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A study concerning Blackwell's example in Markov chain theory

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§ 1. Introduction.

Let S be a countable set and let $P_t(.,.)$ be a (*stationary*) *transition function* on S with a continuous parameter $t \in [0, \infty)$, that is, a system satisfying the following conditions:

(P_1) $P_t(a, b)$ is measurable in $t>0$ for any fixed a and b in S.

(P_2) For each a of S and $t>0$, $\sum_{b\in S} P_t(a, b)=1$.

(P_3) For any $s>0$, $t>0$, $a\in S$ and $b\in S$,

$$P_{s+t}(a, b)=\sum_{c\in S} P_s(a, c)P_t(c, b)$$

(P_4) $P_0(a, b)=\delta(a, b)$ and $\lim_{t\to 0} P_t(a, b)=\delta(a, b)$,

where $\delta(a, a)=1$ and $\delta(a, b)=0$ if $a\neq b$. It is known that there exists the derivative of $P_t(a, a)$ at $t=0$:

$$\lim_{t\to 0}\frac{1}{t}(1-P_t(a, a))=q(a)\leqq +\infty .$$

Following P. Lévy [11], we shall say the state a is *stable or instantaneous* according as $q(a)<+\infty$ or $q(a)=+\infty$.

Usually a well-separable and measurable standard modification $\hat{x}_t$ is chosen [4] as the sample process corresponding to the transition function $P_t(.,.)$. Such $\hat{x}_t$ is suitable for the study of the case where all the states are stable[1]. But if there are instantaneous states, $\hat{x}_t$ has many irregular properties at those states. We have no systematic method to study, for instance, processes with only instantaneous states, several examples of which have been given by Blackwell [1], Dobrusin [5], Feller-McKean [6] and others. Now let us recall Feller-McKean's example. They considered a diffusion y_t with the state space $[0, 1]$ and with some specified transition function $Q_t(.,.)$. Then the restriction of y_t to the set S of all rational points in $[0, 1]$ gives a standard modification $\hat{x}_t$ corresponding to the restriction of $Q_t(.,.)$ to S. In this case, the state space

1) Indeed, even in this case, it is effective to consider an enlarged space (see H. Kunita, J. Math. Soc. Japan, 14 (1962), 66-100).

S of $\hat{x}_t$ has a natural enlarged space and the related properties of $\hat{x}_t$ can be derived from those of a well-studied process y_t defined on the enlarged space.

A few years ago D. Ray [14], starting from any transition function on a measurable space, gave a general method of constructing a strong Markov process with right continuous paths on an enlarged topological space, whose transition function is a natural (unique) extension of the original one. In this paper, by some modification of Ray's method, we shall construct a Markov process x_t defined on the Cantor set S_c which plays the same role for Blackwell's example as y_t for Feller-McKean's one.

As will be seen in Section 2, our approach is applicable for a large class of transition functions on a countable set and the extended process, defined on a compact set, is *strongly-Feller* in the terminology of Girsanov [7]. At this stage it seems to be useful to mention the relationship of Ray's approach with ours, although it is not needed for the later sections. In Section 3 it is proved that the process x_t of the above paragraph is strongly continuous. In Sections 4 and 5, we shall investigate some properties of x_t such as continuity of path functions, recurrence and the existence of invariant measures. In Appendix, the *strong recurrence* will be discussed for a general extended process obtained by Ray's method.

Finally it would be desirable to obtain a rather general condition on the original transition function $P_t(.,.)$ for its extended process to be strongly continuous and also strongly-Feller; this question remains open[2].

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§ 2. General procedure. Definition of a Hunt process.

Let $P_t(a, b)$ be any transition function on a countable set S and let R be the set of all rationals in $(0, \infty)$. Consider a finite measure m on $R \times S$ such that $m(r, b) > 0$ for any $(r, b) \in R \times S$ and define

$$d(a, a') = \sum_{(r,b) \in R \times S} | P_r(a, b) - P_r(a', b) | \, m(r, b) .$$

It follows that d is a metric on S (from (P_4)) and that the completion S^* of S by d is a compact metric space. It is easy to show that S^* is independent of the choice of m. By definition, $P_r(., b)$ has the unique continuous extension $P_r^*(., b)$ to S^*. Let S_R^* be the set of all points ξ in S^* such that $\sum_{b \in S} P_r^*(\xi, b) = 1$ for every $r \in R$. In general, for a topological space X, the set of all real

2) We note that, in some special cases (for instance, Example 3 of [4, Part II, § 20]), the strong continuity of the extended processes is easily verified.

bounded continuous functions is denoted by $\boldsymbol{C}(X)$ and the set of all real bounded Baire functions, by $\boldsymbol{B}(X)$. It is not difficult to prove that, for any $r \in R$, the operator P_r^*, defined by $P_r^* f(\xi) = \sum_{b \in S} P_r^*(\xi, b) f(b)$, maps $\boldsymbol{B}(S_R^*)$ into $\boldsymbol{C}(S_R^*)$. From this it follows that, for any $t \in (0, \infty)$ and $b \in S$, $P_t(., b)$ has a (clearly unique) continuous extension $P_t^*(., b)$ to S_R^*. In fact, the function

$$\sum_{c \in S} P_r^*(\xi, c) P_{t-r}(c, b) \qquad (r \in R,\ r < t) \tag{1.1}$$

is a ξ-continuous function on S_R^* which coincides with $P_t(., b)$ on S. Also (1.1) implies that $P_t^*(\xi, b)$ is right t-continuous on $(0, \infty)$, that $P_t^*(\xi, b)$ satisfies the Chapman-Kolmogorov equation:

$$P_{s+t}^*(\xi, b) = \sum_{c \in S} P_s^*(\xi, c) P_t^*(c, b) \quad \text{for every} \quad s, t > 0\,. \tag{1.2}$$

and that, for any $t > 0$, $P_t^* f(\xi) = \sum_{b \in S} P_t^*(\xi, b) f(b)$ maps $\boldsymbol{B}(S_R^*)$ into $\boldsymbol{C}(S_R^*)$. In other words, the semi-group P_t^* is *strongly-Feller* on S_R^*. Unfortunately, this fact is still insufficient to assure the existence of a well behaved Markov process corresponding to $P_t^*(\xi, b)$. For such discussion, we introduce the following condition:

(A) There is a sequence of finite sets K_n increasing to S such that, for any fixed $r \in R$, the convergence of $\sum_{b \in K_n} P_r(a, b)$ to 1 is uniform in $a \in S$.

With this condition it is obvious that $S^* = S^*_R$. Now, for $\alpha > 0$, define

$$G_\alpha^*(\xi, b) = \int_0^\infty e^{-\alpha t} P_t^*(\xi, b)\, dt\,, \tag{1.3}$$

$$G_\alpha^* f(\xi) = \sum_{b \in S} G_\alpha^*(\xi, b) f(b) = \int_0^\infty e^{-\alpha t} P_t^* f(\xi)\, dt\,. \tag{1.4}$$

Then G_α^* maps $\boldsymbol{B}(S^*)$ into $\boldsymbol{C}(S^*)$ and the function family $\{G_1^*(., b)\,;\, b \in S\}$ separates points of S^* (from the right t-continuity of $P_t^*(\xi, b)$). Therefore, according to Theorem III of [**14**], there is a strong Markov process x_t with right continuous paths whose transition probability is $P_t^*(.,.)$. For short, such a process will be called a *Ray process*[3].

We shall now compare our approach with Ray's completion in his paper [**14**, § 5]. Analogously to (1.3), define

$$G_\alpha(a, b) = \int_0^\infty e^{-\alpha t} P_t(a, b)\, dt, \quad \alpha > 0, \quad a, b \in S \tag{1.5}$$

and consider a finite measure m' on S such that $m'(b) > 0$ for every b in S. Let d' be a metric (from (P_4)) defined by

3) We shall omit the precise description of a Ray process (see [14]).

$$d'(a, a') = \sum_{b \in S} | G_1(a, b) - G_1(a', b) | \, m'(b) .$$

Completing S by d', we get a compact metric space $\hat{S}$ independent of the choice of m' and the continuous extension $\tilde{G}_1(., b)$ of $G_1(., b)$ to $\hat{S}$ for each $b \in S$. Define $\hat{S}_R = \{\xi \in \hat{S}$ such that $\sum_{b \in S} \tilde{G}_1(\xi, b) = 1\}$. Then it follows from the resolvent equation that $G_\alpha(., b)$, $\alpha > 0$, has the continuous extension $\tilde{G}_\alpha(., b)$ to $\hat{S}_R$ and that $\tilde{G}_\alpha$ maps $\boldsymbol{B}(\hat{S}_R)$ to $\boldsymbol{C}(\hat{S}_R)$. Ray proved there is a Ray process $\tilde{x}_t$ on $\hat{S}_R$ whose transition function $\tilde{P}_t(\xi, b)$ is uniquely determined by the relation $\tilde{G}_\alpha(\xi, b) = \int_0^\infty e^{-\alpha t} \tilde{P}_t(\xi, b) dt$ we have

LEMMA 1. *There is a natural continuous mapping φ from S_R^* to $\hat{S}_R$ which is the identity mapping on S. Moreover this mapping is a one to one correspondence between S_R^* and $\varphi(S_R^*)$.* (For the proof, see [Kunita and Nomoto, **10**].)

This lemma implies

THEOREM 1. *Suppose the condition* (A) *is satisfied. Then our system* $\{S^* = S_R^*, P_t^*(\xi, b), G_\alpha^*(\xi, b), x_t\}$ *coincides with Ray's system* $\{\tilde{S} = \hat{S}_R, \tilde{P}_t(\xi, b), \tilde{G}_\alpha(\xi, b), \tilde{x}_t\}$.

We note that this gives a partial answer to the open question which was presented by Ray [**14**, p. 67].

Next we shall assume another condition:

(B) P_t^* is a strongly continuous semi-group. (Under the condition (A), this condition is equivalent to that, for any $f \in \boldsymbol{C}(S^*)$, $P_t^* f(\xi) \to f(\xi)$ as $t \to 0$ for every ξ.)

With (B), our Ray process x_t becomes a *Hunt process* in the terminology of [Blumenthal, Getoor and McKean, **2**]. We shall now give a strict description of Hunt processes, since it will be needed for discussion of the following sections.

Let E be a locally compact space with a countable base and E^* be a space obtained by adding an isolated point ∞ to E. Let W be a set of t-functions $w(t)$ from $[0, \infty]$ to E^* (called paths) which is right continuous and has a left-hand limit and satisfies $w(s) = \infty$ whenever $w(t) = \infty$ and $s \geqq t$. The t-coordinate of a path $w(t)$ is denoted by $x_t(w)$ or simply x_t. Let $\tilde{\mathscr{B}}(\tilde{\mathscr{B}}^*)$ be the topological Borel field over $E(E^*)$. Let $\tilde{\mathscr{F}}_t$ denote the σ-field of subsets of W generated by the sets $\{x_s \in B\}$ with $s \leqq t$ and $B \in \tilde{\mathscr{B}}^*$, and let $\tilde{\mathscr{F}}$ be the σ-field generated by the union of $\tilde{\mathscr{F}}_t$. We suppose given a system of probability measures $P_a(a \in E^*)$ on $\tilde{\mathscr{F}}$ which satisfies the following conditions:

(i) For each $A \in \tilde{\mathscr{F}}$, $P_a(A)$ is $\tilde{\mathscr{B}}^*$ measurable and $P_a\{w : x_0(w) = a\} = 1$.

(ii) For each a in E, $t \geqq 0$, A in $\tilde{\mathcal{F}}_t$ and bounded $\tilde{\mathcal{F}}$ measurable f
$E_a(f(\theta_t w): A) = E_a(E_{x_t} f(w): A)$[4]

where $\theta_t w(s) = w(t+s)$. The system $X = (x_t, \tilde{\mathcal{F}}_t, P_a)$ (simply denoted by X or x_t) is called a *Markov process* over E.

Let μ be a finite measure on $\tilde{\mathcal{B}}$ and we set $P_\mu(A) = \int P_a(A)\mu(da)$ for A in $\tilde{\mathcal{F}}$. Let $\tilde{\mathcal{B}}^\mu$ and $\tilde{\mathcal{F}}^\mu(\tilde{\mathcal{F}}_t^\mu)$ be the completion of $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{F}}(\tilde{\mathcal{F}}_t)$ with respect to μ and P_μ respectively. $\mathcal{B} = \cap \tilde{\mathcal{B}}^\mu$, $\mathcal{F} = \cap \tilde{\mathcal{F}}^\mu$ $(\mathcal{F}_t = \cap \tilde{\mathcal{F}}_t^\mu)$ denote the intersections of $\tilde{\mathcal{B}}^\mu$, $\tilde{\mathcal{F}}^\mu(\tilde{\mathcal{F}}_t^\mu)$ as μ ranges over all finite measures. The measures P_a naturally extended to $\mathcal{F}$ and with this extension $P_a(A)$ is $\mathcal{B}$-measurable in a for each $A \in \mathcal{F}$. The Markov property (ii) remains valid if we assume f is $\mathcal{F}$ measurable and A is in $\mathcal{F}_t$. The function σ from W to $[0, \infty]$ is called a *Markov time* if $\{\sigma < t\} \in \mathcal{F}_t$ for each $t > 0$. Given a Markov time σ we denote by $\mathcal{F}_{\sigma+}$ the σ-field generated by the sets A such that $A \cap \{\sigma < t\} \in \mathcal{F}_t$ for any $t > 0$. A Markov process $X = (x_t, \mathcal{F}_t, P_a)$ is called a *strong Markov process* if for each Markov time σ, $A \in \mathcal{F}_{\sigma+}$ and bounded $\mathcal{F}$ measurable f

(ii)′ $E_a(f(\theta_\sigma w): A, \sigma < \infty) = E_a(E_{x_\sigma} f(w): A, \sigma < \infty)$.

A Markov process X is called a *Hunt process* if it is a strong Markov process and if σ_n is an increasing sequence of Markov times with limit σ then $x(\sigma_n) \to x(\sigma)$ almost everywhere P_a on $\{\sigma < \infty\}$.

§ 3. The Blackwell process.

Let $S^{(n)} = \{0, 1\}$ for every $n = 1, 2, \cdots$ and let $S_c = \prod_{n=1}^{\infty} S^{(n)}$ together with weak topology; S_c is the Cantor set. Points of S_c are denoted by $a, b, \cdots$ and their n-th coordinates by $a(n), b(n), \cdots$:

$$a = (a(1), a(2), \cdots, a(n), \cdots), \text{ where } a(n) = 0 \text{ or } 1.$$

Let S be a subset of S_c whose point has only finitely many 1's in its coordinate $a(n)$.

Blackwell defined a transition function $P_t^{(n)}(a(n), b(n))$ on $S^{(n)}$ as follows:

$$\tag{3.1} \begin{cases} P_t^{(n)}(0, 0) = \dfrac{q^{(n)}(1)}{r(n)} + \dfrac{q^{(n)}(0)}{r(n)} e^{-r(n)t} \\ P_t^{(n)}(1, 1) = \dfrac{q^{(n)}(0)}{r(n)} + \dfrac{q^{(n)}(1)}{r(n)} e^{-r(n)t} \\ P_t^{(n)}(0, 1) = 1 - P_t^{(n)}(0, 0) \\ P_t^{(n)}(1, 0) = 1 - P_t^{(n)}(1, 1), \end{cases}$$

4) $E_a(f(w): A) = \int_A f(w) P_a(dw)$.

where $0 < q^{(n)}(0)$, $q^{(n)}(1) < +\infty$ and $r(n) = q^{(n)}(0) + q^{(n)}(1)$. Then he defined a function $P_t(a, b)$ by

$$(3.2) \qquad P_t(a, b) = \prod_{n=1}^{\infty} P_t^{(n)}(a(n), b(n)) \quad t > 0, \quad a, b \in S\,.$$

Under a condition

$$(\mathrm{C}_1) \qquad \sum_{n=1}^{\infty} \frac{q^{(n)}(0)}{r(n)} < \infty$$

Blackwell showed that $P_t(a, b)$ is a transition function on S and all the states are instantaneous if and only if

$$(3.3) \qquad \sum_{n=1}^{\infty} q^{(n)}(0) = \infty\,.$$

Under the condition (C_1) we can consider the system $\{S_R^*, P_t^*(a, b), G_\alpha^*(a, b)\}$ defined for the transition function $P_t(a, b)$.

Denote by K_M the set of all points a with $a(n) = 0$ for all $n \geqq M+1$. Then the computation in [1] shows that

$$(3.4) \qquad \sum_{b \in K_M} P_t(a, b) = \prod_{n=M+1}^{\infty} P_t^{(n)}(a(n), 0) \geqq \prod_{n=M+1}^{\infty} \frac{q^{(n)}(1)}{r(n)} (1 - e^{-r(n)t})$$

holds for all $a \in S$ and each $t > 0$. We now prove a theorem concerning to this system.

THEOREM 2. *Suppose the condition*

$$(\mathrm{C}_2) \qquad \sum_{n=1}^{\infty} e^{-r(n)t} < +\infty \text{ for each } t > 0$$

then both condition (A) *and* (B) *are satisfied and the Ray's system* $\{\hat{S} = \hat{S}_R, \tilde{P}_t(a, b), \tilde{G}_\alpha(a, b), \tilde{x}_t\}$ *gives a Hunt process* X *over a compact metric space* $\hat{S}$ *which is homeomorphic to the Cantor set* S_c.

The condition (A) follows at once from the estimation (3.4) and the assumption (C_2). To prove the latter parts of the theorem, we shall prepare a lemma.

LEMMA 2. S^* *is homeomorphic to the Cantor set* S_c.

PROOF. 1°. Let φ be the identity mapping from S in S^* to S in S_c. For any point $\xi \in S^*$, we can find a sequence $\{a_p\}$ such that $a_p \in S$, $d(a_p, \xi) \to 0$ as $p \to \infty$. We shall show that a_p $(= \varphi(a_p))$ tends to some point a in S_c. Assume that there are two infinite subseqeuces $\{a_{p_i}\}$ and $\{a_{q_i}\}$ of $\{a_p\}$ and some positive integer k such that $a_{p_i}(k) = 1$ and $a_{q_i}(k) = 0$ for all i.

Let b, b' be two points of S such that $b(k) = 0$, $b'(k) = 1$ and $b(n) = b'(n)$ for $n \neq k$. Then we have

$$\frac{P_t(a_p, b)}{P_t(a_p, b')} = \frac{P_t^{(k)}(a_p(k), b(k))}{P_t^{(k)}(a_p(k), b'(k))}$$

As a_p tends to ξ along a_{p_i} or a_{q_i}, the left-hand side of above equality converges to the same limit so that we have

$$\frac{P_t^{(k)}(1, 0)}{P_t^{(k)}(1, 1)} = \frac{P_t^{(k)}(0, 0)}{P_t^{(k)}(0, 1)}.$$

But by definition of these factors, we can rewrite this relation as $0 = e^{-r(k)t}$ which is impossible. Therefore, for each k, there is some p_k such that

$$a_{p_k}(k) = a_{p_k+1}(k) = a_{p_k+2}(k) = \cdots.$$

We define

$$\varphi(\xi) = (a_{p_1}(1),\ a_{p_2}(2), \cdots).$$

It is easily seen $\varphi(\xi)$ is independent of the choice of $\{a_p\}$ and a continuous map from S^* into S_c.

2°. We now prove that φ is a one to one map. Let ξ, η be two different points in S^*. Then by the definition of $d(.,.)$, there exist some $r \in R$ and some $b \in S$ such that $P_r^*(\xi, b) < P_r^*(\eta, b)$. But since both $P_r^*(\xi,.)$ and $P_r^*(\eta,.)$ are probability measures on S, we can find another point b' from S such that $P_r^*(\xi, b') > P_r^*(\eta, b')$. Let $a_p \to \xi$, $b_p \to \eta$ as $p \to +\infty$ and $b(k) = b'(k) = 0$ for $k \geqq M+1$.

If $\varphi(\xi) = \varphi(\eta)$ then the results in 1° imply that there exists an N such that

$$a_p(k) = b_p(k) \quad (k = 1, 2, \cdots, M),$$

$$P_r(a_p, b) < P_r(b_p, b),\ P_r(a_p, b') > P_r(b_p, b')$$

hold for all $p \geqq N$. Therefore

$$\frac{P_r(a_p, b)}{P_r(a_p, b')} < \frac{P_r(b_p, b)}{P_r(b_p, b')}$$

that is

$$\frac{\prod_{k=1}^{M} P_r^{(k)}(a_p(k), b(k))}{\prod_{k=1}^{M} P_r^{(k)}(a_p(k), b'(k))} < \frac{\prod_{k=1}^{M} P_r^{(k)}(b_p(k), b(k))}{\prod_{k=1}^{M} P_r^{(k)}(b_p(k), b'(k))}$$

for all $p \geqq N$. But this is impossible since $a_p(k) = b_p(k)$ for all $p \geqq N$ and $k = 1, 2, \cdots, M$. That is $\xi \neq \eta$ implies $\varphi(\xi) \neq \varphi(\eta)$. Now, it is obvious that S^* is homeomorphic to S_c since both S^*, S_c are compact.

REMARK. Lemma 2 shows S^* is homeomorphic to $\hat{S}$ so that $\hat{S}$ is homeomorphic to S_c. Moreover, define $\bar{P}_t(a, b) = \prod_{n=1}^{\infty} P_t^{(n)}(a(n), b(n))$ for $a \in S_c$ and $b \in S$. Then it is easily seen that $\bar{P}_t(., b)$ is continuous on S_c and $\bar{P}_t(a, b) = P_t^*(\varphi^{-1}(a), b)$.

PROOF OF THEOREM 2. On account of Lemma 2, we shall not distinguish S^* from S_c in the following arguments. We will also identify $P_t^*(.,.)$ with $\bar{P}_t(.,.)$. Let f be a tame function on S_c:

$$f(\xi)=f(\xi(1), \xi(2), \cdots, \xi(p)).$$

Let a be a point of S and K_M, the set defined before. Then,

$$P_t^* f(a)=\sum_{b\in S} P_t(a, b)f(b)=\lim_{M\to\infty}\sum_{b\in K_M}\Big[\prod_{n=1}^{\infty} P_t^{(n)}(a(n), b(n))\Big]\cdot f(b(1), b(2), \cdots, b(p))$$

$$=\lim_{M\to\infty}\prod_{n=M+1}^{\infty} P_t^{(n)}(a(n),0)\sum_{b(1),\cdots,b(p)}\Big[\prod_{n=1}^{p} P_t^{(n)}(a(n), b(n))\Big]\cdot f(b(1), \cdots, b(p))\times$$

$$\times\sum_{b(p+1),\cdots,b(M)}\prod_{n=p+1}^{M} P_t^{(n)}(a(n), b(n))$$

$$=\lim_{M\to\infty}\prod_{n=M+1}^{\infty} P_t^{(n)}(a(n), 0)\sum_{b(1),\cdots,b(p)}\prod_{n=1}^{p} P_t^{(n)}(a(n), b(n))\cdot f(b(1), \cdots, b(p)),$$

that is,

$$P_t^* f(a)=\sum_{b(1),\cdots,b(p)}\prod_{n=1}^{p} P_t^{(n)}(a(n), b(n))f(b(1), \cdots, b(p)), \text{ for } a\in S.$$

Since both sides are continuous in $a\in S_c$, therefore we have

$$P_t^* f(\xi)=\sum_{b(1),\cdots,b(p)}\prod_{n=1}^{p} P_t^{(n)}(\xi(n), b(n))f(b(1), \cdots, b(p)), \text{ for } \xi\in S_c. \tag{3.5}$$

This implies that

$$\lim_{t\to 0} P_t^* f(\xi)=f(\xi), \quad \text{for each } \xi\in S_c. \tag{3.6}$$

It is known that the set of all continuous tame functions on S_c is dense in $\boldsymbol{C}(S_c)$ in the sense of uniform convergence (see N. Bourbaki [**3**]) so that (3.6) holds for all $f\in\boldsymbol{C}(S_c)$. Therefore $P_t^*\to f$ (strongly) as $t\to 0$. This concludes the proof.

By Theorem 1 and 2, there exists a Hunt process $X=(x_t, \mathcal{F}_t, P_a)$ over S_c corresponding to the system $\{S^*, P_t^*(a, b), G_\alpha^*(a, b)\}$. For short, we shall call this *Blackwell process*.

REMARK. Let f be a tame function in $\boldsymbol{C}(S_c)$ then the expression (3.5) implies that

$$Af(a)=\lim_{t\to 0}\frac{1}{t}(P_t^* f(a)-f(a))$$

$$=\sum_{i=1}^{p}\Big\{f\Big(a(1), \cdots, a(i-1), 1-a(i), a(i+1), \cdots, a(p)\Big)$$

$$-f\Big(a(1), \cdots, a(p)\Big)\Big\}q^{(i)}(a(i)).$$

This suggests us that the Hille-Yoshida generator A of the semi-group P_t^* is of the form $A=\sum_{n=1}^{\infty} A_n$ (symbolically) where A_n is the generator of the semi-group which corresponds to the transition function $P_t^{(n)}(.,.)$ on $S^{(n)}=\{0,1\}$. But it seems to be difficult to determine the domain of the generator A.

§ 4. Sample function properties.

Let W' be the set of all continuous t-functions $w(t)$ over S_c. Then we have

THEOREM 3. *W' can not be taken as the basic space of the Blackwell process.*

PROOF. The topology of the Cantor set S_c is defined by a metric

$$\rho(a, b)=\sum_{n=1}^{\infty}\frac{1}{2^n}\,|\,a(n)-b(n)\,|, \quad a, b \in S_c\,.$$

By a theorem of L. V. Seregin [**15**, Theorem 3.4], it is enough to show that the relation

$$\int_0^{c-h} P_a^*(\rho(x_t, x_{t+h})>\varepsilon)dt=o(h),\ h\downarrow 0$$

is false for some $\varepsilon>0$, $c>0$ and for some a in S.

First, note that

$$\lim_{h\to 0}\frac{1}{h}P_h^*(a, b)=\begin{cases} q^{(n)}(0) \text{ if } a(n)=0,\ b(n)=1 \text{ and } a(k)=b(k)\ (k\neq n)\,,\\ q^{(n)}(1) \text{ if } a(n)=1,\ b(n)=0 \text{ and } a(k)=b(k)\ (k\neq n)\,,\\ 0 \qquad \text{otherwise.}\end{cases}$$

Let ε be $0<\varepsilon<\frac{1}{2}$ and c be any positive constant. Then, for a point a in S,

$$\begin{aligned}\frac{1}{h}\int_0^{c-h} P_a^*(\rho(x_t, x_{t+h})>\varepsilon)dt &\geqq \frac{1}{h}\sum_{b\in S}P_h^*(b,\ V_\varepsilon(b))\int_0^{c-h}P_t^*(a, b)dt\\ &\geqq \frac{1}{h}P_h^*(a,\ V_\varepsilon(a))\int_0^{c-h}P_t^*(a, a)dt\end{aligned}$$

where $V_\varepsilon(b)=\{x : x\in S_c,\ \rho(b, x)\geqq\varepsilon\}$. By definition of the metric ρ, $a'=(1-a(1), a(2), \cdots)$ is in $V_\varepsilon(a)$ and therefore it holds that

$$\liminf_{h\to 0}\frac{1}{h}\int_0^{c-h} P_a^*(\rho(x_t, x_{t+h})>\varepsilon)dt\geqq q^{(1)}(a(1))\int_0^c P_t^*(a, a)dt>0\,.$$

This proves the theorem.

Now assume the condition (3.3) holds then every a in S is instantaneous as was remarked in that place. In this case, the t-set $\mathfrak{z}_a(w)=\{t : x_t(w)=a\}$ is nowhere dense in $(0,\infty)$, which is obtained in P. Lévy [**11**], so that we have $P_a^*\{[0,\infty)-\bigcup_{a\in S}\mathfrak{z}_a(w)$ is everywhere dense in $[0,\infty)\}=1$. This means that the added points S^*-S are essentially used to construct the Blackwell process in § 3. More generally, whenever a in S is instantaneous we see that $\hat{S}-S$ is not empty and the above fact remains valid for the Ray process. On the other hand, for the Blackwell process for which the condition (3.3) is false, we do not know any role of S^*-S.

§ 5. Recurrence and invariant measure.

Let X be the Blackwell process. For an open or closed subset D of S_c, define the hitting time $\sigma_D(w)$ as the infimum of t for which $x_t(w)$ is in D, or $\sigma_D(w)=\infty$ if there are no such t. σ_D is known to be a Markov time. By the estimation analogous to (3.4) and by the continuity of $P_t^*(a, b)$ with respect to a, we have $P_t^*(a, b)>0$ for any $t>0$, $a\in S_c$ and $b\in S$. From this it follows that, for any a of S_c and any open subset U of S_c, $P_a^*(\sigma_U<\infty)>0$. Since x_t is a strongly-Feller process on the compact set S_c and P_t^* is strongly continuous on $\boldsymbol{C}(S_c)$, it follows that, for any compact set K and for any bounded fuction f, $E_a(f(x_{\sigma_K});\sigma_K<+\infty)$ is continuous in a on the set S_c-K (see [7]). Therefore some general results in [**9**; **12**; **16**; **17**] can be applied and we have:

(α) All points of S_c are recurrent in the sense of [**17**], that is, for any a of S_c and for any open neighbourhoods U, V of a such that $\bar{V}\subset U$

$$P_a^*(\sigma_V(\theta_{\sigma_{U^c}}(w))<+\infty/\sigma_{U^c}<+\infty)=1\,.$$

(β) There is a measure m such that, for any function $f\in\boldsymbol{C}(S_c)$,

$$\lim_{t\to\infty}P_t^*f(a)=\int f(b)m(db)\,.$$

This measure m is the unique invariant probability measure, that is, $m(S_c)=1$ and $m(A)=\int_{S_c}m(da)P_t^*(a, A)$, for every $t>0$ and for every $A\subset S_c$. But in our case, we can get a little stronger results than (α) for points in S and also the concrete expression m in (β), which is unique, as will be discussed in the following.

We will call a point a is a *strongly recurrent* if $\bar{P}_a^*(\sigma_a(\theta_{\sigma_{U^c}}(w))<+\infty/\sigma_{U^c}<+\infty)=1$ holds for any open set U containing a, where $\sigma_a=\sigma_{\{a\}}$. Then we have the following theorem.

THEOREM 4. *Each point a of S is strongly recurrent.*

To prove this, we shall give a lemma.

LEMMA 3. *Let a be any point of S and let ξ be any point of S_c. Then $E_\xi(\sigma_a)<+\infty$.*

PROOF. Since $P_t^*(\xi, a)$ is a positive continuous function of ξ on the compact S_c, this has positive minimum α. Therefore, we get the following estimations.

$$\sup_{\xi\in S_c}P_\xi^*(\sigma_a>t)=1-\inf_{\xi\in S_c}P_\xi^*(\sigma_a\leqq t)\leqq 1-\inf_{\xi\in S_c}P_\xi^*(x_t=a)\leqq 1-\alpha\,,$$

$$P_\xi^*(\sigma_a > 2t) = P_\xi^*(\sigma_a > t,\ \sigma_a(\theta_t w) > t) = E_\xi\{P_{x_t}^*(\sigma_a > t) : \sigma_a > t\} \leqq (1-\alpha)^2,\ \cdots,$$

$$P_\xi^*(\sigma_a > nt) \leqq (1-\alpha)^n$$

for arbitrary integers n and $\xi \in S_c$. Thus we get

$$P_\xi^*(\sigma_a = +\infty) = \lim_{n\to\infty} P^*(\sigma_a > nt) = 0\,.$$

Furthermore we have

$$E_\xi(\sigma_a) = \sum_{n=1}^{\infty} \int_{[(n-1)t,nt)} s P_\xi^*(\sigma_a \in ds) \leqq \sum_{n=1}^{\infty} nt\, P_\xi^*(\sigma_a > (n-1)t)$$

$$\leqq t \sum_{n=1}^{\infty} n(1-\alpha)^{n-1} < +\infty\,.$$

This completes the proof.

PROOF OF THEOREM 4. Let a be an arbitrary point of S and U be an open set containing a. Then Lemma 3 implies that

$$P_a^*(\sigma_a(\theta_{\sigma_{U^c}}(w)) < +\infty,\ \sigma_{U^c} < +\infty) = E_a\{P_{x(\sigma_{U^c})}^*(\sigma_a < +\infty) : \sigma_{U^c} < +\infty\}$$

$$= P_a^*(\sigma_{U^c} < +\infty)\,.$$

This means a is strongly recurrent.

Here is an alternative proof of Theorem 4. For any point a of S, it is clear from (C_1) that $P_t^*(a, a) > \alpha$, where α is a positive constant independent of t. Therefore $\int_0^\infty P_t^*(a, a)dt = +\infty$, so that a is strongly recurrent by the Theorem in Appendix.

Note that $P_t^{(n)}(.,.)$ has the (unique) invariant probability measure

$$m^{(n)}(0) = \frac{q^{(n)}(1)}{r(n)},\quad m^{(n)}(1) = \frac{q^{(n)}(0)}{r(n)}\,.$$

Define

$$m(a) = \prod_{n=1}^{\infty} m^{(n)}(a(n)),\quad \text{for any}\quad a \in S_c\,.$$

(C_1) implies that $m(a) > 0$ if $a \in S$, $m(a) = 0$ if $a \notin S$ and $\sum_{a\in S} m(a) = 1$. Therefore

$$m(A) = \sum_{a\in A\cap S} m(a) \tag{5.1}$$

defines a probability measure on S_c.

THEOREM 5. *For any function f of $\boldsymbol{C}(S_c)$,*

$$\lim_{t\to\infty} P_t^* f(a) = \int_{S_c} f(b) m(db),\quad a \in S_c\,. \tag{5.2}$$

Therefore the measure m defined by (5.1), *gives the unique invariant probability measure for the Blackwell process (or for the original transition func-*

tion $P_t(a, b)$ on S).

PROOF. It is enough to show that (5.2) holds for every tame function f. Assume that f is determined by the first p coordinates. Then (3.5) implies that

$$\lim_{t\to\infty} P_t^* f(a) = \lim_{t\to\infty} \sum_{b(1),\cdots,b(p)} \prod_{n=1}^{p} P_t^{(n)}(a(n), b(n)) f(b(1), \cdots, b(p))$$

$$= \sum_{b(1),\cdots,b(p)} f(b(1), \cdots, b(p)) \prod_{n=1}^{p} m^{(n)}(b(n))$$

$$= \int_{S_C} f(b) m(db) .$$

This completes the proof.

Appendix.

Let $\{\tilde{S}_R, \tilde{P}_t(a, b), \tilde{G}_\alpha(a, b), \tilde{x}_t\}$ be the Ray process corresponding to a transition function $P_t(a, b)$ on S. Then we have following theorem.

THEOREM. *The following conditions are equivalent to each other:*

(i) *A point a of S is strongly recurrent.*

(ii) $\int_0^\infty P_t(a, a) dt = +\infty$.

(iii) $\tilde{P}_a\left\{\int_0^\infty \chi_{\{a\}}(x_t(w)) dt = +\infty\right\} = 1$, *where $\chi_{\{a\}}(.)$ is the indicator of the set $\{a\}$.*

(iv) $\tilde{P}_a\{$*the t-set $[t : x_t(w) = a]$ is unbounded*$\} = 1$.

We give a sketch of the proof. The equivalence among (ii), (iii) and (iv) are already obtained in K. L. Chung [4]. (iv) implies (i) clearly so that we show that (i) implies (ii). Let U be an open set containing a. We define

$$\begin{aligned} \tau(w) &= \sigma_{U^c}(w) , \\ \sigma_1(w) &= \tau(w) + \sigma_a(\theta_\tau(w)) && \text{if } \tau(w) < +\infty , \\ &= +\infty && \text{otherwise} , \\ \sigma_n(w) &= \sigma_{n-1}(w) + \sigma_1(\theta_{\sigma_{n-1}}(w)) \ (n \geqq 2) . \end{aligned}$$

We can take U such that $E_a(\sigma_{U^c}) < +\infty$ (see [8]). The assumption (i) and strong Markov property imply that

$$\tilde{P}_a(\sigma_1 < +\infty) = \tilde{P}_a(\tau < +\infty,\ \sigma_a(\theta_\tau(w)) < +\infty)$$
$$= E_a\{\tilde{P}_{x_\tau}(\sigma_a < +\infty) : \tau < +\infty)\} = \tilde{P}_a(\tau < +\infty) = 1$$

and

$$\tilde{P}_a(\sigma_n < +\infty) = [\tilde{P}_a(\sigma_1 < +\infty)]^n = 1 \quad (n \geqq 1)$$

so that

$$E_a(\varphi_\infty)=E_a\{\sum_{n=1}^{\infty}\varphi_{\tau(\theta_{\sigma_{n-1}}w)}(\theta_{\sigma_{n-1}}(w)):\ \sigma_{n-1}<+\infty\}$$
$$=\sum_{n=1}^{\infty}E_a\{E_{x(\sigma_{n-1})}(\varphi_\tau):\ \sigma_{n-1}<+\infty\}$$
$$=\sum_{n=1}^{\infty}E_a(\varphi_\tau)$$

where $\varphi_t(w)=\int_0^t\chi_{\{a\}}(x_s(w))ds$, and $\sigma_0=0$. Since $E_a(\varphi_\tau)>0$ (see [4, Part II, § 5]) so that we have $E_a(\varphi_\infty)=+\infty$.

This proves the theorem.

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References

[1] D. Blackwell, Another countable Markov process with only instantaneous states, Ann. Math. Statist., **29** (1958), 313-316.

[2] R.M. Blumenthal, R.K. Getoor, and H.P. McKean, Jr, Markov processes with identical hitting distributions, Illinois, J. Math., **6** (1962), 402-420.

[3] N. Bourbaki, Topologie Générale, Hermann, 1940-1949.

[4] K.L. Chung, Markov chains with stationary transition probabilities, Springer, 1960.

[5] R.L. Dobrusin, An example of a countable homogeneous Markov process all states of which are instantaneous, Theor. Probability Appl., **1** (1956) 481-485.

[6] W. Feller and H.P. McKean, A diffusion equivalent to a countable Markov chain, Proc. Nat. Acad. Sci. U.S.A., **42** (1956), 351-354.

[7] I.V. Girsanov, Strongly Feller processes I, Theor. Probability Appl., **5** (1960), 7-28.

[8] K. Ito, Stochastic processes, Tokyo, 1957. (Japanese)

[9] R.Z. Khas'minskii, Ergodic properties of recurrent diffusion processes and stabilization of the solution to the Cauchy problem for parabolic equation, Theor. Probability Appl., **5** (1960), 196-214.

[10] H. Kunita and H. Nomoto, A method of compactification concerning Markov processes and its applications, Seminar on Prob., **14** (1962), 1-126. (Japanese)

[11] P. Lévy, Systèmes markoviens et stationnaires. Cas dénombrable., Ann. Sci. École Norm. Sup., (3) **68** (1951), 327-381.

[12] G. Maruyama and H. Tanaka, Ergodic properties of N-dimensional recurrent Markov processes, Mem. Fac. Sci. Kyushu Univ. Ser. A, **13** (1959), 157-172.

[13] M. Nagasawa, The adjoint process of a diffusion with reflecting barrier, Kōdai Math. Sem. Rep., **13** (1961), 235-248.

[14] D. Ray, Resolvents, transition functions and strongly Markovian processes, Ann. of Math., **70** (1959), 43-72.

[15] L.V. Seregin, Continuity conditions for stochastic processes, Theor. Probability Appl., **6** (1961), 3-30.

[16] T. Ueno, On recurrent Markov processes, Kōdai Math. Sem. Rep., **12** (1960), 109-142.

[17] T. Watanabe, Some general properties of Markov processes, J. Inst. Polytech. Osaka City Univ. Ser. A, **9** (1958), 9-29.