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A remark on the groups of type G_2 and F_4

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The problem of classifying Lie algebras over a field of characteristic zero is considered by Landherr, Jacobson and others. Their method is to reduce the problem to the classification of associative algebras with involution or some other kinds of algebras. Using the Galois cohomology theory, A. Weil [6] gave a general proof of such results for classical groups over a field of characteristic zero. In this paper, we make a slight modification of his method so as to make it applicable to some exceptional groups (i.e. to the groups of type G_2 and F_4).

Any group of type G_2 defined over a perfect field k of characteristic >3 is obtained as the automorphism group of an octanion algebra over k, and the automorphism groups are isomorphic over k if and only if the corresponding octanions are. There is a similar relation between the groups of type F_4 and exceptional simple Jordan algebras.

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§1. Let k be a field in the universal domain Ω, and let I be a finite set of indices. Let V and U_i $(i \in I)$ be affine varieties and let α_i $(i \in I)$ be rational maps from $V \times U_i$ to V. We consider the system $\mathbf{V} = (V; U_i, \alpha_i)$. For simplicity we call such a system $\mathbf{V}$ "an AG-variety". An AG-variety $\mathbf{V} = (V; U_i, \alpha_i)$ is called to be defined over k if V, U_i's and α_i's are all defined over k.

Let K be a field containing k. Two AG-varieties defined over k $\mathbf{V} = (V; U_i, \alpha_i)$ and $\mathbf{V}' = (V'; U_i', \alpha_i')$ with the same index set I are called to be isomorphic over K, if there exists a system of rational isomorphisms $F = (f; f_i)$ defined over K such that

$$f: V \to V', \qquad f_i: U_i \to U_i' \ (i \in I),$$

$$f(\alpha_i(v, u_i)) = \alpha_i'(f(v), f_i(u_i)) \qquad (i \in I) \cdots\cdots\cdots\cdots\cdots (1)$$

where v, u_i $(i \in I)$ are respectively generic points of V, U_i $(i \in I)$ over K. An isomorphism from $\mathbf{V}$ onto $\mathbf{V}$ itself is called an automorphism of $\mathbf{V}$. $\mathrm{Aut}(\mathbf{V})$ denotes the group of all automorphisms of $\mathbf{V}$. $\mathrm{Aut}(\mathbf{V})_K$ denotes the group of all automorphisms defined over K of $\mathbf{V}$.

DEFINITION. Let $\mathbf{V}$ be an AG-variety defined over K and $\mathbf{V}'$ an AG-variety defined over k. If $\mathbf{V}'$ is isomorphic to $\mathbf{V}$ over K, then we say that "$\mathbf{V}'$ is a k-form of $\mathbf{V}$".

Now let K be a finite Galois extension of k with Galois group $\mathfrak{G}=\mathfrak{G}(K/k)$ and $\mathbf{V}$ an AG-variety defined over k. A mapping $\Phi_\lambda:\lambda\rightarrow\Phi_\lambda$ $\lambda\in\mathfrak{G}$, $\Phi_\lambda\in\mathrm{Aut}(\mathbf{V})_K$ is called a 1-cocycle if it satisfies $\Phi_\lambda\cdot\Phi_\mu^\lambda=\Phi_{\lambda\mu}$ for any (λ,μ) in $\mathfrak{G}\times\mathfrak{G}$. $C^1(\mathfrak{G},\mathrm{Aut}(\mathbf{V})_K)$ denotes the set of all 1-cocycle of $\mathfrak{G}$ in $\mathrm{Aut}(\mathbf{V})_K$. Two 1-cocycles Φ_λ and Φ'_λ are said to be cohomologous if there exists an element Ψ of $\mathrm{Aut}(\mathbf{V})_K$ satisfying $\Psi\cdot\Phi_\lambda\cdot\Psi^{-\lambda}=\Phi'_\lambda$. $H^1(\mathfrak{G},\mathrm{Aut}(\mathbf{V})_K)$ denotes the quotient set of $C^1(\mathfrak{G},\mathrm{Aut}(\mathbf{V})_K)$ by the cohomologous relation, called the 1-cohomology set of $\mathfrak{G}$ in $\mathrm{Aut}(\mathbf{V})_K$. The following proposition is a direct consequence of the definition.

PROPOSITION 1. *Let $\mathbf{V}_0$ be an AG-variety defined over k, and $\mathbf{V}$ be a k-form over K of $\mathbf{V}_0$, and $F=[f;f_i]$ an isomorphism from $\mathbf{V}_0$ to $\mathbf{V}$ over K. Then $\Phi_\lambda=F^{-1}F^\lambda$ is a 1-cocycle of $\mathfrak{G}(K/k)$ in $\mathrm{Aut}(\mathbf{V}_0)_K$.*

Let $\mathbf{V}'$ be another k-form over K of $\mathbf{V}_0$ with $F':\mathbf{V}_0\rightarrow\mathbf{V}'$, $\Phi'_\lambda=F'^{-1}F'^\lambda$. Then $\mathbf{V}$ is isomorphic to $\mathbf{V}'$ over k if and only if Φ_λ is cohomologous of Φ'_λ.

PROPOSITION 2. *Let Φ_λ be a 1-cocycle of $\mathfrak{G}(K/k)$ in $\mathrm{Aut}(\mathbf{V})_K$. Then there exists a k-form $\mathbf{V}'$ over K of $\mathbf{V}$, and an isomorphism F from $\mathbf{V}$ to $\mathbf{V}'$ defined over K such that $\Phi_\lambda=F^{-1}F^\lambda$.*

PROOF. Let $\Phi_\lambda=[\varphi_\lambda;\varphi_{i\lambda}]$ with $\varphi_\lambda:V\rightarrow V,\varphi_{i\lambda}:U_i\rightarrow U_i$. Then φ_λ and $\varphi_{i\lambda}$ satisfy the relations

$$\varphi_\lambda\varphi_\mu^\lambda=\varphi_{\lambda\mu}\quad \varphi_{i\lambda}\varphi_{i\mu}^\lambda=\varphi_{i\lambda\mu}\quad (\lambda,\mu)\in\mathfrak{G}\times\mathfrak{G},(i\in I).$$

So by Theorem 1 of Weil [5], there exist varieties V',U'_i all defined over k, and there exist isomorphisms f,f_i all defined over K such that

$$\varphi_\lambda=f^{-1}f^\lambda\qquad \varphi_{i\lambda}=f_i^{-1}f_i^\lambda\cdots\cdots\cdots\cdots\cdots\cdots(2)$$

On the system $[V';U'_i]$, we define a structure of an AG-variety by

$$\alpha'_i(v',u'_i)=f(\alpha_i(f^{-1}(v'),f_i^{-1}(u'_i)))\ \cdots\cdots\cdots\cdots\cdots\cdots(3)$$

where $(v'\times(u'_i))$ is a generic point of $V'\times\prod_i U'_i$ over K.

Applying $\lambda\in\mathfrak{G}(K/k)$ to both sides of (3) we get

$$\alpha'^\lambda_i(v'^\lambda,u'^\lambda_i)=f(\alpha^\lambda_i(f^{-\lambda}(v'^\lambda),f_i^{-\lambda}(u'^\lambda_i)))\cdots\cdots\cdots\cdots\cdots\cdots(4)$$

As $\mathbf{V}$ is defined over k, by (2), the right-hand side of (4) is equal to $\alpha_i(\varphi_\lambda^{-1}(f^{-1}(v'^\lambda)),\varphi_{i\lambda}^{-1}(f_i^{-1}(u'^\lambda_i)))$. Then (1) shows $\alpha'^\lambda_i(v'^\lambda,u'^\lambda_i)=\alpha_i(f^{-1}(v'^\lambda),f_i^{-1}(u'^\lambda_i))$. Thus we have shown that α_i $(i\in I)$ are defined over k. $\mathbf{V}'=[V';U'_i,\alpha'_i]$ is an AG-variety defined over k, and $F=[f;f_i]$ has the desired property, q.e.d.

By Propositions 1 and 2, we may say that "there is a canonical one to one correspondence between $H^1(\mathfrak{G}(K/k),\mathrm{Aut}(\mathbf{V})_K)$ and tolality of the isomorphism classes of k-forms over K of $\mathbf{V}$." Next we extend this result to the case of infinite extensions.

Let K be a finite or infinite Galois extension of k (i.e. any element x of K is separablly algebraic over k, and any conjugate over k of x is contained is K) with the Galois group $\mathfrak{G}=\mathfrak{G}(K/k)$.

Let $\mathbf{V}$ be an AG-variety defined over k. We introduce the usual Galois group topology on $\mathfrak{G}$ and the discrete topology on $\mathrm{Aut}(\mathbf{V})_K$, and consider only continuous 1-cocycles of $\mathfrak{G}$ in $\mathrm{Aut}(\mathbf{V})_K$, then we get the continuous 1-cohomology set $H^1(\mathfrak{G}, \mathrm{Aut}(\mathbf{V})_K)$. By the definition and by Propositions 1 and 2, we get:

THEOREM 1. *Let K be a finite or infinite Golois extension of k with the Galois group $\mathfrak{G}$. Let $\mathbf{V}$ be an AG-variety defined over k, and let $H^1(\mathfrak{G}, \mathrm{Aut}(\mathbf{V})_K)$ be the continuous 1-cohomology set of $\mathfrak{G}$ in $\mathrm{Aut}(\mathbf{V})_K$. Then there is a canonical one-to-one correspondence between $H^1(\mathfrak{G}, \mathrm{Aut}(\mathbf{V})_K)$ and totality of the isomorphism classes of k-forms over K of $\mathbf{V}$.*

Now we apply theorem 1 to the classification of semi-simple algebraic groups over a perfect field. C. Chevalley [1] determined all simple groups over Ω. And we can choose, as a representative of each isomorphism class, a simple group which is defined over the prime field. So if we can determine all k-forms of each representative group, then the problem of classifying semi-simple groups over k will be solved.

Let k be a perfect field and $\bar{k}$ the algebraic closure of k in Ω. Let G be a semi-simple algebraic group defined over k. G is clearly an AG-variety ($G=(G; G, \alpha), \alpha:(x, y)\rightarrow xy$). $\mathrm{Aut}(G)$ has a natural structure of an algebraic group, such that the connected component $\mathrm{Aut}_0(G)$ of $\mathrm{Aut}(G)$ is isomorphic to G/center. Theorem 1 is applicable to G, so there is a one to one correspondence between $H^1(\mathfrak{G}(\bar{k}/k), \mathrm{Aut}(G)_{\bar{k}})$ and the totality of k-forms of G. If we can find an AG-variety $\mathbf{V}$ with the property that $\mathrm{Aut}(\mathbf{V})$ has a structure of an algebraic group defined over k, and that $\mathrm{Aut}(\mathbf{V})$ is isomorphic (as an algebraic group) to $\mathrm{Aut}(G)$ over k. Then there is a one to one correspondence between the totality of k-forms of G and that of k-forms of $\mathbf{V}$. In particular, we get:

THEOREM 2. *Let k be a perfect field, $\mathbf{V}$ an AG-variety defined over k and G a connected semi-simple algebraic group without center defined over k. Suppose that $\mathrm{Aut}(\mathbf{V})$ is isomorphic to $\mathrm{Aut}(G)$ as an algebraic group over k. Then any k-form of G can be obtained as the connected component of the automorphism group of some k-form of $\mathbf{V}$. Moreover two k-forms of G, $\mathrm{Aut}_0(\mathbf{V}_1)$ and $\mathrm{Aut}_0(\mathbf{V}_2)$ are isomorphic over k if and only if $\mathbf{V}_1$ is isomorphic to $\mathbf{V}_2$ over k.*

§ 2. (I) For a classical group G other than a few exceptional ones, we can find an involutive algebra which has the property of $\mathbf{V}$ in the theorem 2 (Weil [6]).

For example, let $G=\mathrm{PSO}(n)=\mathrm{SO}(n)$/center. Let $\mathbf{V}=(V; U_i, \alpha_i)$ $(i\in I)$:

$I=\{+, \times, s, t\}$. $V=U_+=U_\times=M_n(\Omega)$, $U_s=\Omega$, $U_t=\{0\}=$ a variety reduced to the one point 0 rational over k, $\alpha_+:(x,y)\rightarrow x+y$, $\alpha_\times:(x,y)\rightarrow xy$, $\alpha_s:(x,a)\rightarrow xa$, $\alpha_t:x\rightarrow {}^t x$, i.e. $\mathbf{V}$ is the involutive algebra $M_n(\Omega)$ with involution $\alpha_t:x\rightarrow {}^t x$ in usual sense. If $n=3$ or $\geqq 5$, G is simple and semi-simple. If $n=3$ or $\geqq 5$ and $\neq 8$, and moreover the characteristic of k is not 2, then $G=\mathrm{Aut}_0(\mathbf{V})$ and G and $\mathbf{V}$ satisfy all the conditions of the theorem 2. So we can get every k-form of G as the connected component of the automorphism group of some k-form of $\mathbf{V}$.

If $n=8$, $\mathrm{Aut}(G)$ has an isogeny to $\mathrm{Aut}(\mathbf{V})$ with the kernel of order 3, but is not isomorphic to $\mathrm{Aut}(\mathbf{V})$, so our method fails. If k has an extension of degree 3, then PO(8) has actually some exceptional k-forms which can not be obtained from involutive algebras.

(II) The exceptional group of type G_2. In this case we may choose as $\mathbf{V}$ an "octanion algebra".

First we sum up the definition and the fundamental properties of octanion algebras after Jacobson [**2**].

Let k be an arbitrary field of characteristic not two, and D a quaternion (not necessarily division) abgebra over k, and α a non-zero element of k. Consider the vector space over k:

$\mathbf{C}=(D,\alpha)=De_0+De$. Define a multiplication in $\mathbf{C}$ by: $e_0=1$ (unit of muliplication) and write $ae_0=a$ for $a\in D$ and $(a+be)(c+de)=(ac+\alpha b\bar{d})+(ad+b\bar{c})e$ where $a\rightarrow\bar{a}$ is the canonical involution of D. Then $\mathbf{C}$ is a non-associative alternative central simple algebra of rank 8 over k. We consider $\mathbf{C}$ as an algebraic variety defined over k. $\mathbf{C}=(D,\alpha)$ with a non-division D_k is called a splitting octanion over k. It can be shown that any two splitting octanions over k are mutually isomorphic (as an algebraic variety over k). An algebra over k which is isomorphic over the algebraic closure of k to the splitting octanion over k, is called an octanion algebra over k. Then the following facts are known: 1) Any octanion algebra over k is isomorphic over k to (D,α) for some quaternion D over k and $\alpha\in k$. 2) An octanion algebra $\mathbf{C}=(D,\alpha)$ has the canonical involution $x\rightarrow\bar{x}$ defined by $\bar{x}=\bar{a}-\bar{b}e$ for $x=a+be\in\mathbf{C}$, having the property that $\bar{x}=x$ means $x\in$ center of $\mathbf{C}$. We use the notations:

$$N(x)=x\bar{x},\quad Tr(x)=x+\bar{x},\quad \mathbf{C}^-=\{x\in\mathbf{C}\,;\,Tr(x)=0\}\,.$$

3) If D and D' are quaternion subalgebras of the octanion $\mathbf{C}$, and if $f:D\rightarrow D'$ is an isomorphism then f may be extended to an automorphism in $\mathbf{C}$.

LEMMA. *Let* $\mathbf{C}$ *be an octanion over a field* k *of characteristic not* 2. *Then* $\mathrm{Aut}(\mathbf{C})$ *is a connected simple and semi-simple algebraic group defined over* k *of dimension* 14.

PROOF. Let $G=\mathrm{Aut}(\mathbf{C})$. G is clearly an algebraic group defined over k (with a representation space $\mathbf{C}$). By (2), any element of G commutes with the canonical involution of $\mathbf{C}$, so any element of G preserves $N(x)$ and $Tr(x)$. We can choose a basis of $\mathbf{C}$ (over Ω), e_i $(i=0,1,\cdots,7)$ such that: $e_0=1, e_i^2=1$, $Tr(e_ie_j)=0$ $(0\neq i\neq j\neq 0)$, that e_i $(i=0,1,2,3)$ generate a quaternion subalgebra D of $\mathbf{C}$, and that $\mathbf{C}=D+De_4=(D,1)$ over Ω. Now we will show:

i) G is connected and of dimension 14. Let $U=\{(x,y)\in \mathbf{C}^-\times\mathbf{C}^-; N(x)=1, N(y)=1, Tr(xy)=0\}$. Then U is isomorphic (as a variety) to the image of the projection π, from $\mathrm{SO}(7)=\{x=(x_{ij})\in \mathrm{GL}(7)\mid {}^txx=1 \ \det x=1\}$ to the first and second rows $\pi: x\to(x_{1j}, x_{2j})$, so U is an irreducible variety of dimension 11. As e_0, x, y $((x,y)\in U)$ generate a quaternion subalgebra isomorphic to D, by (3) G operates transitively on U. Let H be the stability group of (e_1, e_2), $H=\{\sigma\in G; \sigma(e_1)=e_1, \sigma(e_2)=e_2\}$ and $W=\{x\in\langle e_4, e_5, e_6, e_7\rangle_\Omega; N(x)=1\}$. W is isomorphic (as a variety) to a 3-dimensional sphere. As the operation of $\sigma(\in H)$ is completely determined by $\sigma(e_4), \sigma\to\sigma(e_4)\in W$ gives a birational mapping from H onto W, so H is connected and of dimension 3. Therefore G is connected and of dimension 14.

ii) G is simple and seme-simple. As the representation $G\to GL(\mathbf{C}^-)$ is faithful, it is sufficient to show that the representation is semi-simple and G has no center (Lemma 1, exposé 20 of Chevalley [1]).

Let V be an invariant subspace of $\mathbf{C}^-$. If V contains a vector, say ae_1 $(a\in\Omega)$, then V contains e_1, hence all e_i $(i\geqq 1)$ and $V=\mathbf{C}^-$.

Let Z be a center of G, e a vector of $\mathbf{C}^-$, and $G(e)=\{\sigma\in G; \sigma(e)=e\}$, By (3), $G(e)$ fixes only vectors in $\langle e\rangle_\Omega$. If $\tau\in Z$, and $\sigma\in G(e)$, then $\sigma(\tau(e))=\tau(\sigma(e))=\tau(e)$, so $\tau(e)\in\langle e\rangle_\Omega$ and $N(\tau(e))=N(e)$, i.e. $\tau(e)=\pm e$. This means $\tau=1$, in $\mathbf{C}^-$ or $\tau=-1$ in $\mathbf{C}^-$, but in the latter case τ induces an anti-automorphism of $\mathbf{C}$ and not an isomorphism so $\tau=1$ on$\mathbf{C}^-$, i.e. τ is the identity, q.e.d.

There is only one (up to isomorphism) simple group of dimension 14 i.e. the group of type G_2 (Theorem 1, exposé 21 of Chevalley [1]). By the above lemma, $G=\mathrm{Aut}(\mathbf{C})=\mathrm{Aut}_0(\mathbf{C})$ is the group of type G_2, and if the characteristic of Ω is $\neq 3$, we have by Corollary 4 of Chevalley [1] exposé 24, $\mathrm{Aut}(G)\cong G$. G and $\mathbf{C}$ satisfy the condition of theorem 2, and we get:

PROPOSITION. *Any simple group of type G_2 defined over a perfect field k of characteristic >3 is obtained as* $\mathrm{Aut}(\mathbf{C})$ *by some octanion* $\mathbf{C}$ *defined over k. Let* $\mathbf{C}$ *and* $\mathbf{C}'$ *be octanions defined over k, then* $\mathrm{Aut}(\mathbf{C})$ *and* $\mathrm{Aut}(\mathbf{C}')$ *are isomorphic over k if and only if* $\mathbf{C}$ *and* $\mathbf{C}'$ *are isomorphic over k.*

For the field of characteristic 0, this is the result of Jacobson [2]. He also showed that the classification of octanions over a field of characteristic not 2 can be reduced to the classification of quadratic forms of rather special type (norm-form, $N(x)$ of $\mathbf{C}$), and determined all non-isomorphic octanions over some fields (local fields, algebraic number fields).

(III) The exceptional group of type F_4. We may choose as $\mathbf{V}$ a Jordan algebra. Let $\mathbf{C}$ be an octanion over k, $\mathbf{J}=\{x\in M_3(\mathbf{C})\,;{}^t\bar{x}=x\}$. Define a multiplication in $\mathbf{J}$ by $x\circ y=\frac{1}{2}(xy+yx)$ where xy denotes the usual matrix multiplication. With this multiplication, $\mathbf{J}$ is a commutative non-associative algebra defined over k. An algebra over k which is isomorphic to the above $\mathbf{J}$ over Ω, is called an (exceptional simple) Jordan algebra over k. $G=\mathrm{Aut}(\mathbf{J})$ is clearly an algebraic group defined over k. For the field of characteristic >3, it was shown by T. A. Springer that G is the connected simple algebraic group of type F_4 (Springer [4] p. 467, Theorem 3.) As there is only one group of type F_4 (up to isomorphism over Ω) and $\mathrm{Aut}(G)\cong G$ for the group of type F_4, the conditions of theorem 2 are satisfied by $G=\mathrm{Aut}(\mathbf{J})=\mathrm{Aut}_0(\mathbf{J})$ and $V=\mathbf{J}$. We get thus the following result:

"Any exceptional simple algebraic group of type F_4 defined over a perfect field of characteristic>3 is obtained as $\mathrm{Aut}(\mathbf{J})$ for some Jordan algebra over k. $\mathrm{Aut}(\mathbf{J})$ and $\mathrm{Aut}(\mathbf{J}')$ is isomorphic over k if and only if $\mathbf{J}$ and $\mathbf{J}'$ are isomorphic over k."

(IV) The exceptional group of type E_n ($n=6,7,8$). In these cases, we have no convenient $\mathbf{V}$ at hand. But when k is a finite field, we can apply Theorem 1 directly to E_7 and E_8. Using the fact that $\mathrm{Aut}(G)\cong G$ for these groups, and the results of Lang [3] that any cocycle into the connected algebraic group over a finite field splits, we can conclude that each of E_7 and E_8 has only one k-form over a finite field k (i.e. that of the Chevalley type).

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