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A normal space Z with ind Z=0, dim Z=1, Ind Z=2

By Keio NAGAMI

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This paper gives a normal (Hausdorff) space Z for which three basic dimension functions are different from each other: $\operatorname{ind} Z=0$, $\dim Z=1$ and $\operatorname{Ind} Z=2$. As for the definition of three dimension functions see J. Nagata [**7**, p. 9]. The idea developed by P. Vopenka [**9**] as well as the one by C. H. Dowker [**1**] are the main tool in our construction.

Let ω_1 be the first uncountable ordinal, $J=\{\alpha : 0\leqq\alpha<\omega_1\}$ and $J^*=\{\alpha : 0\leqq\alpha\leqq\omega_1\}$, where J and J^* have the usual interval topology. Let I be the unit interval [0,1]. By Dowker [**1**] there exist subsets I_α, $\alpha<\omega_1$, of I such that i) $I_\alpha\subset I_\beta$ if $\alpha<\beta$, ii) $\dim I_\alpha=0$ for each α, iii) $\cup I_\alpha=I$ and iv) each I_α is dense in I. By Nagami [**5**] there exist a separable metric space C with $\dim C=0$ and an open continuous mapping f of C onto I. Let M be a discrete space whose power $|M|$ is $\mathfrak{c}$. Consider the disjoint sum T of I and $C\times M\times I$ and introduce into T the topology due to Vopenka [**9**] as follows:

i) An open set of $C\times M\times I$ with the usual product topology is open in T.

ii) If U is an open set in I and if K is a finite subset of M, then $U\cup(f^{-1}(U)\times(M-K)\times I)$ is open in T.

Then T with the above basic open sets is a Hausdorff space. Set

$$T_\alpha=I_\alpha\cup(f^{-1}(I_\alpha)\times M\times I_\alpha),\ \alpha<\omega_1 .$$

The point set Z is the sum of all $\{\alpha\}\times T_\alpha$, $\alpha<\omega_1$. The topology of Z is the relative topology of the product space $J^*\times T$. We identify T with $\{\omega_1\}\times T$. π' is the projection of $J^*\times T$ onto T. Set $\pi=\pi'\,|\,Z$. ρ' is the projection of $J^*\times T$ onto J^*. Set $\rho=\rho'\,|\,Z$. If E is a subset of Z and J' is a subset of J, then $[E]_{J'}$ denotes the intersection of E and $\rho^{-1}(J')$. If x is a point of I and ε is a positive number, then $S_\varepsilon(x)$ denotes an open ε-sphere in I with the center x.

LEMMA 1. *Let X be a non-empty metric space. Then there exist subsets X_α, $\alpha<\omega_1$ such that* i) $X_\alpha\subset X_\beta$ *if* $\alpha<\beta$, ii) $\dim X_\alpha=0$ *and* iii) $\cup X_\alpha=X$. *If X_α satisfy this condition let Y be the subspace of $J\times X$ which is the sum of all $\{\alpha\}\times X_\alpha$, $\alpha<\omega_1$. Then Y is a normal space such that*

i) $\operatorname{ind} Y=0$.

ii) Ind $Y=\dim Y=\dim X$.

The first half of the lemma is Nagami [6, Theorem 2]. The last half is proved by the analogous argument in Dowker [1]. It is to be noticed that the special case of this lemma where X is separable metric was proved by Yu. M. Smirnov [8].

The following lemma was proved by Vopenka [9, Proposition 1.3] for the case when X is compact. Our proof is nothing but an extraction of his. The author is kind enough for English-reading mathematicians.

LEMMA 2. *Let X be a normal space and Y a non-empty closed set of X which satisfy the following conditions:*

i) *For any open neighborhood U of Y there exists an open and closed neighborhood V of Y with $V\subset U$.*

ii) *There exists a retraction φ of X onto Y*

iii) Ind $Y\leqq m$.

iv) *If F is a closed set of X with $F\cap Y=\phi$, then* Ind $F\leqq n$. *Then* Ind $X\leqq m+n$.

PROOF. We prove this by induction on Ind Y. Since the proof for the starting case when Ind $Y=0$ is completely similar to general case, we merely prove the lemma under the assumption that the lemma is true when Ind $Y<m$.

Let now Ind $Y\leqq m$. Let H be a closed set of X and W be an open set of X with $H\subset W$. Take a relatively open set G of Y with $H\cap Y\subset G\subset\bar{G}\subset W\cap Y$ and with Ind $(\bar{G}-G)\leqq m-1$. Set $X'=\varphi^{-1}(\bar{G}-G)$ and $Y'=\bar{G}-G$. Then it can easily be seen that the condition of the lemma is satisfied if X, Y and m are replaced by X', Y' and $m-1$ respectively. Hence by induction assumption Ind $X'\leqq m+n-1$. Let V be an open and closed neighborhood of Y with

$$V\cap((\varphi^{-1}(\bar{G})-W)\cup(H-\varphi^{-1}(G)))=\phi\,.$$

Let D_1 be an open set of $X-V$ such that

i) $H-V\subset D_1\subset\bar{D}_1\subset W-V$

ii) Ind $(\bar{D}_1-D_1)\leqq n-1$.

Set $D_2=V\cap\varphi^{-1}(G)$. Then $H\cap V\subset D_2\subset\bar{D}_2\subset W\cap V$. Since $\bar{D}_2-D_2\subset X'$, Ind $(\bar{D}_2-D_2)\leqq m+n-1$. Set $D=D_1\cup D_2$. Then $\bar{D}-D$ is the disjoint union of $\bar{D}_1-D_1$ and $\bar{D}_2-D_2$. Hence

$$\operatorname{Ind}(\bar{D}-D)\leqq\max(n-1,\ m+n-1)=m+n-1\,.$$

Moreover $H\subset D\subset\bar{D}\subset W$. Therefore Ind $X\leqq m+n$ and the proof is finished.

The following is the special case of Morita [3, Footnote, p. 164] or Nagami [4, Theorem 3], since a regular space with the Lindelöf property has the star-finite property by Morita [2].

LEMMA 3. *If X is a normal space with the Lindelöf property, then* $\dim X\leqq$ ind X.

Now let us prove that Z is the desired one by several steps.

I) To prove the normality of Z let F and H be a pair of disjoint closed sets of Z. Take an arbitrary point x of I. Both F and H cannot be cofinal on $\pi^{-1}(x)$ at the same time. Suppose that H is not cofinal on $\pi^{-1}(x)$. Then there exists $\alpha(x)$ such that $[\pi^{-1}(x)]_{\alpha(x)} \neq \phi$ and

$$[\pi^{-1}(x)]_{[\alpha(x),\omega_1)} \cap H = \phi .$$

For every β with $\alpha(x) < \beta < \omega_1$ let $\varepsilon(\beta)$ be the largest positive number for which there exist γ with $\alpha(x) \leqq \gamma < \beta$ and a finite subset K_β of M such that

$$[\pi^{-1}(S_{\varepsilon(\beta)}(x) \cup (f^{-1}(S_{\varepsilon(\beta)}(x)) \times (M-K_\beta) \times I))]_{(\gamma,\beta]} \cap H = \phi .$$

Then it is easy to see that

$$\varepsilon(x) = \inf \{\varepsilon(\beta) : \alpha(x) < \beta < \omega_1\}$$

is positive. Set

$$K_x = \{\lambda : \lambda \in M, [\pi^{-1}(f^{-1}(S_{\varepsilon(x)}(x)) \times \{\lambda\} \times I)]_{(\alpha(x),\omega_1)} \cap H \neq \phi\} .$$

To prove K_x is a finite set assume the contrary. Then there exist a countably infinite subset $\{\lambda_1, \lambda_2, \cdots\}$ of K_x and a sequence $\alpha(x) < \alpha_1 \leqq \alpha_2 \leqq \cdots$ such that

$$[\pi^{-1}(f^{-1}(S_{\varepsilon(x)}(x)) \times \{\lambda_i\} \times I)]_{\alpha_i} \cap H \neq \phi .$$

Let $\alpha_0 = \lim \alpha_i$. Then for any δ with $\alpha(x) \leqq \delta < \alpha_0$ and for any finite subset K of M,

$$[\pi^{-1}(f^{-1}(S_{\varepsilon(x)}(x)) \times (M-K) \times I)]_{(\delta,\alpha_0]} \cap H \neq \phi ,$$

which is a contradiction. Thus K_x is finite and

$$[\pi^{-1}(S_{\varepsilon(x)}(x) \cup (f^{-1}(S_{\varepsilon(x)}(x)) \times (M-K_x) \times I))]_{(\alpha(x),\omega_1)}$$

does not meet H.

II) By I) for every $x \in I$ we have a positive number $\varepsilon(x)$, an ordinal $\alpha(x) < \omega_1$ and a finite subset K_x of M such that $[\pi^{-1}(x)]_{\alpha(x)} \neq \phi$ and

$$\{[\pi^{-1}(S_{\varepsilon(x)}(x) \cup (f^{-1}(S_{\varepsilon(x)}(x)) \times (M-K_x) \times I))]_{(\alpha(x),\omega_1)} : x \in I\}$$

refines $\{Z-F, Z-H\}$. Take a finite subset $\{x_1, \cdots, x_n\}$ of I such that

$$\mathfrak{U} = \{S_{\varepsilon(x_i)}(x_i) : i = 1, \cdots, n\}$$

covers I. Set

$$K = \cup \{K_{x_i} : i = 1, \cdots, n\} .$$

Then K is a finite subset of M. Set

$$\beta_0 = \sup \{\alpha(x_i) : i = 1, \cdots, n\} .$$

Let

$$\mathfrak{V} = \{V_1, \cdots, V_m\}$$

be a finite open (in I) covering of I which is a Δ-refinement of $\mathfrak{U}$. We divide

Z into disjoint three parts Z_1, Z_2, Z_3 each of which is open in Z as follows:

$$Z_1=[\pi^{-1}(I\cup(C\times(M-K)\times I))]_{(\beta_0,\omega_1)},$$

$$Z_2=[\pi^{-1}(C\times K\times I)]_{(\beta_0,\omega_1)},$$

$$Z_3=[Z]_{[0,\beta_0]},$$

$$Z=Z_1\cup Z_2\cup Z_3.$$

By construction

$$\bar{\mathfrak{V}}=\{[\pi^{-1}(V_i\cup(f^{-1}(V_i)\times(M-K)\times I))]_{(\beta_0,\omega_1)}:i=1,\cdots,m\}$$

Δ-refines $\{Z-F, Z-H\}$. Let D_1 and G_1 be respectively stars of F and H with respect to $\bar{\mathfrak{V}}$. Then $D_1\cap G_1=\phi$, $D_1\supset F\cap Z_1$ and $G_1\supset H\cap Z_1$. Since

$$[\pi^{-1}(C\times K\times I)]_{(\beta_0,\omega_1)}$$

is normal by Lemma 1, there exist open sets D_2 and G_2 of Z_2 such that $D_2\cap G_2=\phi$, $D_2\supset F\cap Z_2$ and $G_2\supset H\cap Z_2$.

III) Let us prove the normality of Z_3. Let $\mathfrak{W}$ be an arbitrary open covering of Z_3. Consider an arbitrary ordinal α with $0\leqq\alpha\leqq\beta_0$. By perfect separability of I there exist a sequence of open sets $A_1, A_2, \cdots$ of I, a sequence of ordinals $\beta_1, \beta_2, \cdots$ with $\beta_i<\alpha$, $i=1,2,\cdots$, and a sequence of finite subsets $K_1, K_2, \cdots$ of M such that $\cup A_i=I$ and

$$\mathfrak{W}_1=\{[\pi^{-1}(A_i\cup(f^{-1}(A_i)\times(M-K_i)\times I))]_{(\beta_i,\alpha]}:i=1,2,\cdots\}$$

refines $\mathfrak{W}$. Set

$$M_1=\bigcup_{i=1}^{\infty}K_i.$$

Then M_1 is countable. Since $C\times M_1\times I$ is perfectly separable, we can find a countable open collection $\mathfrak{W}_2$ of Z_3 such that i) $\mathfrak{W}_2$ refines $\mathfrak{W}$ and ii) $\mathfrak{W}_2$ covers $[\pi^{-1}(C\times M_1\times I)]_\alpha$. Thus we have a countable open collection $\mathfrak{W}_1\vee\mathfrak{W}_2$ of Z_3 which covers $[Z]_\alpha$ and refines $\mathfrak{W}$. Since $[0,\beta_0]$ contains only a countable number of ordinals, $\mathfrak{W}$ can be refined by a countable open covering of Z_3, which shows that Z_3 has the Lindelöf property. Since Z_3 is evidently regular, Z_3 is normal by Morita [2]. There exist open sets D_3 and G_3 of Z_3 such that $D_3\cap G_3=\phi$, $D_3\supset F\cap Z_3$ and $G_3\supset H\cap Z_3$. Set

$$D=D_1\cup D_2\cup D_3,$$

$$G=G_1\cup G_2\cup G_3.$$

Then D and G are open sets of Z such that $D\cap G=\phi$, $D\supset F$ and $G\supset H$. Thus the normality of Z is established.

IV) It is evident that ind $Z=0$.

V) Let us show dim $Z=1$. dim $Z\geqq1$, because $\pi^{-1}(I)$ is a closed subset of Z and by Lemma 1 we already know that dim $\pi^{-1}(I)=1$. Since $T-I$ is the sum of disjoint open metric subsets and hence $Z-\pi^{-1}(I)$ is a normal space with $\dim(Z-\pi^{-1}(I))=1$ by Lemma 1, $\dim Z \leqq \max(\dim \pi^{-1}(I), \dim(Z-\pi^{-1}(I)))=1$. Thus we have $\dim Z=1$.

VI) Next task is to show $\operatorname{Ind} Z \leqq 2$. For any $\lambda \in M$,

$$\operatorname{Ind} \pi^{-1}(C \times \{\lambda\} \times I)=1$$

by Lemma 1. Here is a closed subset $\pi^{-1}(I)$ of Z with $\operatorname{Ind} \pi^{-1}(I)=1$. If A is any closed subset of Z with $A \cap \pi^{-1}(I)=\phi$, then $\operatorname{Ind} A \leqq 1$. If we can show the condition of Lemma 2 is satisfied, then we have $\operatorname{Ind} Z \leqq 2$. Let U be an arbitrary open set of Z with $U \supset \pi^{-1}(I)$. Set $H=Z-U$. By the same argument for H as in I) there exist a finite subset K of M and an ordinal $\xi<\omega_1$ such that

$$V_1=[\pi^{-1}(I \cup (C \times (M-K) \times I))]_{(\xi,\omega_1)} \subset U .$$

V_1 is open and closed in Z. Since we already knew in III) that $[Z]_{[0,\xi]}$ is a normal space with the Lindelöf property,

$$\dim [Z]_{[0,\xi]} \leqq \operatorname{ind} [Z]_{[0,\xi]}=0 ,$$

which implies

$$\dim [Z]_{[0,\xi]}=0 .$$

Hence there exists an open and closed subset V_2 of $[Z]_{[0,\xi]}$ such that

$$[\pi^{-1}(I)]_{[0,\xi]} \subset V_2 \subset [U]_{[0,\xi]} .$$

If we set $V=V_1 \cup V_2$, then V is an open and closed set of Z with $\pi^{-1}(I) \subset V \subset U$.

We define $\psi: T \rightarrow I$ as follows:

$$\psi(x)=x, \quad \text{if} \quad x \in I ,$$

$$\psi((c, \lambda, x))=f(c), \quad \text{if} \quad (c, \lambda, x) \in C \times M \times I .$$

Then ψ is a retraction of T onto I. Define $\varphi: Z \rightarrow \pi^{-1}(I)$ as follows:

$$\varphi((\alpha, t))=(\alpha, \psi(t)), \quad \text{where} \quad \alpha \in J \quad \text{and} \quad t \in T_\alpha .$$

Then φ is a retraction of Z onto $\pi^{-1}(I)$. By Lemma 2

$$\operatorname{Ind} Z \leqq 2 .$$

VII) Let us show $\operatorname{Ind} Z \geqq 2$. Let 0 and 1 be the terminal points of I. It is to be noticed that there are 0 and 1 which are the first and the second ordinals of J. But there might not be serious confusion. $\pi^{-1}(0)$ and $\pi^{-1}(1)$ are disjoint closed sets of Z. We prove that any closed set separating these two sets has to have $\operatorname{Ind} \geqq 1$, which in turn will imply $\operatorname{Ind} Z \geqq 2$. Let P be an open set of Z with $\pi^{-1}(0) \subset P \subset \bar{P} \subset Z-\pi^{-1}(1)$. Set $B=\bar{P}-P$ and $Z-\bar{P}=Q$. We want to show $\operatorname{Ind} B \geqq 1$. Set

$$C_P=\{x: x\in I,\ P \textit{ is cofinal on } \pi^{-1}(x)\}\,,$$

$$C_Q=\{x: x\in I,\ Q \textit{ is cofinal on } \pi^{-1}(x)\}\,,$$

$$E_B=\{x: x\in I,\ B \textit{ is equifinal on } \pi^{-1}(x)\}\,.$$

VIII) Suppose that $C_P\cap C_Q\neq\phi$. Take $h\in C_P\cap C_Q$. Since $0\notin C_Q$ and $1\notin C_P$, $0<h<1$. For any point $p\in\pi^{-1}(h)\cap P$ there exists a positive integer $i(p)$ such that

i) $1/i(p)<\min\{h, 1-h\}$,

ii) $[\pi^{-1}(S_{1/i(p)}(h))]_{\rho(p)}\subset P$.

Then there exists i such that

$$P_1=\{p: i(p)=i\}$$

is cofinal. For every point $q\in\pi^{-1}(h)\cap Q$ there exists a positive integer $i(q)$ such that

i) $1/i(q)<\min\{h, 1-h\}$,

ii) $[\pi^{-1}(S_{1/i(q)}(h))]_{\rho(q)}\subset Q$.

Then there exists j such that

$$Q_1=\{q: i(q)=j\}$$

is cofinal. Let

$$k=\max\{i, j\}\,.$$

Then

$$B_1=\{z: z\in\pi^{-1}(h),\ [\pi^{-1}(S_{1/k}(h))]_{\rho(z)}\subset B\}$$

is cofinal. Moreover by the closedness of B, $\rho(B_1)$ is closed in J. Hence

$$B_2=\cup\{[\pi^{-1}([h-1/(2k),\ h+1/(2k)])]_{\rho(z)}: z\in B_1\}$$

is a closed subset of B. By Lemma 1 Ind $B_2=1$. Hence

$$\operatorname{Ind} B\geqq \operatorname{Ind} B_2=1\,.$$

IX) It is to be noticed that the above observation contains the assertion: Both C_P and C_Q are open in I. Since $E_B=I-(C_P\cup C_Q)$, E_B is closed in I.

X) Suppose that E_B is not nowhere dense in I. Then by the closedness of E_B, E_B contains a closed interval $I'\subset I$. To prove $\rho(\pi^{-1}(I')\cap P)$ is not cofinal assume the contrary. Then there exists a positive number ε such that

$$\rho(\{p: p\in\pi^{-1}(I')\cap P,\ [\pi^{-1}(S_\varepsilon(\pi(p)))]_{\rho(p)}\subset P\})$$

is cofinal. Then next there exists a closed sub-interval I'' of I' whose length is $\varepsilon/4$ such that

$$\{\alpha: [\pi^{-1}(I'')]_\alpha\subset P\}$$

is cofinal. We have now $I''\subset C_P$ and hence $I''\cap E_B=\phi$, a contradiction. Thus $\rho(\pi^{-1}(I')\cap P)$ is not cofinal. By the same reason $\rho(\pi^{-1}(I')\cap Q)$ is not cofinal.

Hence there exists $\eta \in J$ such that

$$B_3 = [\pi^{-1}(I')]_{(\eta, \omega_1)} \subset B .$$

Since B_3 is closed and $\operatorname{Ind} B_3 = 1$ by Lemma 1, $\operatorname{Ind} B \geqq 1$.

XI) Let us consider the last case when $C_P \cap C_Q = \phi$ and E_B is nowhere dense in I. Set

$$a = \sup C_P ,$$

$$b = \inf C_Q .$$

Since C_P and C_Q are disjoint open sets, $0 < a < 1$ and $0 < b < 1$. If $a < b$, then E_B contains the interval $[a, b]$, a contradiction. Hence $b \leqq a$ and $a \in E_B$.

Let $a_1, a_2, \cdots$ be a monotonically increasing sequence of I such that i) $\sup a_i = a$ and ii) every $a_i \in C_P$. Let $b_1, b_2, \cdots$ be a monotonically decreasing sequence of I such that i) $\inf b_i = a$ and ii) every $b_i \in C_Q$. Such a sequence exists because we are now considering the case when E_B is nowhere dense. Let c be a point of $f^{-1}(a)$. Let η_1 be an ordinal $< \omega_1$ such that $[\pi^{-1}(a)]_{\eta_1} \neq \phi$. Set

$$J_1 = \{\alpha : \eta_1 < \alpha < \omega_1, \ [\pi^{-1}(a)]_\alpha \in \overline{\pi^{-1}(I) \cap P} \cap \overline{\pi^{-1}(I) \cap Q}\} .$$

To see that J_1 is cofinal in J let α_0 be an arbitrary ordinal with $\eta_1 < \alpha_0$. Then there exist a monotonically increasing sequence $\alpha_0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots$, a sequence of points $p_i \in \pi^{-1}(a_i) \cap P$ and a sequence of points $q_i \in \pi^{-1}(b_i) \cap Q$ such that i) $\rho(p_i) = \alpha_i$ for every i and ii) $\rho(q_i) = \beta_i$ for every i. Then $\sup \alpha_i \in J_1$. It is almost evident that J_1 is closed in J.

XII) For every point $p \in \pi^{-1}(I) \cap P$ there exists a finite subset K_p of M such that

$$[\pi^{-1}(f^{-1}(\pi(p)) \times (M - K_p) \times I)]_{\rho(p)} \subset P .$$

For every point $q \in \pi^{-1}(I) \cap Q$ there exists a finite subset K_q of M such that

$$[\pi^{-1}(f^{-1}(\pi(q)) \times (M - K_q) \times I)]_{\rho(q)} \subset Q .$$

Set

$$M_1 = \cup \{K_p : p \in \pi^{-1}(I) \cap P\} ,$$

$$M_2 = \cup \{K_q : q \in \pi^{-1}(I) \cap Q\} .$$

Since $|\pi^{-1}(I)| = \mathfrak{c}$, $|M_1| \leqq \mathfrak{c}$ and $|M_2| \leqq \mathfrak{c}$. Hence

$$M - (M_1 \cup M_2) \neq \phi .$$

Take an arbitrary element μ from $M - (M_1 \cup M_2)$, an arbitrary ordinal γ from J_1 and an arbitrary point x from I_γ. Then $t = (c, \mu, x)$ is a point of $T - I$.

We want to show that

$$[\pi^{-1}(t)]_\gamma \in \bar{P} \cap \bar{Q} .$$

Let U be an arbitrary open neighborhood of c in C, ε an arbitrary positive number and ξ an arbitrary ordinal with $\eta_1 \leqq \xi < \gamma$. Consider a basic neighborhood

$$V=[\pi^{-1}(U\times\{\mu\}\times S_\varepsilon(x))]_{(\xi,\gamma]}$$

of the point $[\pi^{-1}(t)]_\gamma$ in Z and let us prove that V meets both P and Q. Since $f(U)$ is an open neighborhood of a,

$$W=[\pi^{-1}(f(U))]_{(\xi,\gamma]}$$

is a relatively open neighborhood of $[\pi^{-1}(a)]_\gamma$ in $\pi^{-1}(I)$. Hence W meets both P and Q. Take p_0 from $W\cap P$ and q_0 from $W\cap Q$. Then $f^{-1}(\pi(p_0))\cap U\neq\phi$ and $f^{-1}(\pi(q_0))\cap U\neq\phi$. Since $I_{\rho(p_0)}$ and $I_{\rho(q_0)}$ are dense in I, V meets both P and Q. Hence $[\pi^{-1}(t)]_\gamma\subset\bar{P}\cap\bar{Q}=B$. Since x was an arbitrary point of I_γ,

$$[\pi^{-1}(\{c\}\times\{\mu\}\times I)]_\gamma\subset B.$$

Therefore

$$B_4=[\pi^{-1}(\{c\}\times\{\mu\}\times I)]_{J_1}\subset B.$$

Since B_4 is closed in Z,

$$\operatorname{Ind} B \geqq \operatorname{Ind} B_4 = 1.$$

Thus the proof is completely finished.

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Ehime University

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