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A representation theorem for a certain class of regular functions

By Kôichi SAKAGUCHI

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1. Introduction.

Let $f(z)=a_q z^q+\cdots, a_q\neq 0$, be regular in the unit circle and have there exactly p zeros, where multiple zeros are counted in accordance with their multiplicities and $p\geqq q\geqq 1$ or $p>q\geqq 0$. Let γ be a non-negative real constant.
(1) $f(z)$ is said to be a member of the class $C(\gamma;p,q)$, if and only if there exists a positive number ρ such that

$$\int_{\theta_1}^{\theta_2} d\arg f(re^{i\theta})>-\gamma,\quad \theta_1<\theta_2,\quad \rho<r<1.$$

(2) $f(z)$ is said to be a member of the class $C^*(\gamma;p,q)$, if and only if for an arbitrary positive number ε there exists another positive number $\rho(\varepsilon)$ such that

$$\int_{\theta_1}^{\theta_2} d\arg f(re^{i\theta})>-\gamma-\varepsilon,\quad \theta_1<\theta_2,\quad \rho<r<1.$$

It is evident that $C(\gamma;p,q)\subset C^*(\gamma;p,q)$.

The purpose of this paper is to establish a representation theorem for functions belonging to the class $C^*(\gamma;p,q)$ and to show some examples of its applications.

$C(0;p,p)$ and $C(0;p,q)$ are familiar classes of p-valent functions whose members are p-valently starlike with respect to the origin for $|z|<1$. Recently Bender [1] studied the class of functions given by the representation

$$f(z)=\Phi(z)z^{q-p}\prod_{j=1}^{p-q}(1-\alpha_j^{-1}z)(1-\bar{\alpha}_j z),\tag{1.1}$$

where $\Phi(z)\in C(0;p,p), 0<|\alpha_j|<1, j=1,2,\cdots,p-q$, and he proved that it properly contains $C(0;p,q)$ if $p>q$, and its members are also p-valent in $|z|<1$. We shall show that this class of functions studied by Bender is equivalent to $C^*(0;p,q)$.

On the other hand, Umezawa [2] proved that a function $f(z)$ satisfying the condition $zf'(z)\in C(\pi;p,q)$ is at most p-valent in $|z|<1$. Such a function is said to be multivalently close-to-convex of order (p,q) for $|z|<1$ [3]. We shall show that a function $f(z)$ satisfying the condition $zf'(z)\in C^*(\pi;p,q)$ is also at most p-valent in $|z|<1$. Moreover we shall extend at the same time a theorem of Kaplan [4] concerning close-to-convex univalent functions to the case of multivalence.

2. The main theorem.

We owe the following lemma to an idea of Kaplan [4].

LEMMA. *Let $f(z)=a_p z^p+\cdots, a_p\neq 0, p\geqq 1$, be regular for $|z|\leqq 1$ and have no zeros in $0<|z|\leqq 1$. If for a positive constant β, $f(z)$ satisfies the condition*

$$\int_{\theta_1}^{\theta_2} d\arg f(e^{i\theta})>-\beta,\quad \theta_1<\theta_2,$$

then there exists a function $\Phi(z)$ belonging to the class $C(0;p,p)$ such that

$$\left|\arg\frac{f(z)}{\Phi(z)}\right|<\frac{\beta}{2},\quad |z|<1. \tag{2.1}$$

PROOF. The function $g(z)=f(z)^{1/p}=a_p^{1/p}z+\cdots$ is regular for $|z|\leqq 1$ and satisfies the conditions

$$\frac{g(z)}{z}\neq 0,\quad |z|\leqq 1,\quad \text{and}\quad \int_{\theta_1}^{\theta_2} d\arg g(e^{i\theta})>-\frac{\beta}{p},\quad \theta_1<\theta_2.$$

We now choose $P_0(r,\theta)=\arg[g(z)/z], z=re^{i\theta}$, to be single-valued and continuous for $|z|\leqq 1$, and introduce a harmonic function $Q_0(r,\theta)$ by the definitions

$$P(r,\theta)=P_0(r,\theta)+\theta,\quad s(\alpha)=\underset{\theta\leqq\alpha}{\text{l. u. b.}}\, P(1,\theta)-\frac{\beta}{2p},$$

$$Q_0(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r^2)(s(\alpha)-\alpha)}{1+r^2-2r\cos(\alpha-\theta)}d\alpha,\quad r<1.$$

We next take a regular function $h(z)$ whose imaginary part is $Q_0(r,\theta)$, and set $\phi(z)=ze^{h(z)}$. Then it can easily be verified in the same way as used by Kaplan [4] that $\phi(z)\in C(0;1,1)$ and $|\arg[g(z)/\phi(z)]|\leqq\beta/2p$ for $|z|<1$, where the equality sign may appear only when $g(z)/\phi(z)\equiv c$ (constant). Accordingly (2.1) holds for $\Phi(z)=\phi(z)^p\in C(0;p,p)$ or $\Phi(z)=[c\phi(z)]^p\in C(0;p,p)$.

Our main theorem is stated as follows.

THEOREM 1. *A necessary and sufficient condition that $f(z)$ be a member of the class $C^*(\gamma;p,q)$ is that $f(z)$ has a representation of the form*

$$f(z)=\Phi(z)A_\gamma(z)z^{q-p}\prod_{j=1}^{p-q}(1-\alpha_j^{-1}z)(1-\bar{\alpha}_j z), \tag{2.2}$$

where $\Phi(z)\in C(0;p,p)$, $0<|\alpha_j|<1, j=1,2,\cdots,p-q$, and $A_\gamma(z)$ is a non-vanishing regular function in $|z|<1$ such that $|\arg A_\gamma(z)|<\gamma/2$ for $|z|<1$ if $\gamma>0$, and $A_\gamma(z)\equiv 1$ if $\gamma=0$.

PROOF. Every function $f(z)$ given by (2.2) is evidently a member of

$C^*(\gamma\,; p, q)$, since

$$\text{(2.3)} \qquad \underset{\theta_1<\theta_2}{\text{l. u. b.}} \left| \int_{\theta_1}^{\theta_2} \arg \left[(re^{i\theta})^{q-p} \prod_{j=1}^{p-q} (1-\alpha_j^{-1} re^{i\theta})(1-\bar{\alpha}_j re^{i\theta}) \right] \right| \to 0 \quad \text{as} \quad r \to 1 .$$

We suppose conversely that $f(z) \in C^*(\gamma\,; p, q)$ and the zeros of $f(z)$ in $0<|z|<1$ are $\alpha_j, j=1, 2, \cdots, p-q$.

Set

$$g(z) = f(z) z^{p-q} \prod_{j=1}^{p-q} [(1-\alpha_j^{-1} z)(1-\bar{\alpha}_j z)]^{-1} ,$$

then $g(z)$ is a member of the class $C^*(\gamma\,; p, p)$ because of (2.3). Therefore, for a sequence of positive numbers $\{\varepsilon_n\}$ converging to zero, there exists another sequence $\{r_n\}$ such that $0<r_n<1$, $\lim_{n\to\infty} r_n = 1$ and

$$\int_{\theta_1}^{\theta_2} d \arg g(r_n e^{i\theta}) > -\gamma - \varepsilon_n, \quad \theta_1 < \theta_2 .$$

Since $g(r_n z)$ satisfies the assumption of the lemma with $\beta = \gamma + \varepsilon_n$, we can find a function $\Phi_n(z)$ belonging to the class $C(0\,; p, p)$ for which

$$\text{(2.4)} \qquad \left| \arg \frac{g(r_n z)}{\Phi_n(z)} \right| < \frac{\gamma + \varepsilon_n}{2}, \quad |z|<1 ,$$

holds and such that $|\Phi_n(z)/z^p| = 1$ for $z=0$.

The functions $\Phi_n(z), n=1, 2, \cdots$, form a normal family in $|z|<1$, so that there exists a subsequence $\{\Phi_{n_m}(z)\}$ which converges uniformly in every closed disc $|z| \leqq \rho < 1$. Denoting by $\Phi_0(z)$ the limit of this subsequence, $\Phi_0(z)$ is also a member of $C(0\,; p, p)$. Let n in (2.4) take values of the sequence $\{n_m\}$, and let $m \to \infty$, then we have

$$\text{(2.5)} \qquad \left| \arg \frac{g(z)}{\Phi_0(z)} \right| \leqq \frac{\gamma}{2}, \quad |z|<1 .$$

When $\gamma>0$ and $g(z)/\Phi_0(z)$ is not a constant, (2.5) deduces that $|\arg[g(z)/\Phi_0(z)]| < \gamma/2$ for $|z|<1$, whence (2.2) holds for $\Phi(z) = \Phi_0(z)$ and $A_\gamma(z) = g(z)/\Phi_0(z)$. When $\gamma>0$ and $g(z)/\Phi_0(z) \equiv c$ (constant), (2.2) holds for $\Phi(z) = c\Phi_0(z)$ and $A_\gamma(z) \equiv 1$. When $\gamma = 0$, (2.5) deduces $g(z)/\Phi_0(z) \equiv c$ (positive constant), whence (2.2) holds for $\Phi(z) = c\Phi_0(z)$ and $A_\gamma(z) \equiv 1$. Thus the theorem is proved.

3. The relation between the classes $C^*(\gamma\,; p, q)$ and $C(\gamma\,; p, q)$.

THEOREM 2. *The classes $C^*(\gamma\,; p, p)$ and $C(\gamma\,; p, p)$ are equivalent. If $p>q$, then the class $C^*(\gamma\,; p, q)$ properly contains the class $C(\gamma\,; p, q)$, and moreover every function of $C^*(\gamma\,; p, q)$ is the limit of a sequence of functions belonging to $C(\gamma\,; p, q)$.*

PROOF. Let $f(z) \in C^*(\gamma\,; p, p)$, then $f(z)$ has a representation of the form

$f(z)=\Phi(z)A_\gamma(z)$, where $\Phi(z)$ and $A_\gamma(z)$ are subject to the conditions in Theorem 1. From this we see that $f(z)$ belongs to the class $C(\gamma; p, p)$, whence $C^*(\gamma; p, p) \subset C(\gamma; p, p)$. On the other hand, from the definitions, evidently $C^*(\gamma; p, p) \supset C(\gamma; p, p)$. Hence $C^*(\gamma; p, p)=C(\gamma; p, p)$.

We next consider the case $p>q$. Take a positive number M such that $M>(p\pi+2\gamma/\pi)/(p-q)$, and set

$$F(z)=\frac{z^q}{(1-z)^{2p}}\Big(\frac{1+z}{1-z}\Big)^{\gamma/\pi}[(1-\alpha^{-1}z)(1-\bar{\alpha}z)]^{p-q}, \quad \alpha=\frac{1}{M}+i\Big(1-\frac{1}{M}\Big).$$

Since $0<|\alpha|<1, F(z)$ is a member of $C^*(\gamma; p, q)$ from Theorem 1. Now, setting

$$\Phi(z)=z^p/(1-z)^{2p}, \quad A_\gamma(z)=[(1+z)/(1-z)]^{\gamma/\pi},$$
$$G(z)=z^{q-p}[(1-\alpha^{-1}z)(1-\bar{\alpha}z)]^{p-q},$$

we have the following for r (<1) sufficiently near and tending to 1 by brief calculations.

$$\int_{\pi/2}^{3\pi/2} d\arg\Phi(re^{i\theta})=p\int_{\pi/2}^{3\pi/2}\frac{1-r^2}{1-2r\cos\theta+r^2}d\theta$$
$$<p\pi(1-r^2)/(1+r^2)=p\pi[1-r+o(1-r)],$$

$$\int_{\pi/2}^{3\pi/2} d\arg A_\gamma(re^{i\theta})=\Im[\log A_\gamma(-ir)-\log A_\gamma(ir)]$$
$$=(\gamma/\pi)[-\pi+2(1-r)+o(1-r)],$$

$$\int_{\pi/2}^{3\pi/2} d\arg G(re^{i\theta})=\Im[\log G(-ir)-\log G(ir)]$$
$$=-(p-q)[(M+M/(2M^2-2M+1))(1-r)+o(1-r)].$$

Hence

$$\int_{\pi/2}^{3\pi/2} d\arg F(re^{i\theta})<-\gamma-[(p-q)M-p\pi-2\gamma/\pi](1-r)+o(1-r)$$
$$<-\gamma,$$

because of $M>(p\pi+2\gamma/\pi)/(p-q), p>q$. Therefore $F(z)$ is not a member of $C(\gamma; p, q)$. In other words, $C^*(\gamma; p, q)$ properly contains $C(\gamma; p, q)$.

We finally suppose that $f(z)$ is a member of $C^*(\gamma; p, q)$ given by (2.2). Consider the function

$$f_n(z)=\Phi(t_nz)A_\gamma(t_nz)z^{q-p}\prod_{j=1}^{p-q}(1-\alpha_j^{-1}z)(1-\bar{\alpha}_jz), \quad t_n=1-\frac{1}{n+1},$$

where n is a positive integer. (1) When $\gamma>0$, for a suitable positive number δ_n we have

$$\int_{\theta_1}^{\theta_2} d\arg A_\gamma(t_nre^{i\theta})>-\gamma+\delta_n, \quad \theta_1<\theta_2, \quad 0\leqq r<1,$$

which together with (2.3) deduces that there exists a positive number ρ_n such that

$$\int_{\theta_1}^{\theta_2} d \arg f_n(re^{i\theta}) > -\gamma, \quad \theta_1 < \theta_2, \quad \rho_n < r < 1,$$

and so $f_n(z)$ is a member of $C(\gamma; p, q)$. (2) When $\gamma = 0$, we have

$$\Re[zf_n'(z)/f_n(z)] = \Re[t_n z \Phi'(t_n z)/\Phi(t_n z)] > 0, \quad |z| = 1,$$

which with aid of the continuity of $\Re[zf_n'(z)/f_n(z)]$ deduces that there exists a positive number ρ_n such that

$$\Re[zf_n'(z)/f_n(z)] > 0, \quad \rho_n < |z| < 1,$$

and so $f_n(z)$ is a member of $C(0; p, q)$. Since $f(z) = \lim_{n\to\infty} f_n(z)$, $f(z)$ is thus the limit of the sequence $\{f_n(z)\}$ which consists of functions belonging to the class $C(\gamma; p, q)$. This completes the proof of the theorem.

4. Two kinds of multivalent functions.

If $f(z) \in C^*(0; p, q)$, then $f(z)$ is said to be multivalently starlike in a wide sense of order (p, q) with respect to the origin for $|z| < 1$. Theorem 1 certifies that $C^*(0; p, q)$ is equivalent to the class of functions given by the representation (1.1). Therefore by Bender's theorem stated in § 1, every function of $C^*(0; p, q)$ is p-valent in $|z| < 1$. We thus have

THEOREM 3. *Let $f(z)$ be multivalently starlike in a wide sense of order (p, q) with respect to the origin for $|z| < 1$, then $f(z)$ is p-valent in $|z| < 1$.*

Next, if $f(z)$ is a function such that $zf'(z) \in C^*(\pi; p, q)$, then $f(z)$ is said to be multivalently close-to-convex in a wide sense of order (p, q) for $|z| < 1$. From Theorem 2, the class of such functions properly contains the class of functions multivalently close-to-convex of order (p, q) for $|z| < 1$, if $p > q$. Now we have

THEOREM 4. *A necessary and sufficient condition that $f(z)$ be multivalently close-to-convex in a wide sense of order (p, q) for $|z| < 1$ is that $f(z)$ has a representation of the form*

$$f(z) = a_0 + \int_0^z \frac{\Phi^*(z)}{z} A_\pi(z) dz, \tag{4.1}$$

where $\Phi^*(z) \in C^*(0; p, q), p \geqq q \geqq 1$, and $|\arg A_\pi(z)| < \pi/2$ for $|z| < 1$. Moreover, every function $f(z)$ given by (4.1) *is at most p-valent in $|z| < 1$.*

PROOF. Since the former half of the theorem is an immediate consequence of Theorem 1, it suffices to prove only the latter half. Suppose that $f(z)$ has the representation (4.1) with

$$\Phi^*(z) = \Phi(z) z^{q-p} \prod_{j=1}^{p-q} (1 - \alpha_j^{-1} z)(1 - \bar{\alpha}_j z),$$

and set

$$f_n(z)=a_0+\int_0^z \frac{\Phi_n^*(z)}{z}A_\pi(t_nz)dz, \quad t_n=1-\frac{1}{n+1},$$

where n is a positive integer and

$$\Phi_n^*(z)=\Phi(t_nz)z^{q-p}\prod_{j=1}^{p-q}(1-\alpha_j^{-1}z)(1-\bar{\alpha}_jz).$$

Then, as shown in the proof of Theorem 2, $zf_n'(z)$ is a member of the class $C(\pi;p,q)$. Therefore by Umezawa's lemma [2], [3] $f_n(z)$ is at most p-valent in $|z|<1$.

We shall next show that $f_n(z)$ converges to $f(z)$ uniformly in every closed disc $|z|\leqq\rho<1$ as $n\to\infty$. Let ρ be an arbitrary positive number less than 1. Evidently, $\Phi_n^*(z)z^{-1}A_\pi(t_nz)$ converges to $\Phi^*(z)z^{-1}A_\pi(z)$ uniformly in $|z|\leqq\rho$ as $n\to\infty$, i.e. for an arbitrary positive number ε, there exists a positive integer $n_0(\varepsilon)$ such that

$$|\Phi_n^*(z)z^{-1}A_\pi(t_nz)-\Phi^*(z)z^{-1}A_\pi(z)|<\varepsilon, \quad |z|\leqq\rho, \quad n>n_0.$$

Hence

$$|f_n(z)-f(z)|=\left|\int_0^r\Big[\frac{\Phi_n^*(z)}{z}A_\pi(t_nz)-\frac{\Phi^*(z)}{z}A_\pi(z)\Big]e^{i\theta}dr\right|, \quad z=re^{i\theta},$$

$$|f_n(z)-f(z)|<\rho\varepsilon, \quad |z|\leqq\rho, \quad n>n_0,$$

so that $f_n(z)$ converges to $f(z)$ uniformly in $|z|\leqq\rho$ as $n\to\infty$. Consequently $f(z)$ is also at most p-valent in $|z|<1$. We thus complete the proof.

If we put $p=q=1$ in this theorem, we obtain the theorem of Kaplan [4] mentioned in § 1.

5. Properties of functions belonging to the class $C^*(\gamma;p,q)$.

The representation (2.2) permits us to obtain some extremal formulae for functions belonging to the class $C^*(\gamma;p,q)$. For instance, we have

THEOREM 5. *Let $f(z)=a_qz^q+a_{q+1}z^{q+1}+\cdots$ be a member of the class $C^*(\gamma;p,q)$, and let $\alpha_j, j=1,2,\cdots,p-q$, be the zeros of $f(z)$ in $0<|z|<1$. If we set*

$$F(z)=|a_q|\frac{z^q}{(1-z)^{2p}}\Big(\frac{1+z}{1-z}\Big)^\lambda\prod_{j=1}^{p-q}(1+|\alpha_j|^{-1}z)(1+|\alpha_j|z),$$

$$G(z)=q+z\Big[\frac{2p}{1-z}+\frac{2\lambda}{1-z^2}+\sum_{j=1}^{p-q}\Big(\frac{1}{z+|\alpha_j|}+\frac{|\alpha_j|}{1+|\alpha_j|z}\Big)\Big],$$

where $\lambda=\gamma/\pi$, then we have

$$|F(-r)|\leqq|f(z)|\leqq F(r), \quad |f'(z)|\leqq F'(r), \quad |z|=r<1,$$

$$\Re[zf'(z)/f(z)]\geqq G(-r), \quad |z|=r\leqq\min|\alpha_j|,$$

$$|a_{q+1}|\leqq|a_q|\{2p+2\lambda+\sum_{j=1}^{p-q}(|\alpha_j|^{-1}+|\alpha_j|)\}.$$

Moreover, if λ is an integer, then

$$f(z) \ll F(z).$$

The bounds of these estimates are all attained by the function $F(z) \in C^*(\gamma; p, q)$.

PROOF. When $\lambda > 0$, the function $A_\gamma(z) = c_0 + c_1 z + \cdots$ satisfies

$$\Re A_\gamma(z)^{1/\lambda} > 0, \quad |z| < 1,$$

from which we have

$$|c_0| \left(\frac{1-r}{1+r}\right)^\lambda \leqq |A_\gamma(z)| \leqq |c_0| \left(\frac{1+r}{1-r}\right)^\lambda, \quad |z| = r < 1,$$

$$\left| z \frac{A_\gamma'(z)}{A_\gamma(z)} \right| \leqq \frac{2\lambda r}{1-r^2}, \quad |A_\gamma'(z)| \leqq \frac{2\lambda |c_0|}{1-r^2} \left(\frac{1+r}{1-r}\right)^\lambda, \quad |z| = r < 1,$$

and $|c_1| \leqq 2\lambda |c_0|$. Moreover, if λ is an integer, we have

$$A_\gamma(z) \ll |c_0| \left(\frac{1+z}{1-z}\right)^\lambda.$$

Evidently, these estimates are valid also when $\gamma = 0$.

With aid of the above properties of $A_\gamma(z)$ and some known properties of $\Phi(z)$, the representation (2.2) yields easily all the estimates of the theorem. The details of calculations will be omitted.

The extremal formulae of this theorem are generalizations of some results given by Bender [**1**], Goodman [**5**], Robertson [**6**], and the author [**7**].

REMARK. Let $f(z) = a_q z^q + \cdots, a_q \neq 0$, be regular for $|z| < 1$ and have exactly $p - q$ zeros in $0 < |z| < 1$, where $p \geqq q \geqq 1$ or $p > q \geqq 0$. It is easy to see that (1) if for every ρ less than and sufficiently near to 1, the image curve of $|z| = \rho$ under $f(z)$ cuts a straight line through the origin in $2s$ points, then $f(z)$ is a member of the class $C^*((s-p+1)\pi; p, q)$, (2) if for such every ρ, the image curve of $|z| = \rho$ under $f(z)$ cuts a ray starting from the origin in s points, then $f(z)$ is a member of the class $C^*((s-p+2)\pi; p, q)$, and (3) if for an arbitrary positive number ε, there exists another positive number $\rho(\varepsilon)$ such that the total variation of $\arg f(re^{i\theta})$, $\rho < r < 1$, in $0 \leqq \theta \leqq 2\pi$ is smaller than $\beta + \varepsilon$, where β is a positive constant, then $f(z)$ is a member of the class $C^*(\beta/2 - p\pi; p, q)$. Accordingly we can obtain properties of functions of these kinds by putting $\lambda = s-p+1, s-p+2$, and $\beta/2\pi - p$ in this theorem.

COLLORARY. *Let $f(z) \in C(\gamma; p, p)$, then $f(z)$ is p-valent and starlike of order p with respect to the origin for*

$$|z| < \{p + \lambda - \sqrt{\lambda(2p+\lambda)}\}/p. \quad \lambda = \gamma/\pi,$$

and this bound is sharp.

Nara Gakugei University

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