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On characteristic roots of group commutators of non-singular matrices

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§ **1.** The purpose of this note is to give an analogue of the following result of Herstein [**1**].

"For non-singular matrices A, B, if $AB-BA$ commutes with A, then $C=ABA^{-1}B^{-1}-I$ is nilpotent."

We shall give a sufficient condition for the nilpotency of C in terms of the group-commutator $ABA^{-1}B^{-1}$ itself without taking recourse to additive commutators $AB-BA$.

Throughout this note we shall restrict ourselves to a non-singular pair A, B of $n\times n$ matrices for which $C=ABA^{-1}B^{-1}-I$ is supposed to commute with both A and B. We shall also take the scalars to belong to the complex-field always. We shall assume the terminology and results of [**2**].

§ **2.** We shall need the following results:

LEMMA 1. *If $C=ABA^{-1}B^{-1}-I$ commutes with A and B, then $AB^m-B^mA=[(C+I)^m-I]B^mA$, for all positive integers m.*

PROOF. $AB=(ABA^{-1}B^{-1})BA=(C+I)\cdot BA$, and a simple induction on m shows that

$$AB^m=(C+I)^mB^mA\,.$$

Hence, $AB^m-B^mA=[(C+I)^m-I]B^mA$. Q.E.D.

We shall now prove,

THEOREM. *If* (i) *$ABA^{-1}B^{-1}$ commutes with both A and B, and* (ii) *at least one of A and B does not have a complete set of m-*th *roots of any scalar amongst its characteristic roots for any integer m greater than one, then $C=ABA^{-1}B^{-1}-I$ is nilpotent.*

PROOF. We prove the theorem by induction on the degree n of the matrices. Let us suppose that B satisfies the hypothesis (ii) of the statement of the theorem. The result is trivial for $n=1$. Assume the validity of the theorem for all degrees less than n. We divide the proof in three parts.

(a) If all the characteristic roots of C are not identical, then let

$C=\begin{bmatrix} C_{11} & & 0 \\ & \ddots & \\ 0 & & C_{tt} \end{bmatrix}$, be a decomposition of C into primary components C_{kk} belonging to distinct characteristic roots of C. Thus the underlying vector space V decomposes with respect to C,

$$V=V_1\oplus\cdots\oplus V_t$$

such that the restriction of C to V_k is C_{kk}. Since each V_k corresponds to a distinct root of C, and C commutes with A and B, so each V_k is also invariant with respect to both A and B.

This implies that A and B simultaneously decompose into diagonal blocks similar to those of C:

$$A=\begin{bmatrix} A_{11} & & 0 \\ & \ddots & \\ 0 & & A_{tt} \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} B_{11} & & 0 \\ & \ddots & \\ 0 & & B_{tt} \end{bmatrix}.$$

Since each collection of blocks (A_{kk}, B_{kk}, C_{kk}) clearly satisfy both the hypothesis of the theorem, therefore by our induction hypothesis, we may assume that each C_{kk} is nilpotent, contrary to the assumption that each C_{kk} belongs to distinct characteristic root of C.

Therefore we may now assume that C has all its characteristic roots identical. Let λ be this, and put $\omega=\lambda+1$.

(b) By virtue of Lemma 1,

$$AB^r-B^rA=[(C+I)^r-I]B^rA,$$

for all positive integers r. Hence,

$$\text{Trace}\,[\{(C+I)^r-I\}B^r]=\text{Trace}\,(AB^rA^{-1})-\text{Trace}\,B^r=0.$$

Let $\mu_1, \cdots, \mu_m$ denote the distinct characteristic roots of B with multiplicities $x_1, x_2, \cdots, x_m$ respectively. Since C and B commute, hence by a result in [2], we can assume that

$$C+I=\begin{bmatrix} C_{11} & & 0 \\ & \ddots & \\ 0 & & C_{mm} \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} B_{11} & & 0 \\ & \ddots & \\ 0 & & B_{mm} \end{bmatrix},$$

such that

$$C_{kk}=\begin{bmatrix} \omega & & * \\ & \ddots & \\ 0 & & \omega \end{bmatrix} \quad \text{and} \quad B_{kk}=\begin{bmatrix} \mu_k & & * \\ & \ddots & \\ 0 & & \mu_k \end{bmatrix}$$

are $x_k\times x_k$ matrices in the upper-triangular form having identical entries along the diagonal.

Let $A=\begin{bmatrix} A_{11}\cdots A_{1m} \\ \cdots\cdots\cdots \\ A_{m1}\cdots A_{mm} \end{bmatrix}$, be a partition of A conformal to those of B and C. Then from the above trace-relation we have,

$$[(\lambda+1)^r-1]\cdot(x_1\mu_1^r+\cdots+x_m\mu_m^r)=0\,.$$

If $\omega=\lambda+1$ is not a root of unity, then $x_1\cdot\mu_1^r+\cdots+x_m\cdot\mu_m^r=0$, for all positive integers r. Taking $r=1, 2, 3, \cdots, m$, we get a non-trivial solution for the equations,

$$\begin{aligned} x_1\cdot\mu_1+\cdots+x_m\cdot\mu_m&=0\,, \\ x_1\cdot\mu_1^2+\cdots+x_m\cdot\mu_m^2&=0\,, \\ \cdots\cdots\cdots&\cdots\cdots\cdots \\ x_1\cdot\mu_1^m+\cdots+x_m\cdot\mu_m^m&=0\,. \end{aligned}$$

Hence using the Vandermonde-determinant theorem, and our hypothesis on the characteristic roots of B, we conclude that some $\mu_i=0$. But this contradicts the non-singularity of B. Therefore, there exists a minimal positive integer q such that $\omega^q=(\lambda+1)^q=1$. Also, if q is greater than m, then again taking $r=1, 2, 3 \cdots, m$, we may repeat the above argument to reach at the same contradiction. Hence q is less than or equal to m.

If $q=1$, then $\lambda=0$, and C is already nilpotent.

(c) We therefore suppose that $q\neq 1$, $q\leqq m$. Now, since C and B commute, so for every μ_k, there exists a vector $\mathfrak{U}_k$ of the underlying vector-space V of dimension n over the complex-field, such that

$$\mathfrak{U}_k B=\mu_k\cdot\mathfrak{U}_k \quad\text{and}\quad \mathfrak{U}_k\cdot(C+I)=\omega\cdot\mathfrak{U}_k\,.$$

Therefore,

$$\begin{aligned} (\mathfrak{U}_k A)B&=\mathfrak{U}_k(C+I)BA \\ &=\omega(\mathfrak{U}_k B)A \\ &=\omega\mu_k(\mathfrak{U}_k A)\,. \end{aligned}$$

Thus, together with each μ_k, we obtain $\omega\mu_k$ to be another characteristic root of B. Hence multiplication by ω induces a permutation of the set $(\mu_1, \cdots, \mu_m)$. Therefore, this set can be decomposed into disjoint cycles with respect to our permutation. If there were more than one distinct cycles, then let these be $(\mu_1, \cdots, \mu_k), \cdots, (\mu_l, \cdots, \mu_m)$.

Now from the relation, $AB=(C+I)BA$, and the assumption made on the forms of $C+I$, B and A, we get,

$$A_{kl}\cdot B_{ll}=C_{kk}B_{kk}A_{kl}\,.$$

It is well-known that A_{kl} is non-null only if B_{ll} and $C_{kk}B_{kk}$ have common characteristic roots. Hence here A_{kl} is non-null only if the characteristic root of B_{ll} is the characteristic root of $C_{kk}B_{kk}$, i. e., $\mu_l=\omega\mu_k$.

Then from the decomposition of the set $(\mu_1, \cdots, \mu_m)$ defined above, it follows at once that A has the form,

$$A=\left[\begin{array}{cc} \begin{matrix} A_{11}\cdots A_{1k} \\ \cdots\cdots\cdots \\ A_{k1}\cdots A_{kk} \end{matrix} & 0 \\ 0 & \begin{matrix} A_{ll}\cdots A_{lm} \\ \cdots\cdots\cdots \\ A_{ml}\cdots A_{mm} \end{matrix} \end{array}\right].$$

Thus the triple (A, B, C) again decomposes into similar blocks of smaller dimensions. Using induction hypothesis again we obtain that $\lambda=0$, contrary to our assumption.

Therefore, let us assume that the only cycle of permutation is given by, $\mu_1, \omega\cdot\mu_1, \omega^2\cdot\mu_1, \cdots, \omega^{m-1}\cdot\mu_1$.

Since the order of the permutation is the least common multiple of the lengths of its cycles, so the order of ω is m. Hence $q=m$, and the distinct characteristic roots of B are the m distinct m-th roots of the number μ_1^m, contrary to our hypothesis (ii) for B.

Thus in all cases $\lambda=0$, and C is nilpotent. Q. E. D.

§3. In this section we shall show by means of a simple example that the commuting of $ABA^{-1}B^{-1}$ with both A and B is far weaker a condition than the commuting of $AB-BA$ with A and B. Consider

$$A=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $B^2=I$ so that $B=B^{-1}$, and also $A^2=I$, so that $A=A^{-1}$.

Hence $ABA^{-1}B^{-1}-I=\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$, which commutes with both A and B. Now if $AB-BA$ were to commute with A and B, then by Jacobson's lemma in [3], $AB-BA$ will belong to the radical of the polynomial algebra generated by the pair $\{A, B\}$. Since the inverses of A and B can be represented as polynomials in A and B, so we conclude that $(AB-BA)A^{-1}B^{-1} = ABA^{-1}B^{-1}-I$ is also nilpotent. But by the above counterexample we know that it is not always the case.

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References

[1] I. N. Herstein, On a theorem of Putnam and Winter, Proc. Amer. Math. Soc., **9** (1958), 363-364.
[2] N. Jacobson, Lectures on abstract algebra, Vol. II.
[3] N. Jacobson, Rational methods in the theory of Lie algebras, Ann. of Math., **36** (1935), 875-881.