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Local theory in function analysis*

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§ 0. Introduction

The aim of this paper is, as a continuation of the previous papers [**10**], [**11**], to establish the local theory in algebro-topological systems over commutative AW*-algebras.

This paper consists of two parts. In each part, we shall explain how to reduce properties of elements of algebro-topological systems over commutative B*-algebras to those of elements of (classical) algebro-topological systems over the field of complex (or real) numbers by making use of the local theory. In § 1, we shall establish a theorem concerning quasi-ordered linear spaces (Theorem A) and the extension theorem of H. Hahn and S. Banach (Theorem B). In § 2, we deal with a theorem of I. Gelfand (Theorem C) and a theorem of S. Mazur and I. Gelfand (Theorem D). These theorems will be discussed for the case of algebro-topological systems over commutative B*-algebras (for example, quasi-ordered linear spaces over commutative B*-algebras, linear spaces over commutative B*-algebras, Banach algebras over commutative B*-algebras and B*-algebras over commutative B*-algebras). They are, however, essentially valid for the case of algebro-topological systems over commutative AW*-algebras, which were originated by I. Kaplansky [**6**] and investigated by H. Widom [**14**] and M. Nakai [**9**]. Precisely speaking, we consider a compact Hausdorff space Ω and the commutative B*-algebras $C(\Omega)$ (or $R(\Omega)$) of complex- (or real-) valued continuous functions defined on Ω. Suppose there is a theorem concerning an algebro-topological system over $C(\Omega)$ (or $R(\Omega)$). Then we shall say that this theorem is, for instance, of Stonian class if it is valid for the case that the underlying space Ω is Stonian and if further there exists without fail a counter example, that is, an example, for which the theorem does not hold, provided that Ω is not Stonian. In this sense, these theorems are exactly of Stonian class. (L. Nachbin [**8**], D. B. Goodner [**2**], J. L. Kelley [**7**], M. Nakai [**9**], and M. Hasumi [**3**] proved that the extension theorem of H. Hahn and S. Banach is exactly of Stonian class.)

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§ 1. N-Spaces Over N_0

1. Definitions. Let Ω be a compact Hausdorff space. Denote by N_0 the set of real-valued continuous functions on Ω. In the usual way, N_0 constitutes a real normed ring and a semi-ordered linear space. Denote by $(N_0)_+$ the set of non-negative functions in N_0.

We state some definitions and lemmas. The proofs of lemmas in this section are easy and will be omitted.

Definition 1.1: A binary relation $a \leqq b$ between certain elements of a set E is called a quasi-ordering if it satisfies the following conditions: (a) $a \leqq a$ for a in E and (b) $a \leqq b, b \leqq c$ for a, b, c in E imply $a \leqq c$. The relation $a \leqq b$ is also denoted by $b \geqq a$.

Definition 1.2: A module E is called a linear space over N_0 if it has N_0 as an operator domain and the following are satisfied: $(\sigma\tau)a = \sigma(\tau a), (\sigma+\tau)a = \sigma a + \tau a$, and $1a = a$ for σ, τ in N_0, a in E, and where 1 is the function on Ω taking values identically equal to 1.

Definition 1.3: A linear space E over N_0 is called a quasi-ordered linear space over N_0 if it has a quasi-ordering compatible with the linear operation, that is: (a) $a, b \geqq 0$ and a, b in E imply $a+b \geqq 0$, (b) $\sigma, a \geqq 0$ and σ in N_0, a in E imply $\sigma a \geqq 0$, and (c) $a \geqq b$ if $a-b \geqq 0$ for a, b in E.

Definition 1.4: A quasi-ordered linear space N over N_0 is called an N-space over N_0 if it has an order unity e and if $0 \leqq \sigma e \leqq 0$ and σ in N_0 imply $\sigma = 0$, where the order unity e is a positive element in N such that $N = (a\,;\, a \leqq \sigma e$ for some σ in $N_0)$.

Definition 1.5: 1) An N-space N_1 over N_0 is said to be homomorphic onto another N-space N_2 over N_0 if there exists a mapping f of N_1 onto N_2 satisfying the conditions: (a) $f(a+b) = f(a) + f(b)$ for a, b in N_1, (b) $f(\sigma a) = \sigma f(a)$ for σ in N_0, a in N_1, and (c) $a \leqq b$ for a, b in N_1 implies $f(a) \leqq f(b)$.

2) An N-space N_1 over N_0 is said to be isomorphic onto another N-space N_2 over N_0 if there exists an one-to-one mapping f of N_1 onto N_2 satisfying the conditions (a), (b) in 1) and (c)$'$ $f(a) \leqq f(b)$ holds for a, b in N_1 when and only when $a \leqq b$.

Definition 1.6: 1) A submodule E_1 of a linear space E over N_0 is called an N_0-subspace of E if it is admissible with respect to the operator domain N_0, that is: $N_0 E_1 \subseteqq E_1$.

2) An N_0-subspace I of an N-space N over N_0 is called an ideal of N if a, b in I and $a \leqq c \leqq b$ (c in N) imply c in I.

Let N be an N-space over N_0 with order unity e. Then $N_0 e$ is an N-space over N_0 with e as an order unity. The space N_0 is also an N-space over N_0 with the order unity 1.

LEMMA 1.1: *$N_0 e$ is isomorphic onto N_0.*

Let N be an N-space over N_0 and I be its ideal. To mention the natural homomorphism of N onto the quotient space, we add a bar over the elements of N. In N/I, we give a quasi-ordering by defining $\bar{a} \leqq \bar{b}$ if there exists at least one element c in I such that $a+c \leqq b$. With this quasi-ordering we have

LEMMA 1.2: *N/I constitutes a quasi-ordered linear space over N_0.*

DEFINITION 1.7: An ideal I of an N-space N over N_0 is said to be proper if N/I constitutes an N-space over N_0 with $\bar{e}$ as its order unity.

LEMMA 1.3: *An ideal I of an N-space N over N_0 is proper if and only if I does not contain σe for any non-zero element σ in N_0.*

DEFINITION 1.8: 1) An N-space over N_0 is said to be simple if it has no proper ideal except for the zero-ideal (0).

2) An ideal I of an N-space N over N_0 is said to be properly maximal if it is proper and if N/I is simple.

LEMMA 1.4: *Given any proper ideal I of an N-space N over N_0, there exists at least one properly maximal ideal of N containing I.*

DEFINITION 1.9: 1) A linear mapping f of a linear space E over N_0 into N_0 is said to be N_0-linear if it satisfies the condition: $f(\sigma a) = \sigma f(a)$ for σ in N_0 and a in E.

2) A linear mapping f of an N-space N over N_0 into N_0 is said to be positive if it satisfies the condition: $a \geqq 0$ and a in N imply $f(a) \geqq 0$.

3) A positive linear mapping f of an N-space N over N_0 into N_0 is called state of N if it satisfies the condition: $f(e) = 1$.

Here we should like to notice that the simple use of "linear" without "N_0-" never refers to the "N_0-linear" fixed in Definition 1.9, 1).

DEFINITION 1.10: 1) A mapping p of a linear space E over N_0 into N_0 is called a cap of E if it satisfies the following conditions: (a) $p(a+b) \leqq p(a) + p(b)$ for a, b in E and (b) $p(\sigma a) = \sigma p(a)$ for σ in $(N_0)_+$ and a in E. If E is an N-space over N_0, we require further that (c) $p(\sigma e + a) = \sigma + p(a)$ for σ in N_0 and a in E.

2) Let p be a cap of E. A linear mapping f of a linear space E over N_0 into N_0 is said to be compatible with p if $f(a) \leqq p(a)$ for any a in E.

DEFINITION 1.11: An element σ in N_0 is called an (analytic) spectrum of an element a in an N-space N over N_0 if the ideal of N generated by $a - \sigma e$ is proper.

2. Stonian Case. In this section, we assume that Ω is a Stonian space, that is to say, a compact Hausdorff space, in which every lower bounded set of functions in N_0 has its GLB (the greatest lower bound) in N_0. Denote by E_0 the set of projections in N_0. The family $((\omega\,; \omega(e_0) = 1)\,; e_0 \in E_0)$ of open sets in Ω constitutes an open base in Ω, where we denote by $\omega(\sigma)$ the value of a function σ in N_0 at ω. For ω in Ω, we denote by $E_0(\omega)$ the set of projections $e_0(\omega)$ in E_0 satisfying the condition: $\omega(e_0(\omega))=1$. By "almost all points in Ω" we shall mean "points outside a certain first category set in Ω".

Let N be an N-space over N_0. Then N constitutes also an N-space over the field of real numbers R. For a point ω in Ω, denote by $(0)^\omega$ the set of elements a in N such that, given any positive number ε, there exists a projection $e_0(\omega)$ in $E_0(\omega)$ such that $-\varepsilon e_0(\omega)e \leqq e_0(\omega)a \leqq \varepsilon e_0(\omega)e$. It is easy to see that $(0)^\omega$ is an ideal of N as an N-space over R. Set $N_\omega = N/(0)^\omega$. Then N_ω constitutes an N-space over R. Denote by ω the natural homomorphism of N onto N_ω.

LEMMA 1.5: *It holds that $\omega(\sigma a)=\omega(\sigma)\omega(a)$ for σ in N_0 and a in N.*

PROOF: We first prove that $\sigma a \in (0)^\omega$ holds for a non-negative function σ in N_0 such that $\omega(\sigma)=0$ and for a in N. By the property of e we can find a natural number n such that $-ne \leqq a \leqq ne$. Since $\sigma \geqq 0$, we have $-\sigma ne \leqq \sigma a \leqq \sigma ne$. Since $\omega(\sigma)=0$, for any natural number r, there exists a projection $e_0(\omega)$ in $E_0(\omega)$ such that $e_0(\omega)\sigma \leqq r^{-1}e_0(\omega)$. From this it follows that $r^{-1}e_0(\omega)ne \leqq e_0(\omega)\sigma a \leqq r^{-1}e_0(\omega)ne$. This shows that $\sigma a \in (0)^\omega$. For a not necessarily non-negative function σ in N_0, we can find the Jordan decomposition $\sigma=\sigma_1-\sigma_2$ $(\sigma_1 \geqq 0, \sigma_2 \geqq 0, \sigma_1\sigma_2=0)$. Since $\omega(\sigma)=0$, we have $\omega(\sigma_1)=\omega(\sigma_2)=0$. Hence $\sigma_1 a$ and $\sigma_2 a$ are in $(0)^\omega$. Hence σa is in $(0)^\omega$. Now, for a general element σ in N_0, we have $\omega(\sigma a)=\omega(\sigma)\omega(a)+\omega((\sigma-\omega(\sigma))a)=\omega(\sigma)\omega(a)$, thus the proof is completed.

We shall call this N-space N_ω over R the *local N-space of N with respect to ω*. For an ideal I of N, denote by I_ω the ideal of N_ω generated by $\omega(I)$ $(=(\omega(a); a \in I))$; namely $I_\omega=(\omega(c); \omega(a) \leqq \omega(c) \leqq \omega(b)$ for some a, b in $I)$.

LEMMA 1.6: *An ideal I of N is proper if and only if I_ω is proper in N_ω for almost all points ω in Ω.*

PROOF: NECESSITY: Suppose I_ω is not proper for some ω in Ω. Then there exist a, b in I such that $\omega(a) \leqq \omega(e) \leqq \omega(b)$. In other words, there exist a, b in I and a', b' in $(0)^\omega$ such that $a'+a \leqq e \leqq b+b'$. Hence there exists $e_0(\omega)$ in $E_0(\omega)$ such that $e_0(\omega)(-2^{-1}e+a) \leqq e_0(\omega)e \leqq e_0(\omega)(2^{-1}e+b)$. From this it follows that $(2/3)e_0(\omega)a \leqq e_0(\omega)e \leqq 2e_0(\omega)b$. This shows that I is not proper.

SUFFICIENCY: Suppose I is not proper. Then there exist σ $(\neq 0)$ in N_0 and a, b in I such that $a \leqq \sigma e \leqq b$. Since $\sigma \neq 0$, we can assume without loss of generality that there exists a projection e_0 $(\neq 0)$ in E_0 such that $\sigma e_0 \geqq 2^{-1}e_0$. Hence σe_0 has an inverse in e_0N_0. Denote it by τ. Then it holds that $\tau e_0 a \leqq e_0 e \leqq \tau e_0 b$, where τe_0 is in N_0. This shows that I_ω is not proper for any point ω in $(\omega; \omega(e_0)=1, \omega \in \Omega)$. Hence the set of points ω in Ω, for which I_ω is not proper, contains a non-empty open set and so it is not a first category set in Ω. Thus we see that if there exists any point ω in Ω such that I_ω is not proper, then the set of ω's for which I_ω is not proper is not a set of first category. Thus the statement of the lemma is legitimate by our convention on the use of the expression "almost all".

The same reasoning shows the following lemma.

LEMMA 1.7: *An element σ in N_0 is a spectrum of an element a in N if and only if $\omega(\sigma)$ is a spectrum of $\omega(a)$ in N_ω for almost all points ω in Ω.*

We associate $p(a)=\mathrm{GLB}(\sigma\,;a\leqq\sigma e, \sigma\in N_0)$ with each element a in N. This GLB exists, for there is an element τ in N_0 such that $\tau e\leqq a$ and so it holds that $\tau\leqq\sigma$ for those σ in N_0 which satisfies the condition: $a\leqq\sigma e$. Thus we can define a mapping p of N onto N_0. The mapping p satisfies the conditions (a), (b), (c) in Definition 1.10 so that it defines a cap of N as mentioned in the following lemma. Similarly we set $p_\omega(\omega(a))=\inf(\beta\,;\omega(a)\leqq\beta\omega(e), \beta\in R)$ for ω in Ω and $\omega(a)$ in N_ω. We thus obtain a mapping p_ω of N_ω onto R. It is easy to see that p_ω also satisfies the conditions of a cap.

LEMMA 1.8: *p is a cap of N, and for each element a in N, $(*)$ $\omega(p(a))=p_\omega(\omega(a))$ for almost all points ω in Ω.*

PROOF: In order to prove the lemma, we need only to see that $(*)$ holds for non-negative element a in N, because p satisfies (c) in Definition 1.10 and p_ω is a cap of N_ω.

We first see that $\omega(p(a))\leqq p_\omega(\omega(a))$ for all ω in Ω. For a positive number ε, we have $\omega(a)\leqq(p_\omega(\omega(a))+\varepsilon)\omega(e)$. Hence, given any positive number δ, there exists a projection $e_0(\omega)$ in $E_0(\omega)$ such that $e_0(\omega)a\leqq(p_\omega(\omega(a))+\varepsilon+\delta)e_0(\omega)e$. On the other hand, there exists a natural number n such that $a\leqq ne$. Set $\sigma=(p_\omega(\omega(a))+\varepsilon+\delta)e_0(\omega)+n(1-e_0(\omega))$. Then $a\leqq\sigma e$ and so $p(a)\leqq\sigma$. Taking the values at ω, we get $\omega(p(a))\leqq p_\omega(\omega(a))+\varepsilon+\delta$. By making $\delta\downarrow 0$ and then $\varepsilon\downarrow 0$, we reach the desired inequality.

Conversely, denote by Δ the set of those elements σ in N_0 which satisfy the condition: $a\leqq\sigma e$. Given a natural number n, there exist an element σ in Δ and a projection e_0 in N_0 such that $e_0\sigma\leqq(p(a)+n^{-1})e_0$. Hence, by exhaustion method, we can find an orthogonal set $(e_{0\iota}\,;\iota\in I)$ of projections in N_0, whose LUB is equal to 1, and a set $(\sigma_\iota\,;\iota\in I)$ of elements in N_0 such that $e_{0\iota}\sigma_\iota\leqq(p(a)+n^{-1})e_{0\iota}$ for ι in I. Set $\Omega_n=\cup((\omega\,;\omega(e_{0\iota})=1)\,;\iota\in I)$. Then, for each ω in Ω_n, there exists an index ι in I such that $\omega(e_{0\iota})=1$. Since $e_{0\iota}a$ $(\leqq e_{0\iota}\sigma_\iota e)\leqq(p(a)+n^{-1})e_{0\iota}e$, it holds that $\omega(a)\leqq(\omega(p(a))+n^{-1})\omega(e)$, that is, $p_\omega(\omega(a))\leqq\omega(p(a))+n^{-1}$. Set $\Omega_0=\cap(\Omega_n\,;n\in J)$, where we denote by J the set of natural numbers. Then $p_\omega(\omega(a))\leqq\omega(p(a))$ for ω in Ω_0 and so $p_\omega(\omega(a))=\omega(p(a))$ for ω in Ω_0. Since $\mathrm{LUB}(e_{0\iota}\,;\iota\in I)=1$, the closure of Ω_n is Ω. Moreover Ω_n is open. Hence Ω_n^c (the complement of Ω_n in Ω) is non-dense and $\Omega_0^c=\cup(\Omega_n^c\,;n\in J)$ is a first category set in Ω. Thus we have shown that $(*)$ holds, which was to be proved.

LEMMA 1.9: *$p(a)$ is a spectrum of the element a.*

PROOF: LOCAL PROOF: We show that $p_\omega(\omega(a))$ is a spectrum of $\omega(a)$ in N_ω. For this aim, it suffices to show that, when N is an N-space over R with e as order unity, and $a \in N, p(a)$ is a spectrum of a, because N_ω is such one as we are hitherto considering. Suppose the contrary. Then there exists a real number β such that $e \leqq \beta(p(a)e-a)$. From this it follows that $\beta \neq 0$. If $\beta > 0$, we get $a \leqq (p(a)-\beta^{-1})e$. This contradicts the construction of $p(a)$. On the other hand, if $\beta < 0$, we get $(p(a)+(-\beta^{-1}))e \leqq a$. This is also impossible because of the same reason. This implies that $p(a)$ is a spectrum of a for this case.

GLOBAL PROOF: For almost all points ω in $\Omega, \omega(p(a))=p_\omega(\omega(a))$ and so, by the above local proof and Lemma 1.7, $p(a)$ is a spectrum of a. This completes the proof.

We are now in a position to prove the following

THEOREM A: *Every simple N-space over N_0 is isomorphic onto N_0.*

PROOF: For each element a in $N, p(a)$ is a spectrum of a by Lemma 1.9. Hence the ideal of N generated by $a-p(a)e$ is proper and so it must be equal to (0). This implies that $a=p(a)e$. Hence, by Lemma 1.1, we reach the assertion.

Combining Theorem A with Lemma 1.4 and Lemma 1.9, we get the following

LEMMA 1.10: *For any element a in an N-space N over N_0, there exists a state f of N such that $f(a)=p(a)$, where p is the cap of N defined in the paragraph before Lemma 1.8.*

LEMMA 1.11: *If a cap p of an N-space N over N_0 satisfies the following condition: (d) $a \leqq p(a)e$ for a in N, then*

1) *$p(a)=\mathrm{GLB}(\sigma\,; a \leqq \sigma e, \sigma \in N_0)$ for a in N,*

2) *$\omega(p(a))=p_\omega(\omega(a))$ for ω in Ω and a in N, p_ω being the function defined in the paragraph before Lemma 1.8,*

3) *every linear mapping of N into N_0 compatible with p is N_0-linear, and*

4) *a linear mapping f of N into N_0 with $f(\sigma e)=\sigma$ for σ in N_0 is positive if and only if it is compatible with p.*

PROOF: PROOF OF 1): Let p be a cap of N satisfying the condition (d). Suppose $a \leqq \sigma e$ for σ in N_0 and a in N. Then, using (d), we have $0 \leqq \sigma e-a \leqq p(\sigma e-a)e=(\sigma-p(a))e$, or $p(a) \leqq \sigma$. Hence we get $p(a)=\mathrm{GLB}(\sigma\,; a \leqq \sigma e, \sigma \in N_0)$. The other inequality is trivial because of (d). This shows 1).

PROOF OF 2): In the proof of Lemma 1.8, we saw that $\omega(p(a)) \leqq p_\omega(\omega(a))$ for all ω in Ω, where p_ω is the local cap defined in the paragraph before Lemma 1.8. On the other hand, since $a \leqq p(a)e$, we have $\omega(a) \leqq \omega(p(a))\omega(e)$. Namely, we have $p_\omega(\omega(a)) \leqq \omega(p(a))$. This shows 2).

PROOF OF 3): Suppose f is a linear mapping of N into N_0 compatible with p. We first see that

(*) if $\omega(a)=0$ for an element a in N, then $\omega(f(a))=0$. Since $\omega(a)=0$, $p_\omega(\omega(a))=p_\omega(\omega(-a))=0$. Since f is compatible with p, $-p(-a)\leqq f(a)\leqq p(a)$ and so $-\omega(p(-a))\leqq\omega(f(a))\leqq\omega(p(a))$. Combining these equalities with 2), we get (*).

In view of (*), a linear functional f_ω of N_ω can be defined by $f_\omega(\omega(a))=\omega(f(a))$ for a in N. Using this functional, for any σ in N_0, we have $\omega(f(\sigma a))=f_\omega(\omega(\sigma a))=f_\omega(\omega(\sigma)\omega(a))=\omega(\sigma)f_\omega(\omega(a))=\omega(\sigma)\omega(f(a))=\omega(\sigma f(a))$. This implies that $f(\sigma a)=\sigma f(a)$. Therefore f is N_0-linear. Thus we get 3).

PROOF OF 4): Let f be a linear mapping of N into N_0 compatible with p. Suppose a is an element in N such that $a\geqq 0$. Then $-a\leqq 0$ and so, by 1), $-f(a)=f(-a)\leqq p(-a)\leqq 0$, or $f(a)\geqq 0$. This shows that f is positive.

Conversely, suppose f is a positive linear mapping of N into N_0 enjoying the condition: $f(\sigma e)=\sigma$ for σ in N_0. Since $a\leqq p(a)e$ for a in N, we have $f(a)\leqq f(p(a)e)=p(a)$. This shows that f is compatible with p. Thus we get 4).

As an immediate consequence of Theorem A, we shall give an alternative proof of the following generalization of the extension theorem of H. Hahn and S. Banach due to M. Nakai [**9**, Theorem 1].

THEOREM B: *For any linear space E over N_0, for any N_0-subspace E_1 of E, and for any cap p of E, every linear mapping of E_1 into N_0 compatible with p has a linear extension on the whole space E into N_0 compatible with p.*

PROOF: Suppose f_1 is a linear mapping of E_1 into N_0 compatible with p. Construct the direct sum N of N_0 and E, and we shall consider N_0 and E as subspaces of N. Then N will turn out to be an N-space over N_0 with the order unity 1 when introduced a quasi-ordering as follows: $\sigma+a\leqq\tau+b$ for σ,τ in N_0 and for a, b in E holds if and only if $p(a-b)\leqq\tau-\sigma$. Moreover N has a cap $\tilde{p}$ defined by $\tilde{p}(\sigma+a)=\sigma+p(a)$ for σ in N_0, a in E. We notice that $\tilde{p}$ satisfies the condition (d) in Lemma 1.11. Denote by N_1 the N_0-subspace of N generated by N_0 and E_1. Then f_1 can be extended to a linear mapping $\tilde{f}_1$ of N_1 into N_0 such that $\tilde{f}_1(\sigma+a)=\sigma+f(a)$ for σ in N_0, a in E_1. It is easy to see that $\tilde{f}_1$ is compatible with $\tilde{p}$. Therefore, by Lemma 1.11, $\tilde{f}_1$ satisfies the condition: $\tilde{f}_1(\sigma)=\sigma$ for σ in N_0. Hence $\tilde{f}_1$ is positive by Lemma 1.11. Set $I=(a_1;\tilde{f}_1(a_1)=0, a_1\in N_1)$. Then the ideal of N generated by I is proper. In fact, if $a_1\leqq\sigma\leqq b_1$ for σ in N_0 and for a_1, b_1 in I, then, because of the positivity of $\tilde{f}_1$, $0=\tilde{f}_1(a_1)\leqq\tilde{f}_1(\sigma)=\sigma\leqq\tilde{f}_1(b_1)=0$, or $\sigma=0$. Hence, by Lemma 1.4, there exists a properly maximal ideal J of N containing (the ideal of N generated by) I. Since N/J is simple, there is an isomorphism of N/J onto N_0, say φ, by Theorem A. Denote by ψ the natural homomorphism of N onto N/J and set $g=\varphi\circ\psi$. Then g is positive, because it is the composition of positive mappings. Also g satisfies the condition: $g(\sigma)=\sigma$ for σ in N_0. Hence g is compatible with $\tilde{p}$ by Lemma 1.11 and so the restriction f of g on E is compatible with p. Moreover we have $g(a)=0$ for a in I. Since $a-\tilde{f}_1(a)\in I$ for a in E_1, therefore, we get $0=g(a-\tilde{f}_1(a))=g(a)-f_1(a)=f(a)-f_1(a)$; namely $f(a)=f_1(a)$ for a in E_1. This means that f is an extension of f_1. This completes the proof.

3. Converse Theorems. In connection with Theorems A and B, we shall state and prove the following

THEOREM A′: *Ω is Stonian if Theorem A holds for any simple N-space over N_0.*

PROOF: Suppose $(\sigma_\iota; \iota\in I)$ be a bounded below family of elements in N_0. It must be shown that, under the validity of Theorem A, the GLB of this family exists in N_0. Denote by $B(\Omega)$ the set of real-valued bounded functions on Ω. Then $B(\Omega)$ can be considered in a natural way as an N-space over N_0 with order unity 1. Let x be the GLB of σ_ι $(\iota\in I)$ in $B(\Omega)$, and N be the N_0-subspace of $B(\Omega)$ generated by N_0 and x. Let J be a properly maximal ideal of N. Then N/J is simple, and so, by hypothesis, it is isomorphic with N_0. Since 1 is an order unity in N/J, it corresponds to a positive function σ_0. It will then be easy to see that, denoting by σ the image of x under the homomorphism $N\to(N/J\to)\ N_0$, $\sigma_0^{-1}\sigma$ is the GLB in question.

THEOREM B′: *Ω is Stonian if Theorem B holds for any linear space over N_0.*

PROOF: Suppose $(\sigma_\iota; \iota\in I)$ is a bounded below family of elements in N_0. It must be shown that, under the validity of Theorem B, the GLB of this family exists in N_0. Denote by $B(\Omega)$ the set of real-valued bounded functions on Ω and by x the GLB of the family in $B(\Omega)$. Denote by $(\tau_\kappa; \kappa\in K)$ the subset $(\tau; \tau\leqq x)$ of N_0. Also, denote by E_x the N_0-subspace $(\sigma x; \sigma\in N_0)$ of $B(\Omega)$ and by E the restricted direct sum $\sum(E(\iota,\kappa); \iota\in I, \kappa\in K)$ of linear spaces over N_0, whose members $E(\iota,\kappa)$ are all isomorphic to E_x as a linear space over N_0. For the sake of convenience, we assume that these summands are contained in E. Denote by $\varphi(\iota,\kappa)$ the isomorphism of E_x onto $E(\iota,\kappa)$. Any element a in E is written as $a=\sum_{i=1}^{n}\sigma_i\varphi(\iota_i,\kappa_i)(x)$ and we can define a cap p in E such that $p(a)=\sum_{i=1}^{n}((\sigma_i)_+\sigma_{\iota_i}-(\sigma_i)_-\tau_{\kappa_i})=\sum_{i=1}^{n}((\sigma_i)_+(\sigma_{\iota_i}-\tau_{\kappa_i})+\sigma_i\tau_{\kappa_i})$, where $\sigma=(\sigma)_+-(\sigma)_-$ is the Jordan decomposition of an element σ in N_0. Denote by E_1 the N_0-subspace of E spanned by the set of all elements in the form: $\varphi(\iota_1,\kappa_1)(x)-\varphi(\iota_2,\kappa_2)(x)$.

We then show that $p(a)\geqq 0$ for a in E_1. Suppose ω is a point in Ω. First, suppose $\omega(x)\neq 0$. Construct the restricted direct sum E_ω of linear spaces $(E_\omega(\iota,\kappa); \iota\in I, \kappa\in K)$ over R, whose members are all isomorphic to $\omega(E_x)$ $(=(\omega(a_x); a_x\in E_x))$, where $\omega(a_x)$ is the value of a_x at ω. For the sake of convenience, we assume that these summands are contained in E_ω. Denote by ω the natural homomorphism of E_x onto $\omega(E_x)$ and by $\varphi_\omega(\iota, \kappa)$ the isomorphism of $\omega(E_x)$ onto $E_\omega(\iota, \kappa)$. It is easy to see that there exists a homomorphism of E onto E_ω as a linear space over R, whose restriction on $E(\iota, \kappa)$ is equal to $\varphi_\omega(\iota, \kappa) \circ \omega \circ \varphi(\iota, \kappa)^{-1}$. Denote it again by ω. Then, if $\omega(a) = \omega(b)$ for a, b in E, we have $\omega(p(a)) = \omega(p(b))$. In order to see this, we can assume without loss of generality that a and b are in the same $E(\iota, \kappa)$ for some ι in I, κ in K. Suppose $a = \sigma\varphi(\iota, \kappa)(x)$ and $b = \tau\varphi(\iota, \kappa)(x)$. Since $\omega(a) = \omega(b)$, we have $\omega(\sigma) = \omega(\tau)$. If $\omega(\sigma) = \omega(\tau) \geqq 0$, we have $\omega((\sigma)_+) = \omega((\tau)_+)$ and so $\omega(p(a)) = \omega((\sigma)_+\sigma_\iota)$ $= \omega((\tau)_+\tau_\iota) = \omega(p(b))$. The case where $\omega(\sigma) = \omega(\tau) < 0$ will be treated similarly. Thus we get $\omega(p(a)) = \omega(p(b))$. Hence we can define a cap p_ω of E_ω such that $p_\omega(\omega(a)) = \omega(p(a))$ for a in E. Define a linear functional f_ω of E_ω such that $f_\omega(\omega(a)) = \sum_{i=1}^{n} \beta_i \omega(x)$, where $\omega(a) = \sum_{i=1}^{n} \beta_i \varphi_\omega(\iota_i, \kappa_i)(\omega(x))$. Then f_ω is compatible with p_ω. In order to see this, we can assume without loss of generality that $\omega(a)$ is contained in $E_\omega(\iota, \kappa)$ for some ι in I, κ in K. Suppose $\omega(a) = \beta\varphi_\omega(\iota, \kappa)(\omega(x))$. If $\beta > 0$, then $p_\omega(\omega(a)) = \beta\omega(\sigma_\iota)$ and so $f_\omega(\omega(a)) = \beta\omega(x) \leqq p_\omega(\omega(a))$. If $\beta = 0$, then $p_\omega(\omega(a)) = f_\omega(\omega(a)) = 0$. If $\beta < 0$, then $p_\omega(\omega(a)) = -\beta\omega(\tau_\kappa)$ and so $f_\omega(\omega(a)) = (-\beta)$ $(-\omega(x)) \leqq (-\beta)(-\omega(\tau_\kappa)) = p_\omega(\omega(a))$. Hence f_ω is compatible with p_ω. Since $f_\omega(\omega(a)) = 0$ for $\omega(a)$ in $\omega(E_1)$, we have $\omega(p(a)) = p_\omega(\omega(a)) \geqq 0$ for a in E_1 because of the compatibility of f_ω with p_ω. This is the case when $\omega(x) \neq 0$. If $\omega(x)$ $= 0$, we see that $\omega(\sigma_\iota) \geqq 0 \geqq \omega(\tau_\kappa)$ and, for any $a = \sum_{i=1}^{n} \sigma_i \varphi(\iota_i, \kappa_i)(x) \in E$, we have $\omega(p(a)) = \omega(\sum_{i=1}^{n} ((\sigma_i)_+\sigma_{\iota_i} - (\sigma_i)_-\tau_{\kappa_i})) \geqq 0$. We have thus shown that $p(a) \geqq 0$ for all a in E_1.

We then see that the zero functional, say, f_1 of E_1 is compatible with p. According to the validity of Theorem B, f_1 can be extended to a linear mapping, say, f of E into N_0 compatible with p. Denote then by σ the image of $\varphi(\iota, \kappa)(x)$ due to f, which is independent of the choice of indices ι, κ. It is easy to see that σ is the GLB in question. This completes the proof.

4. Connection between the Theorem of L. Nachbin-D. B. Goodner and Theorem of M. Nakai. We say that a normed linear space E_0 has the extension property of L. Nachbin if, for any normed linear space E over R and for any subspace E_1 of E, every linear functional f_1 of E_1 into E_0, whose norm is less than 1, has a linear extension on the whole space E into E_0, whose norm is also less than 1. We also say that the space N_0 of real-valued continuous functions on a compact Hausdorff space has the extension property of M. Nakai if, for any linear space E over N_0 with a cap p and for any N_0-subspace E_1 of E, every linear mapping f_1 of E_1 into N_0 compatible with p has a linear extension on the whole space E into N_0 compatible with p.

We first prove the following

PROPOSITION 1: *Let Ω be a compact Hausdorff space and N_0 be the space of real-valued continuous functions on Ω. If N_0 has the extension property of L. Nachbin, considered as a normed linear space with the usual supremum norm, then it has the extension property of M. Nakai.*

PROOF: Let E be a linear space over N_0 with a cap p and let E_1 be its N_0-subspace. Suppose f_1 is a linear mapping of E_1 into N_0 compatible with p. Consider the N-space constructed from the direct sum N of N_0 and E as in the proof of Theorem B. The spaces E and N_0 then can be regarded as parts of this N-space. As such, the mapping f_1 can be extended as an N-linear mapping F_1 from the N_0-subspace N_1 of N generated by E_1 and N_0, and it is compatible with the cap $\tilde{p}$ of N defined by $\tilde{p}(\sigma+a)=\sigma+p(a)$ $(a\in E)$ as is in the proof of Theorem B.

The norm of an element σ in N_0 being noted as $\|\sigma\|$, write $\|a\| = \mathrm{Max}(\|\tilde{p}(a)\|, \|\tilde{p}(-a)\|)$ for a in N. It is easy to see that $\|a\|$ has the properties of a pseudo-norm: (1) $\|a\|\geqq 0$ for a in N, (2) $\|\beta a\|=|\beta|\,\|a\|$ for β in R and a in N, and (3) $\|a+b\|\leqq\|a\|+\|b\|$ for a, b in N. Let M be the linear subspace over R of N consisting of the elements a such that $\|a\|=0$, and denote by $\tilde{a}$ the image of the element a of N under the natural homomorphism of N onto the quotient space $\tilde{N}=N/M$. The space $\tilde{N}$ is obviously a normed linear space. Let $\tilde{N}_1$ be the subspace of $\tilde{N}$, image of N_1 under the homomorphism $N\to\tilde{N}$.

Now, because F_1 is compatible with p, we have $-\tilde{p}(-a)\leqq F_1(a)\leqq\tilde{p}(a)$ for a in N_1, so that $\|F_1(a)\|\leqq\|a\|$. This inequality permits us to define a mapping $\tilde{F}_1(\tilde{a})$ from $\tilde{N}_1$ into N_0 by setting $\tilde{F}_1(\tilde{a})=F_1(a)$ (naturally $\tilde{a}$ is the image of a), and this satisfies $\|\tilde{F}_1(\tilde{a})\|\leqq\|\tilde{a}\|$. Now by the extension property of L. Nachbin, which we have assumed true for N_0, a linear mapping $\tilde{F}$ from $\tilde{N}$ into N_0 exists which extends $\tilde{F}_1$ and satisfies $\|\tilde{F}(\tilde{a})\|\leqq\|\tilde{a}\|$ for any $\tilde{a}$ in $\tilde{N}$.

Put then $f(a)=\tilde{F}(\tilde{a})$ for a in E. It remains to show that $f(a)\leqq p(a)$ for any a in E. But, if we put $b=p(a)-a$, $b\geqq 0$ by the definition of the quasi-ordering in N, and $f(a)\leqq p(a)$ is equivalent to $\tilde{F}(\tilde{b})\geqq 0$ since $\tilde{F}(\tilde{b})=p(a)-f(a)$. Let $\alpha=2\|a\|$, then $\|\alpha-\tilde{b}\| = \mathrm{Max}(\|\tilde{p}(a-p(a)+\alpha)\|, \|\tilde{p}(-a+p(a)-\alpha)\|) \leqq \mathrm{Max}(\alpha, \|\alpha-(p(a)+p(-a))\|)\leqq\alpha$, and so $\|\alpha-\tilde{F}(\tilde{b})\|=\|\tilde{F}(\alpha-\tilde{b})\|\leqq\|\alpha-\tilde{b}\|\leqq\alpha$, which shows exactly that $\tilde{F}(\tilde{b})\geqq 0$.

As L. Nachbin [8] and D. B. Goodner [2] have shown, the Banach space of continuous functions on a Stonian space has the extension property of L. Nachbin, so the theorem of M. Nakai (our Theorem B) is a consequence of their results by the proposition just shown. We show conversely that the theorem of L. Nachbin and D. B. Goodner follows from the theorem of M. Nakai, so that these two theorems are completely equivalent.

THEOREM (B) (L. Nachbin and D. B. Goodner): *The space of real-valued continuous functions on a Stonian space has the extension property of L. Nachbin.*

PROOF: Let Ω be a Stonian space and N_0 be the set of real-valued continuous functions on Ω. Let E be a normed linear space over R and let E_1 be its linear subspace. Suppose f_1 is a linear mapping of E_1 into N_0 such that $\|f_1(a)\| \leqq \|a\|$ for a in E_1. It must be shown that f_1 has a linear extension f of E into N_0 such that $\|f(a)\| \leqq \|a\|$ for a in E. Denote by E_Ω the set of all symbols $(a_\iota, e_{0\iota}; \iota \in I)$, where $(e_{0\iota}; \iota \in I)$ is an orthogonal set of non-zero projections in N_0, whose LUB (the least upper bound) is 1, and $(a_\iota; \iota \in I)$ is a uniformly bounded set of elements in E. We introduce an equivalence relation in E_Ω as follows: $a=(a_\iota, e_{0\iota}; \iota \in I)$ and $b=(b_\kappa, e_{0\kappa}; \kappa \in K)$ are equivalent if $a_\iota = b_\kappa$ whenever $e_{0\iota}e_{0\kappa} \neq 0$. For simplicity, we do not introduce a new notation to show the equivalence class, but let it represent by one of the elements belonging to the class. We define the addition $a+b$ of a and b in E_Ω, scalar multiplication αa of a in E_Ω by α in R, and norm $\|a\|$ of a in E_Ω by $a+b=(a_\iota+b_\kappa, e_{0\iota}e_{0\kappa}; e_{0\iota}e_{0\kappa} \neq 0, \iota \in I, \kappa \in K)$, $\alpha a=(\alpha a_\iota, e_{0\iota}; \iota \in I)$, and $\|a\| = \sup(\|a_\iota\|; \iota \in I)$, respectively, where $a=(a_\iota, e_{0\iota}; \iota \in I)$ and $b=(b_\kappa, e_{0\kappa}; \kappa \in K)$. It is easy to see that these are well defined and E_Ω constitutes a normed linear space over R. For a in E, $(a, 1)$ is an element of E_Ω and the mapping: $a \to (a, 1)$ gives an isometric isomorphism of E into E_Ω, and it will be convenient in what follows that we identify a with $(a, 1)$. Denote by $\overline{E_\Omega}$ the completion of E_Ω. If $\{a_n\}$ is a Cauchy sequence of elements in E_Ω and if e_0 is a projection in N_0, then $\{e_0 a_n\}$ is also a Cauchy sequence of elements in E_Ω, where, for a projection e_0, and for $a=(a_\iota, e_{0\iota}; \iota \in I)$, we define $e_0 a$ as follows. Put $I_1=$ the set of indices ι in I such that $e_0 e_{0\iota} \neq 0$, and let, for such ι, $e_{0\iota}' = e_0 e_{0\iota}$. Put I_1^0 the set I_1 augmented by one index 0, and let $a_0=0, e_{00}' = 1-e_0$. Then $e_0 a=(a_\iota, e_{0\iota}'; \iota \in I_1^0)$. We define $e_0(\lim a_n)$ by $\lim e_0 a_n$. Denote by $(e_{0,\alpha}(\sigma); -\infty < \alpha < \infty)$ the resolution of identity associated with an element σ in N_0. Then, for σ in N_0 and for a in $\overline{E_\Omega}$, we define σa by $\int_{-\infty}^{\infty} \alpha d(e_{0,\alpha}(\sigma)a)$. (This integral converges in norm.) It is easy to see that $\overline{E_\Omega}$ constitutes a linear space over N_0. Since $\|e_0 a\| \leqq \|a\|$ for a projection e_0 in N_0 and for a in $\overline{E_\Omega}$, we can conclude that $\|\sigma a\| \leqq \|\sigma\| \|a\|$ for σ in N_0 and a in $\overline{E_\Omega}$.

For ω in Ω, set $\|a\|_\omega = \inf(\|e_0 a\|; e_0 \in E_0(\omega))$. It is easy to see that the function: $\omega \to \|a\|_\omega$ is semi-continuous. Hence, by a theorem of Baire and Hausdorff, we can find a continuous function on Ω, which is almost equal to this semi-continuous function. Denote it by $p(a)$. It is not hard to see that p is a cap of $\overline{E_\Omega}$. Moreover, we notice that $p(a) = \|a\|$ for a in E.

Let $(E_1)_\Omega$ be the set of elements $a=(a_\iota, e_{0\iota}; \iota \in I)$ in E_Ω such that $a_\iota \in E_1$ for any $\iota \in I$. If we define $f_1(a)$ for $a=(a_\iota, e_{0\iota}; \iota \in I)$ in $(E_1)_\Omega$ by $f_1(a)$

$=\bigoplus(f_1(a_\iota)e_{0\iota}\,;\,\iota\in I)$ where the right side denotes the element in N_0 uniquely defined to be equal to $f_1(a_\iota)$ in $(\omega\,;\,\omega(e_{0\iota})=1, \omega\in\Omega)$ for any ι, we have $\|f_1(a)\|\leqq\|a\|$, and f_1 can be extended as a linear functional on the closure $\overline{(E_1)_\Omega}$. As we have considered at the end of the last paragraph, we can introduce a N_0-linear structure in $\overline{(E_1)_\Omega}$ and it will be easy to observe that f_1 is actually a N_0-linear functional. The relation $\|f_1(a)\|\leqq\|a\|$ which is true for any a in $\overline{(E_1)_\Omega}$ implies $\|\omega(f_1(a))\|\leqq\|a\|_\omega$ for any ω in Ω, thus we see that $\|f_1(a)\|\leqq p(a)$. Theorem B now asserts that there exists a linear extension f of $\overline{E_\Omega}$ into N_0, compatible with p. Then the restriction of f on E will be the linear extension in question as is easily seen. This completes the proof.

§ 2. Banach Algebras Over R_0.

1. Definitions. Let Ω be a compact Hausdorff space. Denote by R_0 the set of complex-valued continuous functions on Ω. In a usual way R_0 constitutes a commutative Banach algebra over the field of complex numbers C.

We introduce some definitions and state some lemmas.

DEFINITION 2.1: A normed algebra A over C is called a normed algebra over R_0 if it has R_0 as an operator domain and the following are satisfied: $(\sigma\tau)a=\sigma(\tau a)$, $(\sigma+\tau)a=\sigma a+\tau a$, $\sigma(ab)=(\sigma a)b=a(\sigma b)$, $1a=a$, $\|\sigma a\|\leqq\|\sigma\|\|a\|$ for σ, τ in R_0, a, b in A, where 1 is the function on Ω taking values identically equal to 1.

DEFINITION 2.2: An element e in a normed algebra over R_0 is called an R_0-unit if it is a unit and if it satisfies the condition: $\|\sigma e\|=\|\sigma\|\|e\|$ for σ in R_0.

It is easy to see that any normed algebra over R_0 can be extended to a normed algebra over R_0 with an R_0-unit. From now on throughout this section, however, we are concerned only with normed algebras over R_0 with an R_0-unit and denote it by e.

DEFINITION 2.3: A normed algebra over R_0 is called a Banach algebra over R_0 if it is complete.

It is easy to see that the completion of a normed algebra over R_0 becomes a Banach algebra over R_0.

Let A be a Banach algebra over R_0. Denote by N_0 the set of real-valued continuous functions on Ω and by E_0 the set of those functions e_0 in N_0 which satisfies the condition: (a) $0\leqq e_0\leqq 1$ and (b) the set $\gamma(e_0)$ of inner points of $(\omega\,;\,\omega(e_0)=1, \omega\in\Omega)$ is non-empty.

DEFINITION 2.4: A left (right) ideal I of a Banach algebra A over R_0 is said to be proper if $\sigma e\in I$ for some σ in R_0 implies $\sigma=0$.

LEMMA 2.1: *The closure of a proper left ideal I of a Banach algebra A*

over R_0 is also proper.

PROOF: Suppose the contrary. Then there exist a non-zero element σ in R_0 and an element a in I such that $\| \sigma e - a \| < 1/2$. We can assume without loss of generality that σ is in N_0 and $\| \sigma \| = 1$. Then there exists an element e_0 in E_0 such that $\| (\sigma - 1)e_0 e \| < 1/2$. From these inequalities it follows that $\| e_0(e-a) \| < 1$. Hence $e - e_0(e-a)$ has an inverse in A. Denote it by b. Moreover we can find an element e_0' in E_0 such that $e_0' e_0 = e_0'$. Then we get $(e_0' e_0 b) e_0 a = e_0' b(e - e_0(e-a) - (1-e_0)e) = e_0' e$. This means that $e_0' e$ is contained in I. This contradicts the assumption that I is proper. Hence the closure of I must be proper. This completes the proof.

For ω in Ω, denote by $E_0(\omega)$ the set of elements $e_0(\omega)$ in E_0 such that the set $\gamma(e_0(\omega))$ of inner points of $(\rho ; \rho(e_0(\omega)) = 1, \rho \in \Omega)$ contains ω. Set $\| a \|_\omega = \inf (\| e_0(\omega) a \| ; e_0(\omega) \in E_0(\omega))$ for a in A and $(0)^\omega = (a ; \| a \|_\omega = 0, a \in A)$. Then $(0)^\omega$ constitutes a closed ideal of A. Denote by the same ω the natural homomorphism of A onto $A/(0)^\omega$. Then $A/(0)^\omega$ constitutes a Banach algebra over C with norm defined by $\| \omega(a) \| = \| a \|_\omega$ for a in A. Denote it by A_ω and call it the *local Banach algebra of A with respect to ω.*

LEMMA 2.2: *It holds that $\| \omega(\sigma a) \| = | \omega(\sigma) | \, \| \omega(a) \|$ for σ in R_0 and a in A.*

PROOF: It is easy to see that $(\sigma - \omega(\sigma))a \in (0)^\omega$. Hence we get $\| \omega(\sigma a) \| = \| \omega(\omega(\sigma) a) \| = \| \omega(\sigma) \omega(a) \| = | \omega(\sigma) | \, \| \omega(a) \|$. This completes the proof.

LEMMA 2.3: *A left ideal I of a Banach algebra A over R_0 is proper if and only if $\omega(I)$ is proper in A_ω for almost all points ω in Ω.*

PROOF: NECESSITY: Suppose $\omega(I)$ is not proper in A_ω for some point ω in Ω. Then there exists an element a in I such that $\omega(a) = \omega(e)$, that is, $\| e_0(e-a) \| < 1$ for some element e_0 in $E_0(\omega)$. Hence we can find an element b in A such that $b(e - e_0(e-a)) = e$ and an element e_0' in E_0 such that $e_0' e_0 = e_0'$. Then we get $e_0' ba = e_0' e$ as in the proof of Lemma 2.1. This means that I is not proper. Thus we have proved that, if I is proper, then $\omega(I)$ is proper in A_ω for any point ω in Ω. This result contains the assertion.

SUFFICIENCY: Suppose I is not proper. Then there exists a non-zero element σ in R_0 such that $\sigma e \in I$. Since $\sigma \neq 0$, we can find an element e_0 in E_0 such that $\omega(\sigma) \neq 0$ for ω in $\gamma(e_0)$. Since $\sigma e \in I$, we get $\omega(\sigma)\omega(e) = \omega(\sigma e) \in \omega(I)$ and so $\omega(e) \in \omega(I)$. This means that $\omega(I)$ is not proper in A_ω for ω in $\gamma(e_0)$. Hence the set of points ω, for which $\omega(I)$ is not proper in A_ω, is not of first category. This completes the proof.

DEFINITION 2.5: An element σ in R_0 is called a left (right) R_0-spectrum of an element a in a Banach algebra A over R_0 if the left (right) ideal of A generated by $a - \sigma e$ is proper.

Combining Definition 2.5 with Lemma 2.3 we have the following

LEMMA 2.4: *An element σ in R_0 is a left R_0-spectrum of an element a in* *a Banach algebra A over R_0 if and only if $\omega(\sigma)$ is a left spectrum of $\omega(a)$ in A_ω for almost all points ω in Ω.*

DEFINITION 2.6: A left (right) ideal I of a Banach algebra A over R_0 is said to be properly maximal if it is proper and if there is no proper left (right) ideal of A containing I except for I itself.

The proof of the following lemma is easy and will be omitted.

LEMMA 2.5: *Given any proper left ideal I of a Banach algebra A over R_0, there exists at least one properly maximal left ideal of A containing I.*

DEFINITION 2.7: A Banach algebra A over R_0 is said to be left-sidedly (right-sidedly) simple if it has no left (right) proper ideal of A except for the zero ideal (0). A Banach algebra over R_0 is said to be one-sidedly simple if it is left-sidedly simple or right-sidedly simple.

As an immediate consequence of Lemma 2.4, we have the following

LEMMA 2.6: *Let σ be a left R_0-spectrum of an element a in a Banach algebra A over R_0 and let $P(t)=\sigma_0+\sigma_1 t+\cdots+\sigma_n t^n$ be a polynomial of t with $\sigma_i\in R_0$ $(0\leqq i\leqq n)$. We set $P(\sigma)=\sigma_0+\sigma_1\sigma+\cdots+\sigma_n\sigma^n$ for σ in R_0 and $P(a)=\sigma_0 e+\sigma_1 a+\cdots+\sigma_n a^n$ for a in A. Then*

1) $\|\sigma\|\leqq\|a\|$ *and*

2) *$P(\sigma)$ is a left R_0-spectrum of $P(a)$.*

DEFINITION 2.8: An element σ in R_0 is called a mixed R_0-spectrum of an element a in a Banach algebra A over R_0 if $\omega(\sigma)$ is a left or right spectrum of $\omega(a)$ in A_ω for almost all points ω in Ω.

DEFINITION 2.9: For an element a in a Banach algebra A over R_0, we set $\|a\|_\infty=\lim\|a^n\|^{1/n}$ and $\|a\|_0=\sup(\|\sigma\|;\sigma$ being a mixed R_0-spectrum of a). (If there is no mixed R_0-spectrum of a, we set $\|a\|_0=0$. We shall see in Corollary 2 of Theorem C that there exists at least one mixed R_0-spectrum for any element in a Banach algebra over R_0 if Ω is Stonian.)

It is known that $\lim\|a^n\|^{1/n}$ exists, but we shall give an alternative proof of this fact in the next section (Theorem C).

DEFINITION 2.10: A Banach algebra A over R_0 is called a B^*-algebra over R_0 if it is a B^*-algebra in complex scalar case and if it satisfies the condition: $(\sigma a)^*=\sigma^* a^*$ for σ in R_0 and for a in A, where σ^* is the function in R_0 whose value at $\omega\in\Omega$ is the complex conjugate $\overline{\omega(\sigma)}$ of $\omega(\sigma)$.

Suppose A is a B^*-algebra over R_0. Since $(0)^\omega$ for ω in Ω turns out a closed two-sided ideal, it is self-adjoint, and so the local Banach algebra A_ω of A with respect to ω constitutes a B^*-algebra over C by using a result of I. Kaplansky [6, Theorem 7.3]. We call it the *local B^*-algebra of A with respect to ω.*

DEFINITION 2.11: A Banach algebra A over R_0 is said to be regular if $\omega(a)=0$ for almost all points ω in Ω implies $a=0$.

2. Stonian Case. In this section, we assume that Ω is a Stonian space. Denote by E_0 the set of projections in N_0 and by $E_0(\omega)$ for ω in the set Ω of those projections $e_0(\omega)$ in E_0 which satisfies the condition: $\omega(e_0(\omega))=1$ (as in 2, §1). Then one sees easily that we may replace the E_0 used in Section 1 in defining $(0)^{\omega}$, $\| a \|_{\omega}$, and A_{ω} by the E_0 here mentioned.

Let A be a Banach algebra over R_0. Since the function: $\omega \to \| a \|_{\omega}$ of Ω into $B(\Omega)$ (the space of real-valued bounded functions on Ω) is upper semi-continuous, it is equal to a uniquely determined continuous function on Ω outside a certain first category set in Ω by a theorem of Baire and Hausdorff. Denote it by $|a|$. Then the mapping $|\cdot|$ of A into $(N_0)_+$ satisfies the following conditions: (a) $|a| \geqq 0$ for a in A, (b) $|\sigma a| = |\sigma|\,|a|$ for σ in R_0 and a in A, (c) $|a+b| \leqq |a|+|b|$ for a, b in A, and (d) $\omega(|a|) = \| \omega(a) \|$ for almost all ω in Ω, where we denote by $|\sigma|$ the continuous function on Ω, whose value at ω is equal to $|\omega(\sigma)|$ for ω in Ω. The mapping $|\cdot|$ having the properties $(a)-(d)$ is uniquely determined. We shall call it the *pseudo R_0-norm* of A. Moreover, a pseudo R_0-norm is called an *R_0-norm* if it satisfies the further condition: (a') $|a|=0$ for a in A implies $a=0$. If A is a B^*-algebra over R_0, the pseudo R_0-norm satisfies the condition: (e) $|a^*a| = |a|^2$ for a in A. In fact, if A is a B^*-algebra over R_0, A_{ω} is a B^*-algebra for any ω in Ω, and so $\| \omega(a^*a) \| = \| \omega(a) \|^2$ for ω in Ω and a in A. This implies that $|a^*a| = |a|^2$ for all a in A.

DEFINITION 2.12: For an element a of a Banach algebra A over R_0, we set $|a|_{\infty} = \text{order-lim}\, |a^n|^{1/n}$ and $|a|_0 = \text{LUB}(|\sigma|;$ being a mixed R_0-spectrum of $a)$, where by LUB we mean the least upper bound.

We show that order-lim $|a^n|^{1/n}$ exists. In fact, for almost all ω in Ω, $\omega(|a^n|^{1/n}) = \| \omega(a^n) \|^{1/n}$ and so $\omega(\text{order-}\underline{\lim}\, |a^n|^{1/n}) = \underline{\lim}\, \| \omega(a^n) \|^{1/n}$ and $\omega(\text{order-}\overline{\lim}\, |a^n|^{1/n}) = \overline{\lim}\, \| \omega(a^n) \|^{1/n}$. Since $\underline{\lim}\, \| \omega(a^n) \|^{1/n} = \overline{\lim}\, \| \omega(a^n) \|^{1/n}$ for all ω in Ω, we get $\omega(\text{order-}\underline{\lim}\, |a^n|^{1/n}) = \omega(\text{order-}\overline{\lim}\, |a^n|^{1/n})$ for almost all ω in Ω. This implies that $\text{order-}\underline{\lim}\, |a^n|^{1/n} = \text{order-}\overline{\lim}\, |a^n|^{1/n}$. Hence we can conclude that order-lim $|a^n|^{1/n}$ exists.

The proof of the following lemma goes through like in Lemma 2.1 and will be omitted.

LEMMA 2.7: *Suppose A is a Banach algebra over R_0. If a scalar t_0 is not a left spectrum of $\omega(a)$ $(a \in A)$ in A_{ω} for some ω in Ω, then there exist a neighbourhood U at t_0 in the complex plane and a projection e_0 in $E_0(\omega)$ such that $e_0(te-a)$ has a left inverse in e_0A for any t in U.*

Let A be a regular Banach algebra over R_0. It is easy to see that the pseudo R_0-norm of A becomes an R_0-norm.

LEMMA 2.8: *An element a in a regular Banach algebra A over R_0 is contained in R_0e if and only if $\omega(a)$ is contained in $\omega(R_0e)$ for any ω in Ω.*

PROOF: The necessity is obvious, so we need only to see the sufficiency. Since $\omega(a)$ is in $\omega(R_0 e)$ for ω in Ω, we can find a complex number β_ω such that $\omega(a)=\beta_\omega \omega(e)$. Hence, given a natural number n, there exists a projection $e_0(\omega)$ in $E_0(\omega)$ such that $|e_0(\omega)(a-\beta_\omega e)|<1/n$, i. e. $|e_0(\omega)(a-\beta_\omega e)| \leqq (1/n)e_0(\omega)$. Associate $e_0(\omega)$ and β_ω with ω in Ω. Since Ω is Stonian, it is compact and so it contains a finite number of points $\omega_1, \cdots, \omega_m$ such that $\Omega=\cup(\gamma(e_0(\omega_i)); 1\leqq i \leqq m)$. Moreover, we can find an orthogonal set of projections $e_0^{(1)}, \cdots, e_0^{(m)}$ in E_0 such that $e_0^{(i)} \leqq e_0(\omega_i)$ $(1 \leqq i \leqq m)$ and such that $\oplus(e_0^{(i)}; 1\leqq i\leqq m)=1$. Since $e_0^{(i)} \leqq e_0(\omega_i)$ $(1\leqq i\leqq m)$, we have $|e_0^{(i)}(a-\beta_{\omega_i} e)| \leqq (1/n)e_0^{(i)}$ $(1\leqq i\leqq m)$. Summing up these inequalities, we get $|a-\tau_n e| \leqq 1/n$, where $\tau_n=\sum_{i=1}^{n} \beta_{\omega_i} e_0^{(i)}$. Hence $\{\tau_n\}$ is a uniform Cauchy sequence of elements in R_0 and so there is a uniform limit, say, τ in R_0. Hence there exists a natural number n_0 $(\geqq n)$ such that $|\tau e-\tau_{n_0} e| \leqq 1/n$ and so $|a-\tau e| \leqq 2/n$. By making $n\to\infty$, we get $|a-\tau e|=0$. Since A is regular, this implies that $a=\tau e$. This completes the proof.

We are now in a position to prove the following generalization of the spectral radius theorem of I. Gelfand [1, Satz 8′].

THEOREM C: *It holds that* $\| a \|_\infty = \| a \|_0$ *for any element* a *in a Banach algebra* A *over* R_0.

REMARK: The local interpretation of the theorems C, D corresponds to the theorem of I. Gelfand and the theorem of M. Mazur and I. Gelfand, and their proofs given below are the translation of those which the author gave earlier in Japanese [**12**]. C. E. Rickart [**13**] has obtained a similar proof independently.

PROOF OF THEOREM C: LOCAL PROOF: We first give an alternative proof of the theorem of I. Gelfand by making no use of the theory of function of a complex variable as a frame work of S. Kametani [**4**]. Suppose A is a Banach algebra over the field C of complex numbers with an identity e. We know that, if an element a in A has a left inverse and right inverse, then they are unique and equal to each other. Hence a complex number is a (left or right) spectrum of a if and only if it is a spectrum of a in a closed, commutative subalgebra of A containing a. Therefore, we can assume without loss of generality that A is commutative.

First, we see that $\| a \|_0 \leqq \underline{\lim} \| a^n \|^{1/n}$. Indeed, if $\underline{\lim} \| a^n \|^{1/n} < |t|^{-1}$ for a scalar t, then there exists a natural number n such that $\| a^n \|^{1/n} < |t|^{-1}$. Hence the series $t\sum_{m=0}^{\infty}(ta)^m = t(\sum_{m=0}^{n-1}(ta)^m)(\sum_{k=0}^{\infty}(ta)^{nk})$ is convergent and its limit is equal to the inverse of $t^{-1}e-a$. This shows that t is never a spectrum of a and the assertion follows.

In order to see that $\overline{\lim} \| a^n \|^{1/n} \leqq \| a \|_0$, suppose the contrary. Set $\psi(t)$

$=(e-ta)^{-1}$. This function is defined and continuous on $(t;\, \| a \|_0 < | t |^{-1})$. We then use the following Lagrange's formula:

$$(1) \qquad (1/n)\sum_{i=1}^{n}(t\zeta_i)^{-k}\psi(t\zeta_i)=a^k(e-t^na^n)^{-1}$$

for $0 \leqq k \leqq n$ and $\| a \|_0 < | t |^{-1}$, where ζ_i's $(1 \leqq i \leqq n)$ are the n-th roots of 1. From the elementary theory of the Riemann integral of continuous functions it follows that the left-hand side of the above formula (1) has a limit as $n \to \infty$ and hence

$$(2) \qquad (1/2\pi)\int_0^{2\pi}(t\zeta)^{-k}\psi(t\zeta)d\theta = a^k c(t)\,,$$

where $\zeta = e^{\sqrt{-1}\theta}$, $0 \leqq \theta \leqq 2\pi$ and $c(t) = \lim (e-t^na^n)^{-1}$. By making $k=0$, we can see the existence of $c(t)$. Since $\psi(t)$ is continuous with respect to the complex variable t, $c(t)$ is continuous on $(t;\, \| a \|_0 < | t |^{-1})$ with respect to the real variable t. Moreover, one sees easily that

$$(3) \qquad c(t) = e \quad \text{if} \quad \underline{\lim} \, \| a^n \|^{1/n} < | t |^{-1}\,.$$

Therefore, we can select two numbers t, t_1 such that $\| a \|_0 < | t |^{-1} < | t_1 |^{-1} < \overline{\lim} \, \| a^n \|^{1/n}$ and that $c(t)$ has an inverse. Multiplying t_1^k to both sides in (2), we get

$$(4) \qquad (1/2\pi)\int_0^{2\pi} t_1^k (t\zeta)^{-k}\psi(t\zeta)d\theta = t_1^k a^k c(t)\,.$$

Since $(t_1t^{-1})^k \to 0$ as $k \to \infty$, we have $(t_1a)^kc(t) \to 0$ as $k \to \infty$, and, by multiplying $c(t)^{-1}$, we get $(t_1a)^k \to 0$ as $k \to \infty$. This implies that $\overline{\lim} \, \| a^n \|^{1/n} \leqq | t_1 |^{-1}$, which is a contradiction. Hence we must have $\overline{\lim} \, \| a^n \|^{1/n} \leqq \| a \|_0$, which shows, combining with the other inequality already obtained, the validity of the assertion.

GLOBAL PROOF: Return now to the general case. Let A be a Banach algebra over R_0 and a an arbitrary (but fixed) element in A. We can assume without loss of generality that A is commutative by the same reason as in the local proof.

First we see that

$$(5) \qquad \omega(| a |_\infty) = \| \omega(a) \|_\infty \text{ for } \omega \text{ in } \Omega\,.$$

In defining $| a |_\infty$ we used the fact that $\lim \| \omega(a^n) \|^{1/n}$ exists. As this now has been established, the use of $| a |_\infty$ is legitimate. Suppose for a point ω in Ω, we have $\omega(| a |_\infty) < \| \omega(a) \|_\infty$. Take a complex number t such that $\omega(| a |_\infty) < | t | < \| \omega(a) \|_\infty$. Then there exists a projection e_0 in $E_0(\omega)$ such that $| e_0 a |_\infty = e_0 | a |_\infty < | t |$. Hence a natural number n and a projection e_0' in $E_0(\omega)$ can be found such that $|(e_0'a)^n|^{1/n} < | t |$, which implies that, for $| t | < | t' |$, $e_0'(t'^{-1}e - a)$ has an inverse in $e_0'A$ by a similar argument as in the proof of the fact that

$\| a \|_0 \leqq \lim \| a^n \|^{1/n}$ in the local proof. From this it follows that, for $|t| < |t'|$, $\omega(t'^{-1}e-a)$ has an inverse in A_ω, and we have $\| \omega(a) \|_\infty \leqq |t|$. This contradicts the choice of t. Thus we get $\| \omega(a) \|_\infty \leqq \omega(| a |_\infty)$.

On the contrary, suppose $\| \omega(a) \|_\infty < \omega(| a |_\infty)$. Take a real number t such that $\| \omega(a) \|_\infty < t < \omega(| a |_\infty)$. Then there exists a natural number n such that $\| \omega(a^n) \|^{1/n} < t$ and so we can take a projection e_0 in $E_0(\omega)$ such that $|(e_0a)^n|^{1/n} < t$, i.e. $|(t^{-1}e_0a)^n| < 1$. Hence $|(t^{-1}e_0a)^{nk}| < 1$ for all natural number k and so $|(t^{-1}e_0a)^{nk}|^{1/(nk)} < 1$. By making $k \to \infty$, we get $| e_0a |_\infty \leqq t$, that is, $\omega(| a |_\infty) \leqq t$. This is a contradiction. Thus we get $\| \omega(a) \|_\infty = \omega(| a |_\infty)$.

Further we have

$$(6) \qquad \| a \|_\infty = \| \, | a |_\infty \, \| .$$

In fact, for a projection e_0 in $E_0(\omega)$, we have $\| e_0a \| \leqq \| a \|$ and so $\| \omega(a) \| \leqq \| a \|$ for (an arbitrary) a in A and ω in Ω. Hence we get $\| \omega(a) \|_\infty \leqq \| a \|_\infty$; namely, $\| \, | a |_\infty \, \| \leqq \| a \|_\infty$ by (5). On the other hand, any character of A, that is to say, a homomorphism of A onto C, is naturally considered as a character of A_ω for some ω in Ω and therefore, by making use of a theorem of S. Mazur and I. Gelfand [**1**, Satz 3] or the local part of Theorem D (which follows from the local part of Theorem C and the latter was shown to hold), we get $\| a \|_\infty \leqq \sup (\| \omega(a) \|_\infty ; \omega \in \Omega) = \| \, | a |_\infty \, \|$. This completes the proof of (6).

Moreover, we have

$$(7) \qquad \| a \|_0 = \| \, | a |_0 \, \| .$$

In fact, for an R_0-spectrum σ of a, $\| \sigma \| \leqq \| a \|_0$ and so $| \sigma | \leqq \| a \|_0$; namely, $\| \, | a |_0 \, \| \leqq \| a \|_0$. On the other hand, $| \sigma | \leqq | a |_0$ and so $\| \sigma \| \leqq \| \, | a |_0 \, \|$; namely, $\| a \|_0 = \| \, | a |_0 \, \|$. Thus we get (7).

In view of (6) and (7), in order to show Theorem C, it remains only to see that

$$(8) \qquad | a |_0 = | a |_\infty .$$

It is not hard to see that

$$(9) \qquad | a |_0 \leqq | a |_\infty .$$

In fact, due to Lemma 2.4, for an R_0-spectrum σ of $a, \omega(\sigma)$ is a spectrum of $\omega(a)$ in A_ω for almost all points ω in Ω and so we have from (5) $|\omega(\sigma)| \leqq \omega(| a |_\infty)$ for almost all ω in Ω; namely, $| \sigma | \leqq | a |_\infty$. This implies that $| a |_0 \leqq | a |_\infty$.

We see that

(10) if A is a commutative B^*-algebra over R_0, then (8) holds.

Suppose A is a commutative B^*-algebra over R_0. Since $| a^*a | = | a |^2$, we have $| a |_\infty = | a |$. The element $| a^*a |$ in R_0 is an R_0-spectrum of a^*a, because $\omega(| a^*a |)$ $(= \| \omega(a^*a) \|_\infty)$ is a spectrum of $\omega(a^*a)$ for any point ω in Ω. Hence, in view of (9), we have $|a^*a|_0=|a^*a|$. In order to see (10), it remains only to see that $|a^*a|_0=|a|_0^2$.

Take an arbitrary R_0-spectrum σ of a^*a. By Lemma 2.5, there exists a properly maximal ideal (say, J) of A containing $a^*a-\sigma e$. Since J is closed by Lemma 2.1, J is self-adjoint. Hence A/J constitutes a simple B^*-algebra over R_0. Denote by - the natural homomorphism of A onto A/J. For a non-negative hermitian element $\bar{h}$ in A/J, $|\bar{h}|$ is an R_0-spectrum of h. Since A/J is simple, we get $\bar{h}=|\bar{h}|\bar{e}$. Any element in A/J is expressed as a linear combination of non-negative hermitian elements in A/J and therefore, it is contained in $R_0\bar{e}$; especially $\bar{a}=\tau\bar{e}$, say. Then $\bar{a}^*\bar{a}=\tau^*\tau\bar{e}$ and so $\sigma=\tau^*\tau$. Since $a-\tau e\in J,\tau$ is an R_0-spectrum of a in A. Hence we get $|\sigma|=\tau^*\tau\leqq|a|_0^2$. By making $|\sigma|\uparrow|a^*a|_0$, we get $|a^*a|_0\leqq|a|_0^2$. A similar argument shows the other inequality and we reach the assertion (10).

We return again to the case that A is a commutative Banach algebra over R_0. Denote by Γ the set of characters of A considered as an algebra over C. Then Γ constitutes a compact Hausdorff space with the usual Stone topology. The algebra A is homomorphic into the B^*-algebra $C(\Gamma)$ of complex-valued continuous functions on Γ. Denote the homomorphism by ϕ. Since R_0e is isomorphic onto R_0, any character γ of R_0e satisfies the condition: $\gamma(\sigma^*e)=\overline{\gamma(\sigma e)}$ for σ in R_0. Hence we have $\gamma(\sigma^*e)=\overline{\gamma(\sigma e)}$ for σ in R_0 and γ in Γ, and so $\phi(\sigma^*e)=\phi(\sigma e)^*$ for σ in R_0. We set $\sigma x=\phi(\sigma e)x$ for σ in R_0 and x in $C(\Gamma)$. Then we have $(\sigma x)^*=(\phi(\sigma e)x)^*=\phi(\sigma e)^*x^*=\phi(\sigma^*e)x^*=\sigma^*x^*$ for σ in R_0 and x in $C(\Gamma)$, where x^* is the function in $C(\Gamma)$ such that $\gamma(x^*)=\overline{\gamma(x)}$ for γ in Γ. Hence, with this scalar multiplication, $C(\Gamma)$ constitutes a B^*-algebra over R_0. We have from the remark (10)

$$|\phi(a)|_0=|\phi(a)|_\infty. \tag{11}$$

Since the set of characters of A_ω for ω in Ω coincides with the set of characters of $\omega(\phi(A))$ as a subset of Γ, we have $\|\omega(a)\|_\infty=\|\omega(\phi(a))\|_\infty$. In view of (5), thus, we get

$$|a|_\infty=|\phi(a)|_\infty. \tag{12}$$

Moreover, we have

$$|a|_0\geqq|\phi(a)|_0. \tag{13}$$

In fact, an R_0-spectrum σ of $\phi(a)$ in $C(\Gamma)$ is also an R_0-spectrum of a in A and so $|\phi(a)|_0\leqq|a|_0$.

In view of (9) and (11)-(13), we can conclude that $|a|_0=|a|_\infty$. This shows (9) and completes the proof.

COROLLARY 1: *It holds that* $|a|_0=|a|_\infty$ *for any element* a *in a Banach algebra over* R_0.

COROLLARY 2: *There exists at least one mixed R_0-spectrum of an element in a Banach algebra over R_0.*

PROOF: LOCAL PROOF: Let A be a Banach algebra over C and a be an element in A. If $\| a \|_\infty \neq 0$, we have from Theorem C $\| a \|_0 = \| a \|_\infty \neq 0$. This means that there exists a non-zero mixed spectrum of a. Suppose next $\| a \|_\infty = 0$. Then 0 is the only possible mixed spectrum of a. If 0 is not a mixed spectrum of a, then a has an inverse b in A such that $ab = ba = e$. Since b commutes with a, we must have $1 = \| e \|_\infty \leqq \| a \|_\infty \| b \|_\infty = 0$. This is a contradiction. Hence 0 is a mixed spectrum of a if $\| a \|_\infty = 0$.

GLOBAL PROOF: Let A is a Banach algebra over R_0 and a be an element in A. By the same reason as in the local proof, a has a non-zero mixed R_0-spectrum if $\| a \|_\infty \neq 0$. Hence we assume that $\| a \|_\infty = 0$. Then $\| \omega(a) \|_\infty = 0$ for almost all ω in Ω. We have seen that 0 is a mixed spectrum of $\omega(a)$ if $\| \omega(a) \|_\infty = 0$. Hence, by the definition of mixed R_0-spectrum, 0 is a mixed R_0-spectrum of a for this case. This completes the proof.

As a consequence of Theorem C, we shall prove the following generalization of a theorem of S. Mazur and I. Gelfand [**1**, Satz 3].

THEOREM D: *Every one-sidedly simple Banach algebra over R_0 is isomorphic onto R_0.*

PROOF: Let A be a left-sidedly simple Banach algebra over R_0 and a be an element in A. It must be shown that $a \in R_0 e$. By Corollary 2 of Theorem C, there exists a mixed R_0-spectrum σ of a. By taking $a - \sigma e$ instead of a, we can assume that a has 0 as one of its mixed R_0-spectrums.

If there exists no non-zero projection e_0 in E_0 such that $e_0 a$ has a left inverse in $e_0 A$, Lemma 2.7 shows that $\omega(0)$ is a left spectrum of $\omega(a)$ for all ω in Ω, and so 0 is a left R_0-spectrum of a. Hence the left ideal I of A generated by a is proper. Since A is left-sidedly simple, we must have $I = 0$ and so $a = 0$.

In the rest of the proof, we shall show that there exists actually no non-zero projection e_0 in E_0 such that $e_0 a$ has a left inverse in $e_0 A$. Suppose the contrary. Then there exists a non-zero projection e_0 in E_0 such that $e_0 a$ has a left inverse b in $e_0 A$. Since 0 is a mixed R_0-spectrum of $e_0 a$, and since $e_0 a$ has a left inverse in $e_0 A$, 0 is a right R_0-spectrum of $e_0 a$, and so 0 is a right R_0-spectrum of $e_0' a$ for any projection e_0' in $e_0 A$. On the other hand, since $b e_0 a = e_0$, we have $(e_0 ab) a = e_0 a \neq 0$, and so $e_0 ab \neq 0$. Hence, by the left-sided simplicity of A, we can conclude that there exists at least one point ω in Ω such that $\omega(e_0 ab)$ has a left inverse in A_ω. Therefore, by Lemma 2.7, there exists a non-zero projection e_0' in $e_0 A$ such that $e_0' ab$ has a left inverse c in $e_0' A$. Since $(e_0' ab)^2 = e_0' ab$, multiplying c from the left, we get $e_0' ab = e_0'$. This implies that 0 is not a R_0-spectrum of $e_0' a$. This is a contradiction. Hence such a case does not occur. This completes the proof.

3. Converse Theorems. In connection with Theorem C and D, we shall state and prove the following

THEOREM C$'$: *Ω is Stonian if Theorem C holds for any Banach algebra over R_0.*

PROOF: Suppose $(\sigma_\iota; \iota \in I)$ be a bounded below family of elements in N_0. It must be shown that under the validity of Theorem C, the GLB of this family exists in N_0. Denote by $D(\Omega)$ ($B(\Omega)$) the set of complex-valued (real-valued) bounded functions on Ω. Then $D(\Omega)$ constitutes a commutative B^*-algebra over R_0. Denote by x the GLB of the family in $B(\Omega)$ and by A the closed subalgebra of $D(\Omega)$ generated by x and R_0. Then A constitutes a B^*-algebra over R_0. Since A is a commutative B^*-algebra over R_0, it holds that $\| x \|_\infty = \| x \|$. Hence there exists an R_0-spectrum, say, σ of x in A due to the hypothesis that Theorem C holds.

We shall see that σ is the GLB in question. In fact, for each ι in I, being $x \leqq \sigma_\iota$, we have $\| \sigma_\iota - x - \| \sigma_\iota - x \| \| \leqq \| \sigma_\iota - x \|$. Combining this with Lemma 2.6, we get $\| \sigma_\iota - \sigma - \| \sigma_\iota - x \| \| \leqq \| \sigma_\iota - x \|$. This implies that $\sigma_\iota - \sigma \geqq 0$, that is, $\sigma \leqq \sigma_\iota$. On the other hand, for τ in R_0 with $\tau \leqq \sigma_\iota$ for each ι in I, we have $\tau \leqq x$ and the above method can be applied for this case to get $\tau \leqq \sigma$. This shows that σ is the GLB of the family in question. This completes the proof.

THEOREM D$'$: *Ω is Stonian if Theorem D holds for any one-sidedly simple Banach algebra over R_0.*

PROOF: We use the same notations $(\sigma_\iota; \iota \in I)$, $D(\Omega)$, x, and A as in the proof of Theorem C$'$. It must be shown that, under the validity of Theorem D, the GLB of $(\sigma_\iota; \iota \in I)$ exists in N_0. In view of Lemm 2.5, there exists a properly maximal ideal J of A. Since J is closed, it is self-adjoint and so A/J constitutes a simple B^*-algebra over R_0. By use of the validity of Theorem D, A/J is isomorphic onto R_0. Denote by φ the homomorphism of A onto R_0 via A/J and by σ the image of x according to φ.

We shall see that σ is the GLB in question. In fact, for ι in I, being $x \leqq \sigma_\iota$, it holds that $\| \sigma_\iota - x - \| \sigma_\iota - x \| \| \leqq \| \sigma_\iota - x \|$. Applying φ to the inequality, we get $\| \sigma_\iota - \sigma - \| \sigma_\iota - x \| \| \leqq \| \sigma_\iota - x \|$. This implies that $\sigma_\iota - \sigma \geqq 0$, that is, $\sigma \leqq \sigma_\iota$. The same argument leads us to show that, if τ satisfies $\tau \leqq \sigma_\iota$ for all ι in I, or $\tau \leqq x$, it holds that $\tau \leqq \sigma$. This shows that σ is the GLB in question. This completes the proof.

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