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Minimal immersions of Riemannian manifolds

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Introduction

An isometric immersion $M \rightarrow M'$ of a Riemannian manifold M in another manifold M' is called to be *minimal*, if each of its mean curvatures vanishes. In this paper we shall deal with minimal immersions of Riemannian manifolds in a space of constant curvature.

In §1 we shall summarize notations and formulas concerning immersions which are all well-known, and give a criterion for a Riemannian manifold to be immersed minimally in a space of constant curvature (Theorem 1).

In §2 we shall deal with an immersion $x: M \rightarrow R^{m+k}$ of a Riemannian m-manifold in an $(m+k)$-dimensional Euclidean space R^{m+k}. If the image $x(M)$ of M by x is contained in an $(m+k-1)$-dimensional sphere S^{m+k-1} in R^{m+k}, we shall call that *the immersion x realizes an immersion in a sphere.* Since the immersion x can be considered as a vector valued function on M, we can apply Laplace-Beltrami operator Δ to x. Theorem 2 asserts that the immersion x is minimal if and only if $\Delta x=0$, and Theorem 3 asserts that the immersion x realizes a minimal immersion in a sphere if and only if $\Delta x=\lambda x$ for some constant $\lambda \neq 0$ and the radius of the sphere is completely determined by λ. Theorem 2 has been obtained by J. Eells and J. H. Sampson [1].

In §3 we shall give an example of a Riemannian manifold which admits an immersion x in a Euclidean space satisfying $\Delta x=\lambda x$, and prove that the compact homogeneous Riemannian manifold with irreducible linear isotropy group admits a minimal immersion in a sphere. This example is motivated by a work of T. Nagano [2]. The author is grateful to Professors T. Nagano and M. Obata for their many valuable suggestions in this research.

§1. Notations and formulas

Let M and M' be Riemannian manifolds of dimension m and $m+k$ respectively and $\varphi: M \rightarrow M'$ be an isometric immersion of M in M'. In terms of local coordinates $(\xi^1, \cdots, \xi^m)$ of M and $(\eta^1, \cdots, \eta^{m+k})$ of M', the immersion φ is

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locally represented by

$$\eta^A = \eta^A(\xi^1, \cdots, \xi^m) \qquad (A = 1, \cdots, m+k).$$

If we denote $\partial_i \eta^A$ by B^A where $\partial_i = \partial/\partial\xi^i$, we have

$$g_{ji} = B_j^B B_i^A g'_{BA} \text{ **)} \tag{1.1}$$

where g_{ji} and g'_{AB} are the metric tensors of M and M' respectively. Let $\underset{\alpha}{n}$ $(\alpha = 1, \cdots, k)$ be mutually orthogonal unit normals, $\underset{\alpha}{H}_{ji}$ $(\alpha = 1, \cdots, k)$ be the second fundamental tensor and $\underset{\alpha\beta}{H}_j$ $(\alpha, \beta = 1, \cdots, k)$ be the third fundamental tensor of the immersion. Then the following formulas are well-known [3].

$$\nabla_j B_i^A = \sum_{\alpha=1}^{k} \underset{\alpha}{H}_{ji} \underset{\alpha}{N}^A \tag{1.2}$$

$$\nabla_j \underset{\alpha}{N}^A = -\underset{\alpha}{H}_j{}^i B_i^A + \sum_{\beta=1}^{k} \underset{\alpha\beta}{H}_j \underset{\beta}{N}^A \qquad (\alpha = 1, \cdots, k) \tag{1.3}$$

$$K_{kjih} = B_k^D B_j^C B_i^B B_h^A K'_{DCBA} + \sum_{\alpha=1}^{k} (\underset{\alpha}{H}_{kh} \underset{\alpha}{H}_{ji} - \underset{\alpha}{H}_{ki} \underset{\alpha}{H}_{jh}) \tag{1.4}$$

where $\underset{\alpha}{N}^A$ are the components of $\underset{\alpha}{n}$ with respect to the coordinates (η^A) in M', K_{kjih} and K'_{DCBA} are the curvature tensors of M and M' and ∇_j is the so-called van der Waerden-Bortolotti operator of covariant differentiation. If we denote the Christoffel symbols of M and M' by $\left\{ {i \atop kj} \right\}$ and $\left\{ {A \atop CB} \right\}'$ respectively, $\nabla_j B_i^A$ and $\nabla_j \underset{\alpha}{N}^A$ are given by

$$\nabla_j B_i^A = \partial_j B_i^A - \left\{ {h \atop ji} \right\} B_h^A + \left\{ {A \atop BC} \right\}' B_j^C B_i^B$$

$$\nabla_j \underset{\alpha}{N}^A = \partial_j \underset{\alpha}{N}^A + \left\{ {A \atop CB} \right\}' B_j^C \underset{\alpha}{N}^B \qquad (\alpha = 1, \cdots, k).$$

$\underset{\alpha}{h}$ $(\alpha = 1, \cdots, k)$, which are by definition $\underset{\alpha}{H}_{ji} g^{ji}$, are called the *mean curvatures* of the immersion and the immersion is called to be *minimal* if and only if $\underset{\alpha}{h} = 0$ $(\alpha = 1, \cdots, k)$.

If M' is a space of constant curvature c, the curvature tensor of M' has the form

$$K'_{DCBA} = c(g'_{DA} g'_{BC} - g'_{DB} g'_{CA}), \tag{1.5}$$

and so from (1.4) and (1.5) the formula (1.4) are written by

**) In the sequel, we use the summation convention for the Latin indices $h, i, j, k, \cdots = 1, \cdots, m$ and $A, B, C, D, \cdots = 1, \cdots, m+k$. For the Greek indices their values are indicated in each equation.

$$K_{kjih} = c(g_{kh}g_{ji} - g_{ki}g_{jh}) + \sum_{\alpha=1}^{k} (\underset{\alpha}{H}_{kh}\underset{\alpha}{H}_{ji} - \underset{\alpha}{H}_{ki}\underset{\alpha}{H}_{jh}). \tag{1.6}$$

Tranvecting (1.6) with g^{kh}, we obtain

$$K_{ji} = c(m-1)g_{ji} - \sum_{\alpha=1}^{k} \underset{\alpha}{H}_{jk}\underset{\alpha}{H}_{ih}g^{kh},$$

where K_{ji} is Ricci tensor of M. Thus we have

$$c(m-1)g_{ji} - K_{ji} = \sum_{\alpha=1}^{k} \underset{\alpha}{H}_{jk}\underset{\alpha}{H}_{ih}g^{kh}.$$

The right hand member of this equation is positive semi definite. Thus we have the theorem.

THEOREM 1. *If a Riemannian m-manifold M admits a minimal immersion in a space of constant curvature c, the tensor $c(m-1)g-K$ is positive semi-definite where g is metric tensor and K is Ricci tensor of M.*

Now let M'' be a third Riemannian manifold of dimension $m+k'$, and assume that there exist isometric immersions $\varphi' : M - M''$ and $\varphi'' : M'' - M'$ such that $\varphi = \varphi'' \circ \varphi'$. If we take the unit normals $\underset{\alpha}{n}'$ $(\alpha = 1, \cdots, k')$ for the immersion φ' and $\underset{\beta}{n}$ $(\beta = k'+1, \cdots, k)$ for the immersion φ'', then denoting $\underset{\alpha}{n} = d\varphi''(\underset{\alpha}{n}')$ $(\alpha = 1, \cdots, k')$, $\underset{\gamma}{n}$ $(\gamma = 1, \cdots, k)$ are considered as the unit normals for the immersion φ. The following lemma is easily verified.

LEMMA. *Notations being as above, let $\underset{\alpha}{H}'_{ji}$ $(\alpha = 1, \cdots, k')$ be the second fundamental tensor of the immersion φ' with respect to the unit normals n'_α, we have $\underset{\alpha}{H}'_{ji} = \underset{\alpha}{H}_{ji}$ $(\alpha = 1, \cdots, k')$.*

§ 2. The minimal immersion in a sphere

Let M be a Riemannian m-manifold and $x : M \to R^{m+k}$ be an isometric immersion of M in a Euclidean $(m+k)$-space R^{m+k}. Let $(\xi^1, \cdots, \xi^m)$ be a local coordinates in M and $\underset{\alpha}{n}$ $(\alpha = 1, \cdots, k)$ be mutually orthgonal unit normals. Then the formulas (1.2) and (1.3) are written in the vector forms as follows.

$$\nabla_j x_i = \partial_j x_i - \left\{ {h \atop ji} \right\} x_h = \sum_{\alpha=1}^{k} \underset{\alpha}{H}_{ji}\underset{\alpha}{n} \tag{2.1}$$

$$\nabla_j \underset{\alpha}{n} = \partial_j \underset{\alpha}{n} = -\underset{\alpha}{H}_j{}^i x_i + \sum_{\beta=1}^{k} \underset{\alpha\beta}{H}_j \underset{\beta}{n} \qquad (\alpha = 1, \cdots, k) \tag{2.2}$$

where $x_i = \partial_i x$.

By definition $\Delta x = -g^{ji}\nabla_j x_i$ and therefore from (2.1) we have

$$\Delta x = -\sum_{\alpha=1}^{k} \underset{\alpha}{h}\underset{\alpha}{n}. \tag{2.3}$$

The formula (2.3) implies that $\Delta x=0$ if and only if $\underset{\alpha}{h}=0$ $(\alpha=1, \cdots, k)$ which means that the immersion x is minimal. Thus we have the theorem.

THEOREM 2. *An isometric immersion $x: M\rightarrow R^{m+k}$ of a Riemannian m-manifold M in a Euclidean $(m+k)$-space R^{m+k} is minimal if and only if $\Delta x=0$.*

From this theorem it might be natural to ask what the immersion x satisfying $\Delta x=\lambda x$ $(\lambda\neq 0)$ is. The answer is, roughly speaking, that such an immersion realizes a minimal immersion in a sphere and conversely. More precisely,

THEOREM 3. *If an isometric immersion $x: M\rightarrow R^{m+k}$ of a Riemannian m-manifold M in a Euclidean $(m+k)$-space satisfies $\Delta x=\lambda x$ for some constant $\neq 0$, then λ is necessarily positive and x realizes a minimal immersion in a sphere S^{m+k+1} of a radius $\sqrt{m/\lambda}$ in R^{m+k}: Conversely if x realizes a minimal immersion in a sphere of radius a in R^{m+k}, then x satisfies $\Delta x=\lambda x$ up to a parallel displacement in R^{m+k} and $\lambda=m/a^2$.*

PROOF. Assume $\Delta x=\lambda x$ $(\lambda\neq 0)$. Then from (2.3) we have

$$x=-\frac{1}{\lambda}\sum_{\alpha=1}^{k}\underset{\alpha}{h}\underset{\alpha}{n}. \tag{2.4}$$

Differentiating (2.4) by ξ^j and using (2.2) we obtain

$$x_j=\frac{1}{\lambda}\sum_{\alpha=1}^{k}\underset{\alpha}{h}\underset{\alpha}{H}{}_j{}^i x_i-\frac{1}{\lambda}\sum_{\alpha=1}^{k}(\partial_j\underset{\alpha}{h}+\sum_{\beta=1}^{k}\underset{\beta}{h}\underset{\beta\alpha}{H}{}_j)\underset{\alpha}{n}.$$

Thus we have

$$\frac{1}{\lambda}\sum_{\alpha=1}^{k}\underset{\alpha}{h}\underset{\alpha}{H}{}_{ji}=g_{ji}. \tag{2.5}$$

Transvecting this equation with g^{ji}, we have

$$\frac{1}{\lambda}\sum_{\alpha=1}^{k}(\underset{\alpha}{h})^2=m, \tag{2.6}$$

from which it follows that λ must be positive. From (2.4) the length $|x|$ of the position vector x is given by $\frac{1}{\lambda}\sqrt{\sum_{\alpha=1}^{k}(\underset{\alpha}{h})^2}$, hence we have

$$|x|=\sqrt{m/\lambda}=\text{const.}=a.$$

Therefore the image $x(M)$ of M is contained in a sphere of radius a centred at the origin of R^{m+k} which means the immersion x realizes an immersion in a sphere.

Since the vector x is normal to M (precisely $x(M)$) $\underset{k}{n}$ can be chosen as $(1/a)x$ and then $\underset{\alpha}{n}$ $(\alpha=1, \cdots, k-1)$ are tangent to the sphere. Then the formula (2.2) gives

(2.7) $$-\underset{k}{H}_j{}^i x_i+\sum_{\alpha=1}^{k-1}\underset{k\alpha}{H_j}\underset{\alpha}{n}=\frac{1}{a}x_j$$

which implies

(2.8) $$\underset{k}{H}_{ji}=-\frac{1}{a}g_{ji} \quad \text{and} \quad \underset{k\alpha}{H_j}=0 \qquad (\alpha=1,\cdots,k-1)$$

and from which we know $\underset{k}{h}=-m/a=-\sqrt{\lambda m}$. Substituting this in (2.6) we find $\sum_{\alpha=1}^{k-1}(\underset{\alpha}{h})^2=0$ and hence we get $\underset{\alpha}{h}=0$ for $\alpha=1,\cdots,k-1$. Since from the Lemma in § 1, $\underset{\alpha}{H}_{ji}$ $(\alpha=1,\cdots,k-1)$ are equal to the second fundamental tensor of the immersion in S^{m+k-1} induced from x. Thus x realizes a minimal immersion in a sphere.

Conversely assume that x realizes a minimal immersion in a sphere S^{m+k-1} of radius a. By a parallel displacement in R^{m+k}, S^{m+k-1} may be assumed to be centred at the origin of R^{m+k}. Then we can take mutually orthogonal unit normals $\underset{\alpha}{n}$ $(\alpha=1,\cdots.k)$ of the immersion such as $\underset{k}{n}=(1/a)x$ which is normal to S^{m+k-1}. In our case the equation (2.7) and therefore (2.8) are automatically satified, and since $\underset{\alpha}{H}_{ji}$ $(\alpha=1,\cdots,k-1)$ are considered as the second fundamental tensor of the induced immersion in S^{m+k-1}, we have $\underset{\alpha}{h}=0$ $(\alpha=1,\cdots,k-1)$ by the assumption. Then we have

$$\Delta x=-\sum_{\alpha=1}^{k}\underset{\alpha}{h}\underset{\alpha}{n}=-\underset{k}{h}\underset{k}{n}=-(\underset{k}{h}/a)x\,.$$

From (2.8), we know $\underset{k}{h}=-(m/a)$ and therefore we obtain

$$\Delta x=(m/a^2)x\,.$$

This completes the proof of Theorem 3.

§ 3. Application

Let M be a compact homogeneous Riemannian manifold, and assume that the linear isotropy group is irreducible on the tangent space.

For a constant $\lambda\neq 0$ we shall denote by V_λ the set of all functions on M satisfying $\Delta f=\lambda f$. Since M is compact, each V_λ is a finite dimensional vector space over the reals. Assume dim $V\neq 0$ (such a V necessarily exists). The isometries of M act on the space of functions on M in a natural way and leave V_λ invariant. The group G of isometries of M which is transitive on M is compact, and therefore there exists an inner product in V_λ invariant by G. We fix one of them. Let $f_1,\cdots,f_n$ $(n=\dim V_\lambda)$ be an orthonormal basis of V_λ with respect to the inner product. We have a mapping $\tilde{x}:M\rightarrow R^n$ of M in R^n by $x(p)=(f_1(p),\cdots,f_n(p))$ for $p\in M$. We have a covariant tensor field $g=\sum_{i=1}^{n} df_i \cdot df_i$ on M. For an isometry σ, σ^* preserves the inner product of V_λ, we know $\sigma^*(f_j)=\sum_{i=1}^{n}\sigma_{ij}f_i$ and the matrix (σ_{ij}) is an orthogonal matrix. Then the transform $\sigma^*(\tilde{g})$ of $\tilde{g}$ by an isometry σ is calculated as follows.

$$\begin{aligned}\sigma^*(\tilde{g}) &= \sum_{j=1}^{n}\sigma^*(df_j)\cdot\sigma^*(df_j)\\ &= \sum_{j=1}^{n} d(\sigma^* f_j)\cdot d(\sigma^* f_j)\\ &= \sum_{i,j,k=1}\sigma_{ij}^*\sigma_{kj}^* df_i df_k\\ &= \sum_{i=1}^{n} df_i df_i\\ &= \tilde{g}.\end{aligned}$$

Thus $\tilde{g}$ is invariant by all isometries of M. Hence by the assumption of the irreducibility of the linear isotropy group, we obtain

$$\tilde{g}=c^2 g$$

for some constant $c\neq 0$ where g is the metric tensor of M. Therefore the mapping defined by $x(p)=(1/c)\tilde{x}(p)$ gives an isometric immersion of M in R^n satisfying $\Delta x=\lambda x$ for $\lambda\neq 0$. Thus from Theorem 3 we have the theorem.

THEOREM 4. *A compact homogeneous Riemannian manifold with irreducible linear isotropy group admits a minimal immersion in a Euclidean sphere.*

COROLLARY. *An irreducible compact symmetric space admits a minimal immersion in a Euclidean sphere.*

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