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On predicates with constructive infinitely long expressions

Dedicated to Professor Y. Akizuki on his sixtieth birthday.

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In recent years the logic with infinitely long expressions has been considered and developed by Berkeley school. (For this see [1], [2], [7] and also cf. [6].) In this paper, we shall consider 'constructive' infinitely long expressions. In the following we shall give a language for the logic with infinitely long expressions and define a 'formula with constructive infinitely long expressions' as a formula with infinitely long expressions (sometimes called simply a formula) to which a so-called Gödel number is assigned. We shall show that the nesting number of a formula with constructive infinitely long expressions (see below) is less than Church-Kleene's ω_1 (Theorem 1). Moreover we shall establish a correspondence between formulas with constructive infinitely long expressions and predicates in Kleene's analytic hierarchy (cf. [4]). We shall prove that a formula $\mathfrak{A}$ with constructive infinitely long expressions is representable in the $\Sigma^1_{n+1} \cap \Pi^1_{n+1}$-form, if the maximal number of quantifiers nested in $\mathfrak{A}$ is n (cf. n$'(\mathfrak{A})$ defined below) and especially, $\mathfrak{A}$ is representable in Σ^1_n or Π^1_n if the outermost logical symbol of $\mathfrak{A}$ is $\exists$ or $\forall$ (Theorem 2). On the other hand, any predicate expressible in the n-function quantifier form is representable by a formula $\mathfrak{A}$ with constructive infinitely long expressions such that n$'(\mathfrak{A}) = n$ (Theorem 3). We shall also prove that every hyperarithmetical formula is representable by a quantifier-free formula with constructive infinitely long expressions (Theorem 4).

0. In this paper we shall use the following language:

Individual constants $0, 1, 2, \cdots$;

Variables $v_0, v_1, \cdots, v_i, \cdots$ $(i < \omega)$;

The predicate $=$;

Logical symbols $\neg, \vee, \wedge, \exists, \forall$.

Prime formulas are of the form $i = j, i = v_n, v_m = j$ and $v_m = v_n$, where i and j are individual constants. Formulas are composed from prime formulas as follows:

0.1. If $\mathfrak{A}$ is a formula, then $\neg\mathfrak{A}$ is a formula.

0.2. If $\mathfrak{A}_i$ are formulas for all $i<\omega$, $\vee(\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_i, \cdots)$ and $\wedge(\mathfrak{A}_0, \mathfrak{A}_1, \cdots, \mathfrak{A}_i, \cdots)$ are formlas, which are denoted by $\bigvee_{i<\omega} \mathfrak{A}_i$ (or $\bigvee_i \mathfrak{A}_i$) and $\bigwedge_{i<\omega} \mathfrak{A}_i$ (or $\bigwedge_i \mathfrak{A}_i$) respectively in the following. (Ordinary disjunction and conjunction of the formulas $\mathfrak{A}$ and $\mathfrak{B}$ are considered as $\vee(\mathfrak{A}, \mathfrak{B}, \mathfrak{B}, \cdots)$ and $\wedge(\mathfrak{A}, \mathfrak{B}, \mathfrak{B}, \cdots)$, respectively.)

0.3. If $\mathfrak{A}$ is a formula, then $\exists v_{n_0} v_{n_1} \cdots \mathfrak{A}$ and $\forall v_{n_0} v_{n_1} \cdots \mathfrak{A}$ are formulas, where $v_{n_0}, v_{n_1}, \cdots$ is a sequence of variables of order type ω.

1. Let $\mathfrak{A}$ be a formula. We shall define $\mathrm{n}(\mathfrak{A})$ (the nesting number of $\mathfrak{A}$) and $\mathrm{n}'(\mathfrak{A})$ as follows:

1.1. If $\mathfrak{A}$ contains no logical symbol, then $\mathrm{n}(\mathfrak{A})=0$ and $\mathrm{n}'(\mathfrak{A})=0$.

1.2. If $\mathfrak{A}$ is of the form $\neg\mathfrak{B}$, then $\mathrm{n}(\mathfrak{A})=\mathrm{n}(\mathfrak{B})+1$ and $\mathrm{n}'(\mathfrak{A})=\mathrm{n}'(\mathfrak{B})$.

1.3. If $\mathfrak{A}$ is of the form $\bigvee_{i<\omega} \mathfrak{A}_i$ or $\bigwedge_{i<\omega} \mathfrak{A}_i$, then $\mathrm{n}(\mathfrak{A})=\{$the least ordinal number which is greater than $\mathrm{n}(\mathfrak{A}_i)$ for all $i<\omega\}$ and $\mathrm{n}'(\mathfrak{A})=\{$the least ordinal number which is not less than $\mathrm{n}'(\mathfrak{A}_i)$ for all $i<\omega\}$.

1.4. If $\mathfrak{A}$ is of the form $\exists v_{n_0} v_{n_1} \cdots \mathfrak{B}$ or $\forall v_{n_0} v_{n_1} \cdots \mathfrak{B}$, then $\mathrm{n}(\mathfrak{A})=\mathrm{n}(\mathfrak{B})+1$ and $\mathrm{n}'(\mathfrak{A})=\mathrm{n}'(\mathfrak{B})+1$.

2. We shall assign at most one natural number to a formula $\mathfrak{A}$ with infinitely long expressions in the following and call a number the *Gödel number* of $\mathfrak{A}$ (denoted as $\ulcorner\mathfrak{A}\urcorner$) if it is assigned to $\mathfrak{A}$. Then a formula with the Gödel number is called a *formula with constructive infinitely long expressions.* We define the Gödel number of an individual constant i to be 3^{i+1} and the Gödel number of a variable v_j to be 5^{j+1}. The definition of the Gödel number of a formula is as follows:

2.1.1. $\ulcorner i=j\urcorner=2^2\cdot 7^{3^{i+1}}\cdot 11^{3^{j+1}}$.

2.1.2. $\ulcorner i=v_j\urcorner=2^2\cdot 7^{3^{i+1}}\cdot 11^{5^{j+1}}$.

2.1.3. $\ulcorner v_i=j\urcorner=2^2\cdot 7^{5^{i+1}}\cdot 11^{3^{j+1}}$.

2.1.4. $\ulcorner v_i=v_j\urcorner=2^2\cdot 7^{5^{i+1}}\cdot 11^{5^{j+1}}$.

2.2. $\ulcorner\neg\mathfrak{A}\urcorner=2^7\cdot 7^{\ulcorner\mathfrak{A}\urcorner}$ provided that $\ulcorner\mathfrak{A}\urcorner$ is defined.

2.3.1. $\ulcorner\bigvee_i \mathfrak{A}_i\urcorner=2^9\cdot 7^f$, provided that $\ulcorner\mathfrak{A}_i\urcorner$ are defined for all $i<\omega$ and f defines $\ulcorner\mathfrak{A}_i\urcorner$ recursively as the function of i.

2.3.2. $\ulcorner\bigwedge_i \mathfrak{A}_i\urcorner=2^{11}\cdot 7^f$, provided that $\ulcorner\mathfrak{A}_i\urcorner$ are defined for all $i<\omega$ and f defines $\ulcorner\mathfrak{A}_i\urcorner$ recursively as the function of i.

2.4.1. $\ulcorner\exists v_{n_0} v_{n_1} \cdots \mathfrak{A}\urcorner=2^{13}\cdot 7^g\cdot 11^{\llcorner\mathfrak{A}\lrcorner}$, provided that $\ulcorner\mathfrak{A}\urcorner$ is defined, the function $\{g\}(i)$ is defined for all i and $\{g\}(i)=n_i$.

2.4.2. $\ulcorner\forall v_{n_0} v_{n_1} \cdots \mathfrak{A}\urcorner=2^{15}\cdot 7^g\cdot 11^{\ulcorner\mathfrak{A}\urcorner}$, provided that $\ulcorner\mathfrak{A}\urcorner$ is defined, the function $\{g\}(i)$ is defined for all i and $\{g\}(i)=n_i$.

Let ω_1 be the least non-constructive ordinal number.

THEOREM 1. (a) *If $\mathfrak{A}$ is a formula with constructive infinitely long expressions, then* $\mathrm{n}(\mathfrak{A})<\omega_1$.

(b) *If α is an ordinal number less than ω_1, then there exists a formula $\mathfrak{A}$ with constructive infinitely long expressions such that* $\mathrm{n}(\mathfrak{A})=\alpha$.

PROOF. (a) We shall define a partial recursive function φ with the following property: If a is the Gödel number of a formula $\mathfrak{A}$ then $\varphi(a)$ is defined, $\varphi(a)\in O$ and $|\varphi(a)|\geqq \mathrm{n}(\mathfrak{A})$.

Let $\theta(z,y,n)\cong \sum_{x<n} O\{z\}(\{y\}(x))$ (cf. [5] for $\sum_{x<n} O$) and p define $\theta(z,y,n)$ recursively. It suffices for φ to take $\varphi(a)\cong\{e\}(a)$ where e is a solution for z of the following equation given by the recursion theorem (cf. [3. p. 352. Theorem XXVII]):

$$\begin{aligned}\{z\}(a)\cong \mu t((&a=2^2\cdot 7^{(a)_3}\cdot 11^{(a)_4}\wedge t=2)\\ &\vee(a=2^7\cdot 7^{(a)_3}\wedge t=2^{\{z\}((a)_3)})\\ &\vee((a=2^9\cdot 7^{(a)_3}\vee a=2^{11}\cdot 7^{(a)_3})\wedge t=3\cdot 5^{S_1^2(p,z,(a)_3)})\\ &\vee((a=2^{13}\cdot 7^{(a)_3}\cdot 11^{(a)_4}\vee a=2^{15}\cdot 7^{(a)_3}\cdot 11^{(a)_4})\wedge t=2^{\{z\}((a)_4)}))\,.\end{aligned}$$

We can prove that φ has the required property by transfinite induction on the nesting numbers of formulas with constructive infinitely long expressions.

(b) We shall define a partial recursive function ξ with the following property: If $a\in O$, then $\xi(a)$ is defined and is the Gödel number of a formula with constructive infinitely long expressions.

Let $\zeta(z,y,n)\cong\{z\}(\{y\}(n_o))$ and q define $\zeta(z,y,n)$ recursively. It suffices for ξ to take $\xi(a)\cong\{f\}(a)$ where f is a solution for z of the following equation given by the recursion theorem:

$$\begin{aligned}\{z\}(a)\cong \mu t((&a=1\wedge t=2^2\cdot 7^3\cdot 11^3)\\ &\vee(a=2^{(a)_0}\wedge (a)_0\neq 0\wedge t=2^7\cdot 7^{\{z\}((a)_0)})\\ &\vee(a=3\cdot 5^{(a)_2}\wedge t=2^9\cdot 7^{S_1^2(q,z,(a)_2)}))\,.\end{aligned}$$

Then we can prove that $\xi(a)$ has the required property by induction on $a\in O$.

THEOREM 2. *Let $\mathfrak{A}$ be a formula with constructive infinitely long expressions with $\mathrm{n}'(\mathfrak{A})<\omega$. If $\mathrm{n}'(\mathfrak{A})=n$ $(n\geqq 0)$, $\mathfrak{A}$ is representable as a predicate in $\Sigma^1_{n+1}\cap\Pi^1_{n+1}$ in Kleene hierarchy. Moreover if the outermost logical symbol of $\mathfrak{A}$ is a quantifier, $\mathfrak{A}$ is representable in Σ^1_n- or Π^1_n-form keeping the outermost quantifier in the same kind.*

PROOF. We shall define inductively a predicate $Q_m(a,\alpha)$ satisfying the following condition: If a is the Gödel number of a formula $\mathfrak{A}$ with infinitely long expressions such that $\mathrm{n}'(\mathfrak{A})\leqq m$, then $Q_m(a,\alpha)$ represents $\mathfrak{A}$ in the theory of recursive functions $(m\geqq 0)$. Moreover we shall give primitive recursive predicates $R_m(a,u,u_0,\cdots,u_m)$ and $S_m(a,u,u_0,\cdots,u_m)$ satisfying the following condition: If a is the Gödel number of a formula $\mathfrak{A}$ with infinitely long expressions such that $\mathrm{n}'(\mathfrak{A})\leqq m$,

$$Q_m(a, \alpha) \rightleftarrows \sharp\beta_0 \cdots \forall\beta_m \exists x R_m(a, \bar{\alpha}(x), \bar{\beta}_0(x), \cdots, \bar{\beta}_m(x))$$
$$\rightleftarrows \natural\beta_0 \cdots \exists\beta_m \forall x S_m(a, \bar{\alpha}(x), \bar{\beta}_0(x), \cdots, \bar{\beta}_m(x))$$

where $\sharp$ is $\forall$ or $\exists$ according as m is even or odd and $\natural$ is $\exists$ or $\forall$ according as $\sharp$ is $\forall$ or $\exists$.

First we shall give some auxiliary notations. Let $E(a, \alpha)$ be

$$((a)_3 = 3^{(a)_{3,1}} \wedge (a)_4 = 3^{(a)_{4,1}} \wedge (a)_{3,1} \neq 0 \wedge (a)_{4,1} \neq 0 \wedge (a)_{3,1} = (a)_{4,1})$$
$$\vee ((a)_3 = 3^{(a)_{3,1}} \wedge (a)_4 = 5^{(a)_{4,2}} \wedge (a)_{3,1} \neq 0 \wedge (a)_{4,2} \neq 0 \wedge (a)_{3,1} \dot{-} 1 = \alpha((a)_{4,2} \dot{-} 1))$$
$$\vee ((a)_3 = 5^{(a)_{3,2}} \wedge (a)_4 = 3^{(a)_{4,1}} \wedge (a)_{3,2} \neq 0 \wedge (a)_{4,1} \neq 0 \wedge \alpha((a)_{3,2} \dot{-} 1) = (a)_{4,1} \dot{-} 1)$$
$$\vee ((a)_3 = 5^{(a)_{3,2}} \wedge (a)_4 = 5^{(a)_{4,2}} \wedge (a)_{3,2} \neq 0 \wedge (a)_{4,2} \neq 0 \wedge \alpha((a)_{3,2} \dot{-} 1) = \alpha((a)_{4,2} \dot{-} 1)).$$

Then $E(a, \alpha) \rightleftarrows \exists x E_0(a, \bar{\alpha}(x)) \rightleftarrows \forall x E_1(a, \bar{\alpha}(x))$ for some primitive recursive E_0 and E_1.

The inductive definition of $Q_0(a, \alpha)$ is given as follows:

$$
\begin{aligned}
Q_0(a, \alpha) \rightleftarrows\ & (a = 2^2 \cdot 7^{(a)_3} \cdot 11^{(a)_4} \wedge E(a, \alpha)) \\
& \vee (a = 2^7 \cdot 7^{(a)_3} \wedge \neg Q_0((a)_3, \alpha)) \\
& \vee (a = 2^9 \cdot 7^{(a)_3} \wedge \forall u \exists v T_1((a)_3, u, v) \\
& \qquad \wedge \exists z \forall u (\neg T_1((a)_3, z, u) \vee Q_0(U(u), \alpha))) \\
& \vee (a = 2^{11} \cdot 7^{(a)_3} \wedge \forall u_1 \exists v T_1((a)_3, u_1, v) \\
& \qquad \wedge \forall u_0 \forall u_1 (\neg T_1((a)_3, u_0, u_1) \vee Q_0(U(u_1), \alpha))).
\end{aligned} \tag{0}
$$

By substituting in (0) for "$Q_0(a, \alpha)$" the predicate expressions "$\forall\beta\exists x R_0(a, \bar{\alpha}(x), \bar{\beta}(x))$" and "$\exists\beta\forall x S_0(a, \bar{\alpha}(x), \bar{\beta}(x))$" where $R_0(a, u, v)$ and $S_0(a, u, v)$ remain to be selected, we obtain the following equivalences (1) and (2).

$$
\begin{aligned}
\forall\beta\exists x R_0(a, \bar{\alpha}(x), \bar{\beta}(x)) \rightleftarrows\ & (a = 2^2 \cdot 7^{(a)_3} \cdot 11^{(a)_4} \wedge \exists x E_0(a, \bar{\alpha}(x))) \\
& \vee (a = 2^7 \cdot 7^{(a)_3} \wedge \forall\beta\exists x \neg S_0((a)_3, \bar{\alpha}(x), \bar{\beta}(x))) \\
& \vee (a = 2^9 \cdot 7^{(a)_3} \wedge \forall u \exists v T_1((a)_3, u, v) \\
& \qquad \wedge \exists z \forall u (\neg T_1((a)_3, z, u) \vee \forall\beta\exists x R_0(U(u), \bar{\alpha}(x), \bar{\beta}(x)))) \\
& \vee (a = 2^{11} \cdot 7^{(a)_3} \wedge \forall u_1 \exists v T_1((a)_3, u_1, v) \\
& \qquad \wedge \forall u_0 \forall u_1 (\neg T_1((a)_3, u_0, u_1) \vee \forall\beta\exists x R_0(U(u_1), \bar{\alpha}(x), \bar{\beta}(x)))).
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
\exists\beta\forall x S_0(a, \bar{\alpha}(x), \bar{\beta}(x)) \rightleftarrows\ & (a = 2^2 \cdot 7^{(a)_3} \cdot 11^{(a)_4} \wedge \forall x_0 E_1(a, \bar{\alpha}(x_0))) \\
& \vee (a = 2^7 \cdot 7^{(a)_3} \wedge \exists\beta\forall x_1 \neg R_0((a)_3, \bar{\alpha}(x_1), \bar{\beta}(x_1))) \\
& \vee (a = 2^9 \cdot 7^{(a)_3} \wedge \forall u \exists v T_1((a)_3, u, v) \\
& \qquad \wedge \exists z \forall u (\neg T_1((a)_3, z, u) \vee \exists\beta\forall x_2 S_0(U(u), \bar{\alpha}(x_2), \bar{\beta}(x_2)))) \\
& \vee (a = 2^{11} \cdot 7^{(a)_3} \wedge \forall u_1 \exists v T_1((a)_3, u_1, v) \\
& \qquad \wedge \forall u_0 \forall u_1 (\neg T_1((a)_3, u_0, u_1) \vee \exists\beta\forall x_3 S_0(U(u_1), \bar{\alpha}(x_3), \bar{\beta}(x_3)))).
\end{aligned} \tag{2}
$$

We shall define $R_0(a, u, v)$ and $S_0(a, u, v)$ so that they will be primitive recursive and (1) and (2) will be true. To find primitive recursive $R_0(a, u, v)$ and $S_0(a, u, v)$ satisfying (1) and (2), we begin by observing that in the right member of (1) the quantifiers can be advanced to give an equivalent of the form

$$\exists z \forall u \forall u_0 \forall u_1 \forall \beta \exists v \exists x F^{R_0, S_0}(a, z, u, u_0, u_1, v, \bar{\alpha}(x), \bar{\beta}(x)) \tag{1.1}$$

where $F^{R_0, S_0}(a, z, u, u_0, u_1, v, \bar{\alpha}(x), \bar{\beta}(x))$ is exactly the right member of (1) with its quantifiers omitted. Next (1.1) is equivalent to

$$\begin{aligned} &\forall \beta \exists x (F^{R_0, S_0}(a, (x)_0, \beta(2^{(x)_0}), \beta(2^{(x)_0}\cdot 3), \beta(2^{(x)_0}\cdot 3^2), \\ &\qquad ((x)_1 \dot{-} 3)_0, \bar{\alpha}(((x)_1 \dot{-} 3)_1), \bar{\beta}^*_{(x)_0}(((x)_1 \dot{-} 3)_1)) \wedge (x)_1 \geqq 3) \end{aligned} \tag{1.2}$$

where β^*_u stands for $\lambda i \beta(2^u \cdot 3^{i+3})$.

Then we can see that (1.2) takes the form $\forall \beta \exists x M^{R_0, S_0}(a, \bar{\alpha}(x), \bar{\beta}(x))$. Similarly as above we can reduce (2) to the form $\exists \beta \forall x N^{R_0, S_0}(a, \bar{\alpha}(x), \bar{\beta}(x))$. We can obtain a partial recursive function $\tau(a, u, v)$ by using the recursion theorem such that

$$(\tau(a, u, v))_0 = 0 \rightleftarrows M^{\lambda xyz(\tau(x,y,z))_0=0,\, \lambda xyz(\tau(x,y,z))_1=0}(a, u, v)$$

$$\wedge \operatorname{Seq}(u) \wedge \operatorname{Seq}(v) \wedge \operatorname{lh}(u) = \operatorname{lh}(v),$$

$$(\tau(a, u, v))_1 = 0 \rightleftarrows N^{\lambda xyz(\tau(x,y,z))_0=0,\, \lambda xyz(\tau(x,y,z))_1=0}(a, u, v)$$

$$\wedge \operatorname{Seq}(u) \wedge \operatorname{Seq}(v) \wedge \operatorname{lh}(u) = \operatorname{lh}(v).$$

This is seen to be primitive recursive by course-of-values induction on u. (We omit here to give M^{R_0, S_0}, N^{R_0, S_0} and $\tau(a, u, v)$ exactly, because it is not difficult but laborious and needs too much space.) Now we propose to define R_0 and S_0 by $R_0(a, u, v) \rightleftarrows (\tau(a, u, v))_0 = 0$ and $S_0(a, u, v) \rightleftarrows (\tau(a, u, v))_1 = 0$ respectively. Then $R_0(a, u, v)$ and $S_0(a, u, v)$ are primitive recursive,

$$\forall \beta \exists x R_0(a, \bar{\alpha}(x), \bar{\beta}(x)) \rightleftarrows \forall \beta \exists x M^{R_0, S_0}(a, \bar{\alpha}(x), \bar{\beta}(x))$$

and

$$\exists \beta \forall x S_0(a, \bar{\alpha}(x), \bar{\beta}(x)) \rightleftarrows \exists \beta \forall x N^{R_0, S_0}(a, \bar{\alpha}(x), \bar{\beta}(x)).$$

We can prove

$$Q_0(a, \alpha) \rightleftarrows \forall \beta \exists x R_0(a, \bar{\alpha}(x), \bar{\beta}(x)) \rightleftarrows \exists \beta \forall x S_0(a, \bar{\alpha}(x), \bar{\beta}(x)) \tag{3}$$

under the presupposition that a is the Gödel number of a formula $\mathfrak{A}$ with infinitely long expressions such that $\mathrm{n}'(\mathfrak{A}) = 0$ by transfinite induction on the nesting number of $\mathrm{n}(\mathfrak{A})$. The hypothesis of induction states that, for any b, if b is the Gödel number of a formula $\mathfrak{B}$ with infinitely long expressions such that $\mathrm{n}'(\mathfrak{B}) = 0$ and $\mathrm{n}(\mathfrak{B}) < \mathrm{n}(\mathfrak{A})$, then

$$Q_0(b, \alpha) \rightleftarrows \forall \beta \exists x R_0(b, \bar{\alpha}(x), \bar{\beta}(x)) \rightleftarrows \exists \beta \forall x S_0(b, \bar{\alpha}(x), \bar{\beta}(x)).$$

By our presupposition one of four cases applies.

Case 1. $n(\mathfrak{A})=0$, i. e. $a=2^2\cdot 7^{(a)_3}\cdot 11^{(a)_4}$. This is obvious by (0), (1) and (2).

Case 2. $\mathfrak{A}$ is of the form $\neg\mathfrak{B}$, i. e. $a=2^7\cdot 7^{(a)_3}$ where $(a)_3$ is the Gödel number of $\mathfrak{B}$. By the hypothesis of induction

$$Q_0((a)_3, \alpha) \rightleftarrows \forall\beta\exists x R_0((a)_3, \bar{\alpha}(x), \bar{\beta}(x)) \rightleftarrows \exists\beta\forall x S_0((a)_3, \bar{\alpha}(x), \bar{\beta}(x)),$$

from which follows

$$\neg Q_0((a)_3, \alpha) \rightleftarrows \forall\beta\exists x \neg S_0((a)_3, \bar{\alpha}(x), \bar{\beta}(x)) \rightleftarrows \exists\beta\forall x \neg R_0((a)_3, \bar{\alpha}(x), \bar{\beta}(x)).$$

From (0), (1), (2) and this follows (3).

Case 3. $\mathfrak{A}$ is of the form $\bigvee_i \mathfrak{A}_i$, i. e. $a=2^9\cdot 7^{(a)_3}$ where $(a)_3$ defines recursively the Gödel numbers of $\mathfrak{A}_i$ as a function of i. If $Q_0(a, \alpha)$ then $\forall u\exists v T_1((a)_3, u, v)$ and $Q_0(\{(a)_3\}(i), \alpha)$ for some i, which implies $\forall\beta\exists x R_0(\{(a)_3\}(i), \bar{\alpha}(x), \bar{\beta}(x))$ and $\exists\beta\forall x S_0(\{(a)_3\}(i), \bar{\alpha}(x), \bar{\beta}(x))$ by the hypothesis of induction. From above we see easily that $\forall\beta\exists x R_0(a, \bar{\alpha}(x), \bar{\beta}(x))$ and $\exists\beta\forall x S_0(a, \bar{\alpha}(x), \bar{\beta}(x))$. Conversely if $\forall\beta\exists x R_0(a, \bar{\alpha}(x), \bar{\beta}(x))$, $\forall u\exists v T_1((a)_3, u, v)$ and $\forall\beta\exists x R_0(\{(a)_3\}(i), \bar{\alpha}(x), \bar{\beta}(x))$ for some i. By the hypothesis of induction this implies $Q_0(\{(a)_3\}(i), \alpha)$, whence follows $Q_0(a, \alpha)$. Similary for the dual form.

Case 4. $\mathfrak{A}$ is of the form $\bigwedge_i \mathfrak{A}_i$. This case can be treated in the same way as Case 3. Thus the proof is completed for $m=0$.

Now we assume that $Q_k(a, \alpha)$, $R_k(a, u, u_0, \cdots, u_k)$ and $S_k(a, u, u_0, \cdots, u_k)$ have been defined to satisfy the conditions stated at the beginning of this proof. We are to consider the case $m=k+1$. Let $D(a, \alpha, \beta)$ be

$$\forall u(\forall v\forall w(T_1(a, v, w) \vdash u \neq U(w)) \vdash \alpha(u)=\beta(u)).$$

Then $D(a, \alpha, \beta) \rightleftarrows \forall\gamma\exists x D_0(a, \bar{\alpha}(x), \bar{\beta}(x), \bar{\gamma}(x)) \rightleftarrows \exists\gamma\forall x D_1(a, \bar{\alpha}(x), \bar{\beta}(x), \bar{\gamma}(x))$ for some primitive recursive D_0 and D_1. The inductive definition of $Q_{k+1}(a, \alpha)$ is given as follows:

$$
\begin{aligned}
Q_{k+1}(a, \alpha) \rightleftarrows & (a=2^2\cdot 7^{(a)_3}\cdot 11^{(a)_4} \wedge E(a, \alpha)) \\
& \vee (a=2^2\cdot 7^{(a)_3} \wedge \neg Q_{k+1}((a)_3, \alpha)) \\
& \vee (a=2^9\cdot 7^{(a)_3} \wedge \forall u\exists v T_1((a)_3, u, v) \\
& \qquad \wedge \exists z\forall u(\neg T_1((a)_3, z, u) \vee Q_{k+1}(U(u), \alpha))) \\
& \vee (a=2^{11}\cdot 7^{(a)_3} \wedge \forall u_1\exists v T_1((a)_3, u_1, v) \\
& \qquad \wedge \forall u_0\forall u_1(\neg T_1((a)_3, u_0, u_1) \vee Q_{k+1}(U(u_1), \alpha))) \\
& \vee (a=2^{13}\cdot 7^{(a)_3}\cdot 11^{(a)_4} \wedge \forall u_2\exists v T_1((a)_3, u_2, v) \\
& \qquad \wedge \exists\gamma_0(D((a)_3, \alpha, \gamma_0) \wedge Q_k((a)_4, \gamma_0))) \\
& \vee (a=2^{15}\cdot 7^{(a)_3}\cdot 11^{(a)_4} \wedge \forall u_3\exists v T_1((a)_3, u_3, v) \\
& \qquad \wedge \forall\gamma_1(\neg D((a)_3, \alpha, \gamma_1) \vee Q_k((a)_4, \gamma_1))).
\end{aligned} \tag{4}
$$

By substituting in (4) for "$Q_{k+1}(a, \alpha)$" the predicate expressions

$$\text{“}\sharp\beta_0\natural\beta_1\cdots\exists\beta_k\forall\beta_{k+1}\exists x R_{k+1}(a,\bar{\alpha}(x),\bar{\beta}_0(x),\bar{\beta}_1(x),\cdots,\bar{\beta}_k(x),\bar{\beta}_{k+1}(x))\text{”}$$

and

$$\text{“}\natural\beta_0\sharp\beta_1\cdots\forall\beta_k\exists\beta_{k+1}\forall x S_{k+1}(a,\bar{\alpha}(x),\bar{\beta}_0(x),\bar{\beta}_1(x),\cdots,\bar{\beta}_k(x),\bar{\beta}_{k+1}(x))\text{”}$$

where $R_{k+1}(a,u,u_0,\cdots,u_{k+1})$ and $S_{k+1}(a,u,u_0,\cdots,u_{k+1})$ remain to be selected, we obtain the following equivalences (5) and (6). (We shall abbreviate $R_m(a,\bar{\alpha}(x),\bar{\beta}_0(x),\cdots,\bar{\beta}_m(x))$ and $S_m(a,\bar{\alpha}(x),\bar{\beta}_0(x),\cdots,\bar{\beta}_m(x))$ as $\bar{R}_m(a,\alpha,\beta_0,\cdots,\beta_m,x)$ and $\bar{S}_m(a,\alpha,\beta_0,\cdots,\beta_m,x)$ respectively.)

$$\begin{aligned}
&\sharp\beta_0\cdots\forall\beta_{k+1}\exists x\bar{R}_{k+1}(a,\alpha,\beta_0,\cdots,\beta_{k+1},x)\\
&\quad\rightleftarrows(a=2^2\cdot7^{(a)_3}\cdot11^{(a)_4}\wedge\exists xE_0(a,\bar{\alpha}(x)))\\
&\quad\vee(a=2^7\cdot7^{(a)_3}\wedge\sharp\beta_0\cdots\forall\beta_{k+1}\exists x\neg\bar{S}_{k+1}((a)_3,\alpha,\beta_0,\cdots,\beta_{k+1},x))\\
&\quad\vee(a=2^9\cdot7^{(a)_3}\wedge\forall u\exists vT_1((a)_3,u,v)\\
&\qquad\wedge\exists z\forall u(\neg T_1((a)_3,z,u)\vee\sharp\beta_0\cdots\forall\beta_{k+1}\exists x\bar{R}_{k+1}(U(u),\alpha,\beta_0,\cdots,\beta_{k+1},x)))\\
(5)&\quad\vee(a=2^{11}\cdot7^{(a)_3}\wedge\forall u_1\exists vT_1((a)_3,u_1,v)\\
&\qquad\wedge\forall u_0\forall u_1(\neg T_1((a)_3,u_0,u_1)\vee\sharp\beta_0\cdots\forall\beta_{k+1}\exists x\bar{R}_{k+1}(U(u_1),\alpha,\beta_0,\cdots,\beta_{k+1},x)))\\
&\quad\vee(a=2^{13}\cdot7^{(a)_3}\cdot11^{(a)_4}\wedge\forall u_2\exists vT_1((a)_3,u_2,v)\\
&\qquad\wedge\exists\gamma_0(\forall\beta_{m+1}\exists yD_0((a)_3,\bar{\alpha}(y),\bar{\gamma}_0(y),\bar{\beta}_{m+1}(y))\\
&\qquad\quad\wedge\natural\beta_1\cdots\exists\beta_k\forall\beta_{k+1}\exists x\bar{R}_k((a)_4,\gamma_0,\beta_1,\cdots,\beta_{k+1},x)))\\
&\quad\vee(a=2^{15}\cdot7^{(a)_3}\cdot11^{(a)_4}\wedge\forall u_3\exists vT_1((a)_3,u_3,v)\\
&\qquad\wedge\forall\gamma_1(\forall\gamma\exists x\neg D_1((a)_3,\bar{\alpha}(x),\bar{\gamma}_1(x),\bar{\gamma}(x))\\
&\qquad\quad\vee\natural\beta_1\cdots\exists\beta_k\forall\beta_{k+1}\exists x\bar{R}_k((a)_4,\gamma_1,\beta_1,\cdots,\beta_{k+1},x))).
\end{aligned}$$

$$\begin{aligned}
&\natural\beta_0\cdots\exists\beta_{k+1}\forall x\bar{S}_{k+1}(a,\alpha,\beta_0,\cdots,\beta_{k+1},x)\\
&\quad\rightleftarrows(a=2^2\cdot7^{(a)_3}\cdot11^{(a)_4}\wedge\forall x_0E_1(a,\bar{\alpha}(x_0)))\\
&\quad\vee(a=2^7\cdot7^{(a)_3}\wedge\natural\beta_0\cdots\exists\beta_{k+1}\forall x_1\neg\bar{R}_{k+1}((a)_3,\alpha,\beta_0,\cdots,\beta_{k+1},x_1))\\
&\quad\vee(a=2^9\cdot7^{(a)_3}\wedge\forall u\exists vT_1((a)_3,u,v)\\
&\qquad\wedge\exists z\forall u(\neg T_1((a)_3,z,u)\vee\natural\beta_0\cdots\exists\beta_{k+1}\forall x_2\bar{S}_{k+1}(U(u),\alpha,\beta_0,\cdots,\beta_{k+1},x_2)))\\
(6)&\quad\vee(a=2^{11}\cdot7^{(a)_3}\wedge\forall u_1\exists vT_1((a)_3,u_1,v)\\
&\qquad\wedge\forall u_0\forall u_1(\neg T_1((a)_3,u_0,u_1)\vee\natural\beta_0\cdots\exists\beta_{k+1}\forall x_3\bar{S}_{k+1}(U(u_1),\alpha,\beta_0,\cdots,\beta_{k+1},x_3)))\\
&\quad\vee(a=2^{13}\cdot7^{(a)_3}\cdot11^{(a)_4}\wedge\forall u_2\exists vT_1((a)_3,u_2,v)\\
&\qquad\wedge\exists\gamma_0(\exists\gamma\forall x_4D_1((a)_3,\bar{\alpha}(x_4),\bar{\gamma}_0(x_4),\bar{\gamma}(x_4))\\
&\qquad\quad\wedge\sharp\beta_1\cdots\forall\beta_k\exists\beta_{k+1}\forall x_4\bar{S}_k((a)_4,\gamma_0,\beta_1,\cdots,\beta_{k+1},x_4)))\\
&\quad\vee(a=2^{15}\cdot7^{(a)_3}\cdot11^{(a)_4}\wedge\forall u_3\exists vT_1((a)_3,u_3,v)\\
&\qquad\wedge\forall\gamma_1(\exists\beta_{k+1}\forall y\neg D_0((a)_3,\bar{\alpha}(y),\bar{\gamma}_1(y),\bar{\beta}_{k+1}(y))\\
&\qquad\quad\vee\sharp\beta_1\cdots\forall\beta_k\exists\beta_{k+1}\forall x_5\bar{S}_k((a)_4,\gamma_1,\beta_1,\cdots,\beta_{k+1},x_5))).
\end{aligned}$$

In the right members of (5) and (6) the quantifiers can be advanced to give equivalents of the following (7.0) and (8.0), respectively or (7.1) and (8.1), respectively, according as $\sharp$ is $\forall$ or $\exists$:

(7.0) $$\exists z\forall u\forall u_0\forall u_1\forall \gamma_1\forall \beta_0\exists \gamma_0\exists \beta_1 \cdots \exists \beta_k\forall u_2\forall u_3\forall \gamma\forall \beta_{k+1}\exists v\exists y\exists x P_0$$

where P_0 is exactly the right member of (5) with its quantifiers omitted;

(8.0) $$\exists z\forall u\forall u_0\forall u_1\exists \gamma_0\exists \beta_0\forall \gamma_1\forall \beta_1 \cdots \forall \beta_k\forall u_2\forall u_3\exists v\exists \gamma\exists \beta_{k+1}\forall y\forall x_0 \cdots \forall x_5 P_1$$

where P_1 is exactly the right member of (6) with its quantifiers omitted.

(7.1) $$\exists z\forall u\forall u_0\forall u_1\exists \gamma_0\exists \beta_0\forall \gamma_1\forall \beta_1 \cdots \exists \beta_k\forall u_2\forall u_3\forall \gamma\forall \beta_{k+1}\exists v\exists y\exists x P_0;$$

(8.1) $$\exists z\forall u\forall u_0\forall u_1\forall \gamma_1\forall \beta_0\exists \gamma_0\exists \beta_1 \cdots \forall \beta_k\forall u_2\forall u_3\exists v\exists \gamma\exists \beta_{k+1}\forall y\forall x_0 \cdots \forall x_5 P_1.$$

Then in the same way as in the case $m=0$, we can define $R_{k+1}(a, u, u_0, \cdots, u_{k+1})$ and $S_{k+1}(a, u, u_0, \cdots, u_{k+1})$ satisfying the required condition. By mathematical induction on m we can complete the proof of the first part of Theorem 2.

Now let $\mathfrak{A}$ be a formula with constructive infinitely long expressions such that the outermost logical symbol of $\mathfrak{A}$ is $\exists$ (or $\forall$), $a = \ulcorner \mathfrak{A} \urcorner$ and $\mathrm{n}'(\mathfrak{A}) = m < \omega$. $\mathfrak{A}$ is representable by

$$a = 2^{13}\cdot 7^{(a)_3}\cdot 11^{(a)_4} \wedge \exists \gamma(D((a)_3, \alpha, \gamma) \wedge Q_{m-1}((a)_4, \gamma))$$

$$(\text{or } a = 2^{15}\cdot 7^{(a)_3}\cdot 11^{(a)_4} \wedge \forall \gamma(\neg D((a)_3, \alpha, \gamma) \vee Q_{m-1}((a)_4, \gamma))).$$

Since $Q_{m-1}((a)_4, \gamma)$ is expressible in $\Sigma^1_m \cap \Pi^1_m$, this is expressible in Σ^1_m (or Π^1_m). Thus we complete the proof.

THEOREM 3. *Any predicate expressible in the n-function quantifier form is representable by a formula* $\mathfrak{A}$ *with constructive infinitely long expressions such that* $\mathrm{n}'(\mathfrak{A}) = n$.

PROOF. We shall consider that the theory of recursive functions is constructed from the following primitives which are correlated Gödel numbers as follows:

Primitives: $0, 1, a_n, +, \times, \alpha_m, =, \neg, \vee, \wedge, \exists, \forall$

Correlated Gödel numbers: $3, 5, 7^{n+1}, 9, 11, 13^{m+1}, 15, 17, 19, 21, 23, 25$.

Then the Gödel numbers of formulas and terms of the theory of recursive functions are defined in the same way as in [3]. Since no confusion is to be feared, we shall denote the Gödel number of a formula or a term A of the theory of recursive functions as $\ulcorner A \urcorner$. We shall fix two primitive recursive functions $\lambda i\{c\}(i)$ and $\lambda ij\{g\}(i, j)$ and correlate the n-th variable a_n to the variable $v_{\{c\}(n)}$ and the m-th function variable α_m supplied by i (i.e. $\alpha_m(i)$) to $v_{\{g\}(m,i)}$. We shall denote $\{c\}(i)$ and $\{g\}(i, j)$ by $c(i)$ and $g(i, j)$, respectively.

Let A be a formula in the theory of recursive functions. We shall denote the number of logical symbols in A by $\mathrm{l}(A)$. In the end we shall obtain partial recursive functions φ_n such that $\varphi_{\mathrm{l}(A)}(\ulcorner A \urcorner)$ is defined and is the Gödel number of a formula with infinitely long expressions, which represents A in the language with infinitely long expressions. We shall first explain the outline to obtain φ_n. The formula of the form $T_1=T_2$ contained in A is regarded as $\bigvee_i(i=T_1\wedge i=T_2)$. (Though the expression of the form '$i=T$' is not a formula with infinitely long expressions, we shall make use of this and similar rough wording for the convenience of explanation.) In the following let J and K stand for sequences $j_1,\cdots,j_m$ and $k_1,\cdots,k_m$ $(0\leqq m)$, respectively. We shall obtain partial recursive functions $\psi^m(i,t,J,K)$ for $m\geqq 0$ satisfying the following condition:

1) If t is the Gödel number of a term T, $\psi^m(i,t,J,K)$ is defined and is the Gödel number of a formula with infinitely long expressions which is denoted as $[i=T]$ sometimes and which means $i=T_0$, where T_0 is obtained from T by replacing every occurrence of the variable a_{j_l} by the individual constant k_l for each l $(1\leqq l\leqq m)$. Roughly speaking, $[i=T]$ is obtained as follows:

(1.1) If T is 0, $[i=T]$ is $i=0$.

(1.2) If T is 1, $[i=T]$ is $i=1$.

(1.3.1) If T is a_j where $j\neq j_l$ $(1\leqq l\leqq m)$, $[i=T]$ is $i=v_{c(j)}$.

(1.3.2) If T is a_{j_l} $(1\leqq l\leqq m)$, $[i=T]$ is $i=k_l$.

(1.4) If T is $\alpha_j(T_1)$, $[i=T]$ is $\bigvee_k(i=v_{g(j,k)}\wedge[k=T_1])$.

(1.5) If T is T_1+T_2 or $T_1\cdot T_2$, $[i=T]$ is

$$\bigvee_{h_1}\bigvee_{h_2}(i=h_1+h_2\wedge[h_1=T_1]\wedge[h_2=T_2])$$

or

$$\bigvee_{h_1}\bigvee_{h_2}(i=h_1\cdot h_2\wedge[h_1=T_1]\wedge[h_2=T_2])$$

where h_1+h_2 or $h_1\cdot h_2$ means an individual constant which is the value of h_1+h_2 or $h_1\cdot h_2$ for individual constants h_1 and h_2.

Then we shall obtain partial recursive functions $\varphi_n^m(a,J,K)$ satisfying the following condition:

(2) If a is the Gödel number of a formula A with $1(A)\leqq n$, $\varphi_n^m(a,J,K)$ is defined and is the Gödel number of a formula $\mathfrak{A}$ with infinitely long expressions obtained as follows:

(2.0.1) Every variable a_j where $j\neq j_l$ $(1\leqq l\leqq m)$ in A is correlated to $v_{c(j)}$.

(2.0.2) a_{j_l} $(1\leqq l\leqq m)$ in A is correlated to the individual constant k_l.

(2.1) If A is of the form $T_1=T_2$, $\mathfrak{A}$ is $\bigvee_i(\wedge([i=T_1],[i=T_2],[i=T_2],\cdots))$,

(2.2) If A is of the form $\urcorner A_1$, $\mathfrak{A}$ is $\urcorner\mathfrak{A}_1$ where $\mathfrak{A}_1$ corresponds to A_1 by the relation $\ulcorner\mathfrak{A}_1\urcorner=\varphi_{n-1}^m(\ulcorner A_1\urcorner,J,K)$.

(2.3) If A is of the form $A_1\vee A_2$ (or $A_1\wedge A_2$), $\mathfrak{A}$ is $\vee(\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_2,\cdots)$ (or $\wedge(\mathfrak{A}_1,\mathfrak{A}_2,\mathfrak{A}_2,\cdots)$) where $\mathfrak{A}_i$ correspond to A_i by the relation $\ulcorner\mathfrak{A}_i\urcorner=\varphi_{n-1}^m(\ulcorner A_i\urcorner,J,K)$ for $i=1,2$.

(2.4) If A is of the form $\exists a_jA_1$ (or $\forall a_jA_1$), $\mathfrak{A}$ is $\bigvee_k\mathfrak{A}_1$ (or $\bigwedge_k\mathfrak{A}_1$) where $\mathfrak{A}_1$ corresponds to A_1 by the relation $\ulcorner \mathfrak{A}_1 \urcorner = \varphi_{n-1}^{m+1}(\ulcorner A_1 \urcorner, J, j, K, k)$.

(2.5) If A is of the form $\exists \alpha_j A_1$ (or $\forall \alpha_j A_1$), $\mathfrak{A}$ is

$$\exists v_{g(j,0)} v_{g(j,1)} \cdots \mathfrak{A}_1$$

$$(\text{or } \forall v_{g(j,0)} v_{g(j,1)} \cdots \mathfrak{A}_1)$$

where $\mathfrak{A}_1$ corresponds to A_1 by the relation $\ulcorner \mathfrak{A}_1 \urcorner = \varphi_{n-1}^{m}(\ulcorner A_1 \urcorner, J, K)$. $\varphi_n(a)$ is defined to be $\varphi_n^0(a)$. It is easy to see that, if A is in Σ_n^1 or in Π_n^1, $\mathrm{n}'(\mathfrak{A}) = n$ for the formula $\mathfrak{A}$ which corresponds to A by the relation $\ulcorner \mathfrak{A} \urcorner = \varphi_{1(A)}(\ulcorner A \urcorner)$.

Next we shall obtain the functions ψ^m and φ_n^m in practice. Let

$$\rho_0^m(w, i, l, x, J, K, j, v) \cong \begin{cases} 2^2 \cdot 7^{3^{i+1}} \cdot 11^{5^{g(l,j)+1}} & \text{if } v = 0 \\ \{w\}(j, x, J, K) & \text{otherwise} \end{cases}$$

and $p_{0,m}$ define ρ_0^m recursively. Let

$$\rho_1^m(w, i, l, x, J, K, j) = 2^{11} \cdot 7^{S_1^{2m+5}(p_{0,m}, w, i, l, x, J, K, j)}$$

and $p_{1,m}$ define ρ_1^m recursively. Let

$$\rho_2^m(w, i, x, y, J, K, i_1, i_2, v) \cong \begin{cases} 2^2 \cdot 7^{3^{i+1}} \cdot 11^{3^{i_1+i_2+1}} & \text{if } v = 0, \\ \{w\}(i_1, x, J, K) & \text{if } v = 1, \\ \{w\}(i_2, y, J, K) & \text{otherwise} \end{cases}$$

and $p_{2,m}$ define ρ_2^m recursively. Let

$$\rho_3^m(w, i, x, y, J, K, i_1, i_2) \cong 2^{11} \cdot 7^{S_1^{2m+6}(p_{2,m}, w, i, x, y, J, K, i_1, i_2)}$$

and $p_{3,m}$ define ρ_3^m recursively. Let

$$\rho_4^m(w, i, x, y, J, K, i_1) \cong 2^9 \cdot 7^{S_1^{2m+5}(p_{3,m}, w, i, x, y, J, K, i_1)}$$

and $p_{4,m}$ define ρ_4^m recursively. Let ρ_5^m be obtained from ρ_2^m by replacing the part $i_1 + i_2$ by $i_1 \cdot i_2$, ρ_6^m be obtained from ρ_3^m by replacing $p_{2,m}$ by $p_{5,m}$, $p_{6,m}$ define ρ_6^m recursively, ρ_7^m be obtained from ρ_4^m by replacing $p_{3,m}$ by $p_{6,m}$ and $p_{7,m}$ define ρ_7^m recursively. Then it suffices for ψ^m to take $\psi^m(i, t, J, K) \cong \{r_m\}(i, t, J, K)$, where r_m is a solution for w of the following equation given by the recursion theorem:

$$\begin{aligned} &\{w\}(i, t, J, K) \\ &\cong \mu u((t = 3 \wedge u = 2^2 \cdot 7^{3^{i+1}} \cdot 11^3) \\ &\quad \vee (t = 5 \wedge u = 2^2 \cdot 7^{3^{i+1}} \cdot 11^{3^2}) \\ &\quad \vee (t = 7^{(t)_3} \wedge (t)_3 \neq 0 \\ &\qquad \wedge (\forall v(0 < v \leqq m \vdash (t)_3 \dot{-} 1 \neq j_v) \wedge u = 2^2 \cdot 7^{3^{i+1}} \cdot 11^{5^{c((t)_3 \dot{-} 1)+1}}) \\ &\qquad \vee \exists v(0 < v \leqq m \wedge (t)_3 \dot{-} 1 = j_v \wedge u = 2^2 \cdot 7^{3^{i+1}} \cdot 11^{3^{k_v+1}}))) \end{aligned}$$

$$\vee (t = 2^{13^{(t)_{0,5}}} \cdot 3^{(t)_1} \wedge (t)_{0,5} \neq 0$$

$$\wedge\, u = 2^9 \cdot 7^{S_1^{2m+4}(p_{1,m}, w, i, (t)_{0,5} \dot{-} 1, (t)_1 J, K)})$$

$$\vee (t = 2^9 \cdot 3^{(t)_1} \cdot 5^{(t)_2} \wedge u = 2^9 \cdot 7^{S_1^{2m+4}(p_{4,m}, w, i, (t)_1, (t)_2, J, K)}) .$$

$$\vee (t = 2^{11} \cdot 3^{(t)_1} \cdot 5^{(t)_2} \wedge u = 2^9 \cdot 7^{S_1^{2m+4}(p_{7,m}, w, i, (t)_1, (t)_2, J, K)})) .$$

Let

$$\theta_0^m(x, y, J, K, i, v) \cong \begin{cases} \psi^m(i, x, J, K) & \text{if} \quad v = 0 , \\ \psi^m(i, y, J, K) & \text{otherwise} \end{cases}$$

and $q_{0,m}$ define θ_0^m recursively. Let

$$\theta_1^m(x, y, J, K, i) = 2^{11} \cdot 7^{S_1^{2m+3}(q_{0,m}, x, y, J, K, i)}$$

and $q_{1,m}$ define θ_1^m recursively. Then let

$$\varphi_0^m(a, J, K) = 2^9 \cdot 7^{S_1^{2m+2}(q_{1,m}, (a)_1, (a)_2, J, K)} .$$

and $e_{0,m}$ define φ_0^m recursively. We are to obtain φ_{n+1}^m under the hypothesis of induction that φ_q^p have been obtained for all $p \geqq 0$ and q $(0 \leqq q \leqq n)$ and possess the required property. Let $e_{q,p}$ define φ_q^p recursively for all $p \geqq 0$ and q $(0 \leqq q \leqq n)$. Let

$$\zeta_n^m(x, y, J, K, v) \cong \begin{cases} \varphi_n^m(x, J, K) & \text{if} \quad v = 0 , \\ \varphi_n^m(y, J, K) & \text{otherwise} \end{cases}$$

and $r_{n,m}$ define ζ_n^m recursively. Let

$$\varphi_{n+1}^m(a, J, K)$$

$$\cong \mu u((a = 2^{15} \cdot 3^{(a)_1} \cdot 5^{(a)_2} \wedge u = \varphi_n^m(a, J, K))$$

$$\vee (a = 2^{17} \cdot 3^{(a)_1} \wedge u = 2^7 \cdot 7^{\varphi_n^m((a)_1, J, K)})$$

$$\vee (a = 2^{19} \cdot 3^{(a)_1} \cdot 5^{(a)_2} \wedge u = 2^9 \cdot 7^{S_1^{2m+2}(r_{n,m}, (a)_1, (a)_2, J, K)})$$

$$\vee (a = 2^{21} \cdot 3^{(a)_1} \cdot 5^{(a)_2} \wedge u = 2^{11} \cdot 7^{S_1^{2m+2}(r_{n,m}, (a)_1, (a)_2, J, K)})$$

$$\vee (a = 2^{23} \cdot 3^{(a)_1} \cdot 5^{(a)_2}$$

$$\wedge (((a)_1 = 7^{(a)_{1,3}} \wedge (a)_{1,3} \neq 0 \wedge \forall v(0 < v \leqq m \vdash (a)_{1,3} \dot{-} 1 \neq j_v)$$

$$\wedge\, u = 2^9 \cdot 7^{S_1^{2m+2}(e_{n,m+1}, (a)_2, J, (a)_{1,3} \dot{-} 1, K)})$$

$$\vee ((a)_1 = 13^{(a)_{1,5}} \wedge (a)_{1,5} \neq 0$$

$$\wedge\, u = 2^{13} \cdot 7^{S_1^1(g, (a)_{1,5} \dot{-} 1)} \cdot 11^{\varphi_n^m((a)_2, J, K)})))$$

$$\vee (a = 2^{25} \cdot 3^{(a)_1} \cdot 5^{(a)_2}$$

$$\wedge (((a)_1 = 7^{(a)_{1,3}} \wedge (a)_{1,3} \neq 0 \wedge \forall v(0 < v \leqq m \vdash (a)_{1,3} \dot{-} 1 \neq j_v)$$

$$\wedge\, u = 2^{11} \cdot 7^{S_1^{2m+2}(e_{n,m+1}, (a)_2, J, (a)_{1,3} \dot{-} 1, K)})$$

$$\vee ((a)_1 = 13^{(a)_{1,5}} \wedge (a)_{1,5} \neq 0$$

$$\wedge\, u = 2^{15} \cdot 7^{S_1^1(g, (a)_{1,5} \dot{-} 1)} \cdot 11^{\varphi_n^m((a)_2, J, K)})))$$

and $e_{n+1,m}$ define φ_{n+1}^m recursively. Take $\varphi_n(a) \cong \varphi_n^0(a)$. By induction on n we can see that φ_n^m (and φ_n) satisfy the required conditions.

THEOREM 4. *Any hyperarithmetical formula is representable by a quantifier-free formula with constructive infinitely long expressions.*

PROOF. To prove the theorem we shall first obtain a partial recursive function $\chi^m(a, n_1, \cdots, n_m, i_1, \cdots, i_m)$ satisfying the following condition: If a is the Gödel number of a quantifier-free formula $\mathfrak{A}$ with infinitely long expressions, then $\chi^m(a, n_1, \cdots, n_m, i_1, \cdots, i_m)$ is defined and is the Gödel number of the formula obtained from $\mathfrak{A}$ by substituting individual constants $i_1, \cdots, i_m$ for variables $v_{n_1}, \cdots, v_{n_m}$ respectively in $\mathfrak{A}$. Let N and I stand for sequences $n_1, \cdots, n_m$ and $i_1, \cdots, i_m$, respectively. Let

$$\tau(w, d, N, I, x) \cong \{w\}(\{d\}(x), N, I)$$

and t define τ recursively. It suffices to take $\chi^m(a, N, I) \cong \{k_m\}(a, N, I)$ where k_m is a solution for w of the following equation given by the recursion theorem:

$$\{w\}(a, N, I)$$

$$\cong \mu u((a = 2^2 \cdot 7^{(a)_3} \cdot 11^{(a)_4}$$

$$\wedge ((((((a)_3 = 3^{(a)_{3,1}} \wedge (a)_{3,1} \neq 0)$$

$$\vee ((a)_3 = 5^{(a)_{3,2}} \wedge (a)_{3,2} \neq 0$$

$$\wedge \forall v(0 < v \leqq m \vdash (a)_{3,2} \dot{-} 1 \neq n_v)))$$

$$\wedge (((a)_4 = 3^{(a)_{4,1}} \wedge (a)_{4,1} \neq 0)$$

$$\vee ((a)_4 = 5^{(a)_{4,2}} \wedge (a)_{4,2} \neq 0$$

$$\wedge \forall v(0 < v \leqq m \vdash (a)_{4,2} \dot{-} 1 \neq n_v)))$$

$$\wedge\, u = a)$$

$$\vee ((((a)_3 = 3^{(a)_{3,1}} \wedge (a)_{3,1} \neq 0)$$

$$\vee ((a)_3 = 5^{(a)_{3,2}} \wedge (a)_{3,2} \neq 0$$

$$\wedge \forall v(0 < v \leqq m \vdash (a)_{3,2} \dot{-} 1 \neq n_v)))$$

$$\wedge (a)_4 = 5^{(a)_{4,2}} \wedge (a)_{4,2} \neq 0$$

$$\wedge \exists v(0 < v \leqq m \wedge (a)_{4,2} \dot{-} 1 = n_v \wedge u = 2^2 \cdot 7^{(a)_3} \cdot 11^{3^{i_v+1}}))$$

$$\vee ((a)_3 = 5^{(a)_{3,2}} \wedge (a)_{3,2} \neq 0$$
$$\wedge (((a)_4 = 3^{(a)_{4,1}} \wedge (a)_{4,1} \neq 0)$$
$$\vee ((a)_4 = 5^{(a)_{4,2}} \wedge (a)_{4,2} \neq 0$$
$$\wedge \forall v(0 < v \leqq m \vdash (a)_{4,2} \dot{-} 1 \neq n_v)))$$
$$\wedge \exists v(0 < v \leqq m \wedge (a)_{3,2} \dot{-} 1 = n_v \wedge u = 7^{3^{iv+1}} \cdot 11^{(a)_4}))$$
$$\vee ((a)_3 = 5^{(a)_{3,2}} \wedge (a)_{3,2} \neq 0 \wedge (a)_4 = 5^{(a)_{4,2}} \wedge (a)_{4,2} \neq 0$$
$$\wedge \exists v \exists w(0 < v \leqq m \wedge 0 < w \leqq m \wedge (a)_{3,2} \dot{-} 1 = n_v \wedge (a)_{4,2} \dot{-} 1 = n_w$$
$$\wedge u = 2^2 \cdot 7^{3^{iv+1}} \cdot 11^{3^{iw+1}}))))$$
$$\vee (a = 2^7 \cdot 7^{(a)_3} \wedge u = 2^7 \cdot 7^{\{w\}((a)_3, N, I)})$$
$$\vee (a = 2^9 \cdot 7^{(a)_3} \wedge u = 2^9 \cdot 7^{S_1^{2m+2}(t, w, (a)_3, N, I)})$$
$$\vee (a = 2^{11} \cdot 7^{(a)_3} \wedge u = 2^{11} \cdot 7^{S_1^{2m+2}(t, w, (a)_3, N, I)})).$$

By transfinite induction on n($\mathfrak{A}$) we see that χ^m satisfies the required condition.

Let $y \in O$ and $3 \cdot 5^z \in O$. Then

$$H_1(a_0) \rightleftarrows a_0 = a_0 ;$$
$$H_{2^y}(a_0) \rightleftarrows \exists a_1 \exists a_2 (a_2 = \prod_{a_3 < a_1} p_{a_3}^{(a_2)_{a_3}} \wedge T_1^1(a_2, a_0, a_0, a_1)$$
$$\wedge \forall a_3 (a_3 < a_1 \vdash (H_y(a_3) \wedge (a_2)_{a_3} = 0)$$
$$\vee (\neg H_y(a_3) \wedge (a_2)_{a_3} = 1))) ;$$
$$H_{3 \cdot 5^z}(a_0) \rightleftarrows H_{\{z\}((a_0)_1)_O}((a_0)_0)$$
$$\rightleftarrows \exists a_1 \exists a_2 (a_1 = (a_0)_0 \wedge a_2 = (a_0)_1 \wedge H_{\{z\}(a_2)_O}(a_1)).$$

We shall obtain a partial recursive function $\eta(y)$ satisfying the following condition: If $y \in O$, then $\eta(y)$ is defined and is the Gödel number of a quantifier-free formula with infinitely long expressions which means $H_y(a_0)$.

We shall take $\eta(y) \cong \{h\}(y)$ where h is a solution for w of an equation given below by the recursion theorem. To give the equation we shall treat several cases separately.

Case 1. $y = 1$. Let

$$\{w\}(y) = 2^2 \cdot 7^{5^{c(0)+1}} \cdot 11^{5^{c(0)+1}}.$$

Case 2. $y = 2^{(y)_0} \wedge (y)_0 \neq 0$. We shall translate $H_{2^z}(a_0)$ changing a_1, a_2, a_3 to i, j, k, respectively. Let

$$a_2 = \prod_{a_3 < a_1} p_{a_3}^{(a_2)_{a_3}} \rightleftarrows A_1(a_1, a_2),$$
$$\neg (a_3 < a_1) \rightleftarrows A_2(a_1, a_3),$$
$$(a_2)_{a_3} = 0 \rightleftarrows A_3(a_2, a_3),$$

$$(a_2)_{a_3}=1 \rightleftarrows A_4(a_2, a_3),$$
$$T_1^1(a_2, a_0, a_0, a_1) \rightleftarrows A_5(a_0, a_1, a_2)$$

where $A_1, \cdots, A_5$ are arithmetical and expressible by a quantifier-free formula with constructive infinitely long expressions by Theorem 3. Let

$$\delta_1(i, j) \cong \chi^2(\varphi_n(\ulcorner A_1(a_1, a_2)\urcorner), c(1), c(2), i, j),$$
$$\delta_2(i, k) \cong \chi^2(\varphi_n(\ulcorner A_2(a_1, a_3)\urcorner), c(1), c(3), i, k),$$
$$\delta_3(j, k) \cong \chi^2(\varphi_n(\ulcorner A_3(a_2, a_3)\urcorner), c(2), c(3), j, k),$$
$$\delta_4(j, k) \cong \chi^2(\varphi_n(\ulcorner A_4(a_2, a_3)\urcorner), c(2), c(3), j, k),$$
$$\delta_5(i, j) \cong \chi^2(\varphi_n(\ulcorner A_5(a_0, a_1, a_2)\urcorner), c(1), c(2), i, j)$$

where $n \geqq l(A_m)$ $(1 \leqq m \leqq 5)$. Let

$$\kappa_0(w, z, j, k, v) \cong \begin{cases} \chi^1(\{w\}(z), c(0), k) & \text{if } v=0, \\ \delta_3(j, k) & \text{otherwise} \end{cases}$$

and r_0 define κ_0 recursively. Let

$$\kappa_1(w, z, j, k) = 2^{11} \cdot 7^{S_1^4(r_0, w, z, j, k)}$$

(which corresponds to $H_z(a_3) \wedge (a_2)_{a_3}=0$). Proceeding with this way successively we can obtain $\theta_0(w, z, i, j)$ corresponding to $\forall a_3(a_3 < a_1 \vdash (H_z(a_3) \wedge (a_2)_{a_3}=0) \vee (\neg H_z(a_3) \wedge (a_2)_{a_3}=1))$. Let

$$\theta_1(w, z, i, j, v) \cong \begin{cases} \delta_1(i, j) & \text{if } v=0, \\ \delta_5(i, j) & \text{if } v=1, \\ \theta_0(w, z, i, j) & \text{otherwise} \end{cases}$$

and s_1 define θ_1 recursively. Let

$$\theta_2(w, z, i, j) = 2^{11} \cdot 7^{S_1^4(s_1, w, z, i, j)}$$

and s_2 define θ_2 recursively. Let

$$\sigma(w, z, i) = 2^9 \cdot 7^{S_1^3(s_2, w, z, i)}$$

and s define σ recursively. Then take

$$\{w\}(2^{(y)_0}) = 2^9 \cdot 7^{S_1^2(s, w, (y)_0)}.$$

As is easily seen this corresponds to $H_{2^{(y)_0}}(a_0)$.

Case 3. $y = 3 \cdot 5^{(y)_2}$. Let

$$a_1 = (a_0)_0 \wedge a_2 = (a_0)_1 \rightleftarrows B(a_0, a_1, a_2)$$

where B is arithmetical and expressible by a quantifier-free formula with constructive infinitely long expressions by Theorem 3. Let

$$\pi_0(w, z, i, j, v) \cong \begin{cases} \chi^2(\varphi_n(\ulcorner B(a_0, a_1, a_2)\urcorner), c(1), c(2), i, j) & \text{if } v=0, \\ \chi^1(\{w\}(\{z\}(j_O)), c(0), i) & \text{otherwise} \end{cases}$$

(where $n \geqq l(B(a_0, a_1, a_2))$) and d_0 define π_0 recursively. Let

$$\pi_1(w, z, i, j) = 2^{11} \cdot 7^{S_1^4(d_0, w, z, i, j)}$$

and d_1 define π_1 recursively. Let

$$\pi_2(w, z, i) = 2^9 \cdot 7^{S_1^3(d_1, w, z, i)}$$

and d define π_2 recursively. Then let

$$\{w\}(3 \cdot 5^{(y)_2}) \cong 2^9 \cdot 7^{S_1^2(d, w, (y)_2)}.$$

Conclusion by the definition by cases: Let

$$\{w\}(y) \cong \begin{cases} 2^2 \cdot 7^{5^{c(0)+1}} \cdot 11^{5^{c(0)+1}} & \text{if } y=1, \\ 2^9 \cdot 7^{S_1^2(s, w, (y)_0)} & \text{if } y=2^{(y)_0} \wedge (y)_0 \neq 0, \\ 2^9 \cdot 7^{S_1^2(d, w, (y)_2)} & \text{if } y=3 \cdot 5^{(y)_2}. \end{cases}$$

We can prove that $\eta(y)$ satisfies the required condition by induction on $y \in O$.

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