same way as before. Then, by the definition, $\mathcal{D}(A_\lambda(0)) = H_2 \cap \overset{\circ}{H}_1$ and $A_\lambda(0)v = P\Big(\frac{\partial}{\partial x}, 0\Big)v$. Hence the resolvent set of $A_\lambda(0)$ contains the origin for large λ. Therefore, by the same reason as in Corollary 1, u_λ vanishes in $U\times(-\infty, \infty)$. Thus u vanishes in $U\times(-\infty, \infty)$.

§ 2. Some lemmas for the proof of the theorem.

LEMMA 1. *Let us assume that for z_1 on γ the resolvent set of $A(z_1)$ contains the origin. Then there exists a neighborhood Δ of γ such that for z in Δ the resolvent set of $A(z)$ contains the origin and $A(z)^{-1}$ is holomorphic in Δ.*

PROOF. If $v \in \mathcal{D}(A(z)) = H_2 \cap \overset{\circ}{H}_1$, we have

$$A(z)v - A(z_1)v = ((iz)^2 - (iz_1)^2)a_{00}v - 2(iz - iz_1)\sum_{j=1}^{n} a_{0j}\frac{\partial v}{\partial x_j} - (iz - iz_1)a_0 v, \tag{15}$$

so that

$$\begin{aligned}\|[A(z)-A(z_1)]v\| \leqq\; & |z^2 - z_1^2|(\sup|a_{00}(x)|)\|v\| \\ & + 2|z - z_1|\sum_{j=1}^{n}(\sup|a_{0j}(x)|)\Big\|\frac{\partial v}{\partial x_j}\Big\| \\ & + |z - z_1|(\sup|a_0(x)|)\|v\|.\end{aligned}$$

Hence, by putting $c_8 = 2\max\{\sup_{x\in G}|a_{00}(x)|, \sup_{x\in G}|a_{0j}(x)|, \sup_{x\in G}|a_0(x)|\}$ and $c_9 = \|A(z_1)^{-1}\|$, we obtain, for $z_1 \in \gamma$,

$$\begin{aligned}\|[A(z)-A(z_1)]v\| & \leqq |z - z_1|(|z + z_1| + 3)c_8\|v\|_2 \\ & \leqq |z - z_1|(|z + z_1| + 3)c_8 c_1(\|A(z_1)v\| + \|v\|), \qquad \text{(from (7))}, \\ & \leqq |z - z_1|(|z + z_1| + 3)c_8 c_1(1 + c_9)\|A(z_1)v\|,\end{aligned}$$

where c_1 and c_9 depend on $z_1 \in \gamma$. Let us take a $\delta(z_1)$ such that

$$|z-z_1|(|z+z_1|+3)c_8c_1(1+c_9)<\frac{1}{2} \text{ for all } z \text{ in } |z-z_1|<\delta(z_1),$$

then we have, for each $z_1 \in \gamma$ and for any z in $|z-z_1|<\delta(z_1)$,

$$\|[A(z)-A(z_1)]v\|<\frac{1}{2}\|A(z_1)v\|.$$

Since γ is a compact set, there exists a finite covering $\varDelta=\bigcup_{j=1}^{N}\{z\,;\,|z-z_j^*|<\delta(z_j^*),\ z_j^*\in\gamma\}$ of γ such that

$$\|[A(z)-A(z_j^*)]v\|<\frac{1}{2}\|A(z_j^*)v\|$$

for any z in $\varDelta$ and some z_j^*. Hence, we see that for any z in $\varDelta$, $A(z)^{-1}$ exists as a bounded operator in $L^2(G)$ and

$$\|A(z)^{-1}\|\leqq 2\|A(z_j^*)^{-1}\| \qquad \text{for some } z_j^*. \tag{16}$$

From (15), we have, for $v\in\mathcal{D}(A(z))$ and any z_1 in $\varDelta$,

$$\left\|\frac{A(z)-A(z_1)}{z-z_1}v-2iz_1a_{00}v+2i\sum_{j=1}^{n}a_{0j}\frac{\partial v}{\partial x_j}+ia_0v\right\|\leqq|z-z_1|(\sup|a_{00}(x)|)\|v\|. \tag{17}$$

Since the right hand side of (17) tends to zero as $z\to z_1$,

$$s\text{-}\lim_{z\to z_1}\frac{A(z)-A(z_1)}{z-z_1}v \text{ exists for } v\in\mathcal{D}(A(z_1))=H_2\cap\mathring{H}_1.$$

Thus

$$s\text{-}\lim_{z\to z_1}\frac{A(z)-A(z_1)}{z-z_1}A(z_1)^{-1}w \text{ exists for any } z_1 \text{ in } \varDelta \text{ and any } w \text{ in } L^2(G). \tag{18}$$

We see, by (18), that

$$s\text{-}\lim_{z\to z_1}(A(z)A(z_1)^{-1}w-w)=0. \tag{19}$$

By (16) and (19), we have

$$s\text{-}\lim_{z\to z_1}(A(z)^{-1}w-A(z_1)^{-1}w)=-s\text{-}\lim_{z\to z_1}A(z)^{-1}(A(z)A(z_1)^{-1}w-w)=0 \tag{20}$$

for any z_1 in $\varDelta$ and any w in $L^2(G)$. From (18) and (20), it follows that

$$w\text{-}\lim_{z\to z_1}\frac{A(z)^{-1}-A(z_1)^{-1}}{z-z_1}w=-w\text{-}\lim_{z\to z_1}A(z_1)^{-1}\frac{A(z)-A(z_1)}{z-z_1}A(z_1)^{-1}w$$
$$-w\text{-}\lim_{z\to z_1}[A(z)^{-1}-A(z_1)^{-1}]\frac{A(z)-A(z_1)}{z-z_1}A(z_1)^{-1}w$$

exists for any z_1 in $\varDelta$ and any w in $L^2(G)$, proving that $A(z)^{-1}$ is holomorphic in $\varDelta$.

LEMMA 2. *Let us define the Fourier-Laplace transforms of $u(x, t)$ satisfying the conditions in the theorem, as follows:*

$$\hat{u}^+(z)=\hat{u}^+(x,\, z)=\int_0^\infty e^{-izt}u(x,\, t)dt\,,$$

$$\hat{u}^-(z)=\hat{u}^-(x,\, z)=\int_{-\infty}^0 e^{izt}u(x,\, t)dt\,.$$

We have

(a) $\hat{u}^+(z)$ *and* $\hat{u}^-(z)$ *belong to* $L^2(G)$ *for* $\operatorname{Im} z < -\operatorname{Im} z_0$,

(b) *for any* φ *in* $C_0^\infty(U)$,

$(\hat{u}^-(z),\, \varphi)=\int_{-\infty}^0 e^{izt}(u,\, \varphi)dt$ *is holomorphic in the lower half-plane* $\{z\,;\ \operatorname{Im} z< -\operatorname{Im} z_0\}$ *and* $(\hat{u}^+(z),\, \varphi)=\int_0^\infty e^{-izt}(u,\, \varphi)dt$ *is holomorphic in the lower half-plane* $\{z\,;\ \operatorname{Im} z<\operatorname{Im} z_0+\varepsilon\}$ *for some positive constant* $\varepsilon=\varepsilon(\varphi)$,

(c) $\hat{u}^+(z)\in \mathcal{D}(A(-z))$, $A(-z)\hat{u}^+(z)=f(-z)$,

and

$$\hat{u}^-(z)\in \mathcal{D}(A(z)),\ A(z)\hat{u}^-(z)=-f(z)$$

for z *with* $\operatorname{Im} z< -\operatorname{Im} z_0$ *and for the function* $f(z)$ *defined by*

$$f(z)=f(x,\, z)=a_{00}(x)\frac{\partial u(x,\, 0)}{\partial t}$$
$$+2\sum_{j=1}^n a_{0j}(x)\frac{\partial u(x,\, 0)}{\partial x_j}-iza_{00}(x)u(x,\, 0)+a_0(x)u(x,\, 0)\,.$$

PROOF. By (8), we have

$$\| e^{-izt}u(\cdot,\, t)\| \leqq e^{\operatorname{Im} z\cdot t}\| u(\cdot,\, t)\| \leqq c_2 e^{\operatorname{Im} z\cdot t}e^{\operatorname{Im} z_0\cdot |t|}\,,$$

so that

$$\int_0^\infty \| e^{-izt}u(\cdot,\, t)\| dt<\infty \qquad \text{for} \quad \operatorname{Im} z< -\operatorname{Im} z_0\,,$$

which shows that $\hat{u}^+(z)\in L^2(G)$ for $\operatorname{Im} z< -\operatorname{Im} z_0$. Similarly we have $\hat{u}^-(z)\in L^2(G)$ for $\operatorname{Im} z< -\operatorname{Im} z_0$. Thus we have proved (a). Let φ be any function in $C_0^\infty(U)$. Then we have, by (8), $|(u,\, \varphi)|\leqq c_2 e^{-\operatorname{Im} z_0\cdot t}\|\varphi\|$ for some $c_2>0$ and any $t\leqq 0$. Hence $\int_{-\infty}^0 e^{izt}(u,\, \varphi)dt$ is holomorphic in $\operatorname{Im} z< -\operatorname{Im} z_0$. By (9), we have $|(u,\, \varphi)|\leqq c_3 e^{-(\operatorname{Im} z_0+\varepsilon)t}\|\varphi\|$ for some $c_3=c_3(\varphi)>0$ and $\varepsilon=\varepsilon(\varphi)>0$ and for any $t\geqq 0$. Hence $\int_0^\infty e^{-izt}(u,\, \varphi)dt$ is holomorphic in $\{z\,;\ \operatorname{Im} z<\operatorname{Im} z_0+\varepsilon\}$. Thus we have proved (*b*). Let $h(s)$ be a real-valued function in $C_0^\infty(E^1)$ such that $h(s)=1$ for $|s|<1$ and $h(s)=0$ for $|s|>2$. Let $h_j(s)=h(j^{-1}s)$. For any v in $\mathcal{D}(A'(-z))=H_2\cap \mathring{H}_1$ and any z with $\operatorname{Im} z< -\operatorname{Im} z_0$, we have

$$(\hat{u}^+(z),\, A'(-z)v)=\int_0^\infty e^{-izt}(u,\, A'(-z)v)dt$$
$$=\lim_{j\to\infty}\int_0^{2j} e^{-izt}h_j(t)(u,\, A'(-z)v)dt$$
$$=\lim_{j\to\infty}\int_0^{2j}(u,\, A'(-z)(\overline{e^{-izt}}h_j(t)v))dt\,.$$

Remembering that $\overline{e^{-i\bar{z}t}h_j(t)v} \in (H_2 \cap \mathring{H}_1)(G\times(-\infty, \infty))$ and that u is a solution of (1) with the zero Dirichlet boundary condition, we have

$$(\hat{u}^+, A'(-z)v)=\left(a_{00}\frac{\partial u(\cdot, 0)}{\partial t}+2\sum_{j=1}^{n} a_{0j}\frac{\partial u(\cdot, 0)}{\partial x_j}+iz\,a_{00}u(\cdot, 0)+a_0 u(\cdot, 0), v\right).$$

In other words,

$$(\hat{u}^+, A'(-z)v)=(f(-z), v)$$

for any z with $\operatorname{Im} z<-\operatorname{Im} z_0$ and any $v\in\mathcal{D}(A'(-z))$. Since $\hat{u}^+(z)$ and $f(-z)$ lie in $L^2(G)$ for z with $\operatorname{Im} z<-\operatorname{Im} z_0$, we have, by $A'(-z)^*=A(-z)$, $\hat{u}^+(z)\in\mathcal{D}(A(-z))$ and $A(-z)u=f(-z)$ for $\operatorname{Im} z<-\operatorname{Im} z_0$. Similarly we have the latter part of (c).

§3. Proof of the theorem.

Let us fix an arbitrary function φ in $C_0^\infty(U)$. Consider the equation in $L^2(G)$:

$$A(-z)v=f(-z) \quad \text{for } z \text{ with } -z \text{ in } \Delta, \tag{21}$$

where Δ is a neighborhood of γ, defined in Lemma 1, and $f(-z)$ is defined in Lemma 2. Then we have, by the assumption, $f(-z)=f(x, -z)\in L^2(G)$ and by Lemma 1, the resolvent set of $A(-z)$ for z with $-z$ in Δ contains the origin, so that the equation (21) has a unique solution $v(z)$ in $L^2(G)$ for z with $-z$ in Δ. By Lemma 2, we know that

$$\hat{u}^+(z)\in\mathcal{D}(A(-z)) \text{ and } A(-z)\hat{u}^+(z)=f(-z) \text{ for } z \text{ with } \operatorname{Im} z<-\operatorname{Im} z_0.$$

Hence

$$\hat{u}^+(z)=v(z) \text{ for any } z \text{ with } \operatorname{Im} z<-\operatorname{Im} z_0 \text{ and } -z \text{ in } \Delta. \tag{22}$$

Such a z really exists since Δ is an open set and z_0 is contained in Δ. By Lemma 2, $(\hat{u}^+(z), \varphi)$ is holomorphic in the lower half-plane $\{z;\ \operatorname{Im} z<\operatorname{Im} z_0+\varepsilon(\varphi)\}$, and, by Lemma 1, $(v(z), \varphi)=(A(-z)f(-z), \varphi)$ is holomorphic in $-\Delta$. It follows from (22) and the unique continuation theorem in the complex analysis that

$$(\hat{u}^+(z), \varphi)=(v(z), \varphi) \tag{23}$$

for any z with $\operatorname{Im} z<\operatorname{Im} z_0+\varepsilon(\varphi)$ and $-z$ in Δ. Let us define $\hat{u}(z)=v(-z)+\hat{u}^-(z)$ for $z\in\Omega$, where $\Omega=\{z;\ \operatorname{Im} z<-\operatorname{Im} z_0,\ z\in\Delta\}$. Then, remembering that $\hat{u}^-(z)\in\mathcal{D}(A(z))$ and $A(z)\hat{u}^-(z)=-f(z)$ by Lemma 2, we have

$$\hat{u}(z)\in\mathcal{D}(A(z)) \quad\text{and}\quad A(z)\hat{u}(z)=0 \qquad \text{for } z \text{ in } \Omega.$$

By Lemma 1, the resolvent set of $A(z)$ for z in Ω contains the origin, so that

$$\hat{u}(z)=0 \qquad \text{for } z \text{ in } \Omega. \tag{24}$$

Hence, we have, by (23) and (24),

$$(\hat{u}^+(-z)+\hat{u}^-(z),\varphi)=(v(-z)+\hat{u}^-(z),\varphi)=(\hat{u}(z),\varphi)=0 \tag{25}$$

for z with z in Ω and $-\operatorname{Im} z_0-\varepsilon(\varphi)<\operatorname{Im} z<-\operatorname{Im} z_0$. By Lemma 2, $(\hat{u}^+(-z)+\hat{u}^-(z),\varphi)$ is holomorphic in the strip $\{z;\ -\operatorname{Im} z_0-\varepsilon(\varphi)<\operatorname{Im} z<-\operatorname{Im} z_0\}$. Therefore, it follows from (25) that

$$(\hat{u}^+(-z)+\hat{u}^-(z),\varphi)=0$$

for any z in the strip $\{z;\ -\operatorname{Im} z_0-\varepsilon(\varphi)<\operatorname{Im} z<-\operatorname{Im} z_0\}$.

Hence,

$$\begin{aligned}(\hat{u}^+(-z),\varphi)+(\hat{u}^-(z),\varphi)&=\int_0^\infty e^{izt}(u(\cdot,t),\varphi)dt+\int_{-\infty}^0 e^{izt}(u(\cdot,t)\varphi)dt\\&=\int_{-\infty}^\infty e^{izt}(u(\cdot,t),\varphi)dt\\&=\int_{-\infty}^\infty e^{i\operatorname{Re} z\cdot t}e^{-\operatorname{Im} z\cdot t}(u(\cdot,t),\varphi)dt\\&=0\end{aligned} \tag{26}$$

for any z in the strip $\{z;\ -\operatorname{Im} z_0-\varepsilon(\varphi)<\operatorname{Im} z<-\operatorname{Im} z_0\}$.

On the other hand, we have, by (8), the estimate $|(u(\cdot,t),\varphi)|\leqq c_2e^{\operatorname{Im} z_0|t|}\|\varphi\|$ for any $t\leqq 0$, and, by (9), the estimate $|(u(\cdot,t),\varphi)|\leqq c_3e^{-(\operatorname{Im} z_0+\varepsilon)t}\|\varphi\|$ for any $t\geqq 0$, where c_2, $c_3=c_3(\varphi)$ and $\varepsilon=\varepsilon(\varphi)$ are positive constants in t. Thus, for any z in the strip $\{z;\ -\operatorname{Im} z_0-\varepsilon(\varphi)<\operatorname{Im} z<-\operatorname{Im} z_0\}$, $e^{-\operatorname{Im} z\cdot t}(u(\cdot,t),\varphi)$ is a square integrable function of t on $(-\infty,\infty)$. Therefore, by (26), it follows from the uniqueness of the Fourier transformation that $e^{-\operatorname{Im} z\cdot t}(u(\cdot,t),\varphi)=0$ so that $(u(\cdot,t),\varphi)=0$. Since φ is an arbitrary function in $C_0^\infty(U)$, $u(\cdot,t)$ must be 0 in $U\times(-\infty,\infty)$. Thus the theorem is proved.

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