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Applications of Martin boundaries to instantaneous return Markov processes over a denumerable space

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0. Introduction.

Countable Markov processes with right continuous path functions have been discussed by many authors under somewhat different formulations. For example, the *process whose transitions are well ordered in time* was studied by J.L. Doob [**1**], the process of "*type transfini*" by P. Lévy [**9**] and the *instantaneous return process* by W. Feller [**5**].

Let X be a countable state space with the discrete topology and $x_t(w)$ $(t \geqq 0)$, a Markov process over X having the right continuous path functions. $P_x(\cdot)$ is the Markovian probability measure determining the behavior of the paths which start at x. We can define the jumping times as follows:

$$\tau_1(w) = \inf\{t\,;\, x_t(w) \neq x_0(w)\}\,, \cdots, \tau_n(w) = \tau_{n-1}(w) + \tau_1(w^+_{\tau_{n-1}})\,, \cdots,$$
$$\tau_\omega(w) = \lim_{n\to\infty} \tau_n(w)\,, \cdots, \tau_{\omega n}(w) = \tau_{\omega(n-1)}(w) + \tau_\omega(w^+_{\tau_{\omega(n-1)}})\,, \cdots,$$
$$\tau_{\omega^2}(w) = \lim_{n\to\infty} \tau_{\omega n}(w)\,,$$

where the shifted path $w^+_{\tau_{n-1}}$ is defined by $x_t(w^+_{\tau_{n-1}}) = x_{t+\tau_{n-1}}(w)$ as well as $w^+_{\tau_{\omega(n-1)}}$ by $x_t(w^+_{\tau_{\omega(n-1)}}) = x_{t+\tau_{\omega(n-1)}}(w)$. The expectation of τ_1 relative to $P_x(\cdot)$ is denoted by q_x^{-1}, and the distribution of x_{τ_1} relative to $P_x(\cdot)$ is denoted by $\Pi(x, \cdot)$.

As is usual in the theory of Markov processes, we shall define the Green operator G_α by

$$G_\alpha f(x) = E_x\Big(\int_0^\infty e^{-\alpha t} f(x_t)dt\Big)\,.$$

Let $\mathfrak{D}(\mathfrak{G})$ be the range of the set of all bounded functions by G_α. The generator $\mathfrak{G}$ is defined for $u \in \mathfrak{D}(\mathfrak{G})$ as

$$\mathfrak{G}u = (\alpha - G_\alpha^{-1})u\,.$$

It is well known that a Markov process is completely determined by specifying the generator $\mathfrak{G}$ together with its domain $\mathfrak{D}(\mathfrak{G})$. In our case $\mathfrak{G}$ can be expressed in terms of $\Pi(x, \cdot)$ and q_x as

$$\mathfrak{G}u(x) = \sum_{y\in X} q_x \Pi(x, y)u(y) - q_x u(x)$$

for $u \in \mathfrak{D}(\mathfrak{G})$. In order to determine $\mathfrak{D}(\mathfrak{G})$ we should introduce the boundary ∂X and the jumping-in probability $\tilde{\Pi}(b, \cdot)$, and establish the boundary condition that determines $\mathfrak{D}(\mathfrak{G})$.

J. L. Doob [**1**] and W. Feller [**5**] discussed this problem in case ∂X consists of a finite number of points and P. Levy [**9**], the general case in an intuitive way. The aim of this paper is to discuss it rigorously in full generality, making use of the theory of Martin boundary introduced by T. Watanabe [**11, 12**], J. L. Doob [**3**] and G. A. Hunt [**6**].

It should be noted that the boundary condition for the reflecting barrier is not contained in our discussion since any path of our process describes a right continuous curve *in* X by our definition of the Markov process.

The outline of this paper is as follows. In § 1 we shall mention preliminary known results in the theory of Markov processes with countable number of states. In § 2 we shall introduce superharmonic functions and Martin boundaries for Markov processes following [**11, 12**]. § 3 is devoted to the classification of boundary points. In § 4 we discuss the representation of bounded $x_\alpha^0(t)$-harmonic functions by means of α-order harmonic measures, while the Martin representation of a wider class of $x_\alpha^0(t)$-harmonic functions which are not necessarily bounded, are obtained in § 3. In § 5 we shall determine the boundary condition for the first instantaneous return process, i. e. the process satisfying the condition $P_x(\tau_{\omega^2} \geqq \sigma_\infty) = 1$, where σ_∞ is the killing time. To discuss the higher order instantaneous return process such that $P_x(\tau_{\omega^2} \geqq \sigma_\infty) < 1$, we should introduce the higher order boundaries and corresponding boundary conditions besides the above boundary conditions. This problem will be discussed in § 6. In § 7, we shall construct the paths of the Markov process corresponding to the given $\mathfrak{G}$ (including the boundary conditions). In § 8, we shall give an example of a higher order instantaneous return process by modifying the dyadic branching scheme, together with some other examples. In Appendix, we shall discuss the instantaneous return process satisfying $P_x(\tau_{\omega(k+1)} = \infty) = 1$ and show that the instantaneous return processes treated in [**1**] and [**5**] satisfy $P_x(\tau_{\omega^2} = \infty) = 1$.

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1. Definitions and preliminary results of Markov processes.

Let X be a countable state space with discrete topology and ∞ an extra point to be added to X as an isolated point. $X \cup \{\infty\}$ is denoted by X^* and the set of all subsets of X^* by $\mathfrak{B}_{X^*}$. Let T be a continuous parameter set $[0, +\infty]$. Any function from T into X^* is denoted by w and its value at $t \in T$ by $x(t, w)$ or $x_t(w)$. The set of all the w's which satisfy the following conditions (W.1)~(W.3) is denoted by W and each element of W is called a *path-function.*

(W.1) $$x_{+\infty}(w) = \infty\,.$$

(W.2) There exists a mapping $\sigma_\infty(w)$ from W into T and:

$$x_t(w) = \infty \qquad \text{for} \quad t \geqq \sigma_\infty(w) \qquad \text{and,}$$
$$x_t(w) \in X \qquad \text{for} \quad t < \sigma_\infty(w)\,.$$

(W.3) $x_t(w)$ is right continuous with respect to t.
We shall denote by $\mathfrak{B}_W$ the Borel field generated by the sets $\{w\,; x_t(w) \in E\}$, where E runs over $\mathfrak{B}_{X^*}$ and t over T. For any $w \in W$ and any *random time* $\sigma(w)$, i.e. a measurable function from W into T, we shall define the *stopped path* w_σ^-, and the *shifted path* w_σ^+ as follows:

(1.1) $$x_t(w_\sigma^-) = x_{\min(t,\sigma)}(w) \quad (t < +\infty) \text{ and } = \infty \quad (t = +\infty)\,,$$
$$x_t(w_\sigma^+) = x_{\sigma+t}(w)\,.$$

We can easily show that both w_σ^- and w_σ^+ belong to W and that $\varphi_\sigma(w) = w_\sigma^-$ and $\psi_\sigma(w) = w_\sigma^+$ are measurable mappings from $(W, \mathfrak{B}_W)$ into itself. Therefore $(\varphi_\sigma^-)\mathfrak{B}_W = \mathfrak{B}_\sigma$ is the Borel field of $\mathfrak{B}_W$. Especially $\mathfrak{B}_t$ for the constant random time t coincides with the Borel field generated by the sets $\{w\,; x_s(w) \in E\}$ for $s \leqq t$. Further $\bigcap_{n=1}^{\infty} \mathfrak{B}_{\sigma+\frac{1}{n}}$ is denoted by $\mathfrak{B}_{\sigma+}$. Finally a random time σ is a *Markov time* if

(1.2) $$\{w\,; \sigma(w) < t\} \in \mathfrak{B}_t \qquad \text{for any} \quad t \in T\,.$$

Lemma 1.1. (1) *Let $\{\sigma_n\,; n = 1, 2, \cdots\}$ be a sequence of Markov times. If $\sigma_n \uparrow (\downarrow)\, \sigma$, σ is also a Markov time.*
(2) *If $\sigma(w)$ and $\tau(w)$ are Markov times, $\theta(w) = \sigma(w) + \tau(w_\sigma^+)$ is a Markov time.*
(3) *If $\sigma(w)$ is a Markov time, $\sigma(w)$ is a $\mathfrak{B}_{\sigma+}$ measurable function.*

The proof will be given in K. Itô and H. P. McKean [8] and omitted here.

The *jumping times* are defined as follows:

$$\tau_1(w) = \inf\{t\,; x_t(w) \neq x_0(w)\} \qquad \text{(the } \textit{first jumping time}\text{)}\,,$$
$$\tau_n(w) = \tau_{n-1}(w) + \tau_1(w^+_{\tau_{n-1}}) \qquad \text{(the } \textit{n-th jumping time}\text{)}\,,$$
$$\tau_\omega(w) = \lim_{n\to\infty} \tau_n(w)\,, \qquad \tau_{\omega+n}(w) = \tau_\omega(w) + \tau_n(w^+)\,, \cdots\,,$$

(1.3) $$\tau_{\omega n} = \tau_{\omega(n-1)} + \tau_\omega(w^+_{\tau_{\omega(n-1)}})\,, \cdots\,,$$

$$\tau_{\omega n+m}=\tau_{\omega n}+\tau_m(w^+_{\tau_{\omega n}}),\cdots, \qquad \tau_{\omega^2}=\lim_{n\to\infty}\tau_{\omega n},$$
$$\tau_{\omega^2 n}=\tau_{\omega^2(n-1)}+\tau_{\omega^2}(w^+_{\tau_{\omega^2(n-1)}}),\cdots,$$
$$\tau_{\omega^2 n+\omega m+l}=\tau_{\omega^2 n}+\tau_{\omega m+l}(w^+_{\tau_{\omega^2 n}}),\cdots, \qquad \tau_{\omega^3}=\lim_{n\to\infty}\tau_{\omega^2 n},$$

and so on. The first jumping time is a Markov time, because, using (W.2) and (W.3) we have

$$\{w\,;\tau_1\geqq t\}=\bigcup_{x\in X^*}\{w\,;x_r(w)=x \text{ for any rational } r<t\}\in\mathfrak{B}_t.$$

Therefore, applying Lemma 1.1, all jumping times are Markov times.

A *countable Markov process* is a system $\boldsymbol{M}\equiv(X^*, W, \mathfrak{B}_W, P_x\,; x\in X^*)$ satisfying the following conditions.

(P.1) For any fixed x, $P_x(\,\cdot\,)$ is a probability measure over $(W, \mathfrak{B}_W)$.

(P.2) $P_x(w\,; x_0(w)=x)=1$ for any $x\in X^*$.

(P.3) (Markov Property). For any $x\in X^*$, $t\in T$ and $B\in\mathfrak{B}_W$,

$$P_x(w\,; w^+_t\in B/\mathfrak{B}_t)=P_{x_t}(B) \quad \text{with } P_x\text{-probability 1.} \tag{1.4}$$

The Markov process $\boldsymbol{M}$ is also denoted briefly by $x(t)$.

Let $\mathfrak{B}(X)$ be the family of all bounded functions on X^* such that $f(\infty)=0$. We can consider $\mathfrak{B}(X)$ as a Banach space by introducing the norm $\|f\|=\sup_{x\in X}|f(x)|$. *Semi-group* H_t and *Green operator* G_α $(\alpha>0)$ are bounded linear operators from $\mathfrak{B}(X)$ into itself defined by

$$H_tf(x)=E_x(f(x_t)). \tag{1.5}$$

$$G_\alpha f(x)=E_x\left(\int_0^\infty e^{-\alpha t}f(x_t)dt\right)=\int_0^\infty e^{-\alpha t}H_tf(x)dt. \tag{1.6}$$

H_t and G_α have the following properties.

(H.1) $H_tf\geqq 0$ for any $f\geqq 0$.

(H.2) $H_t1\leqq 1$.

(H.3) $H_{t+s}f(x)=H_tH_sf(x)$ (Semi-Group Property).

(G_α.1) $G_\alpha f\geqq 0$ for any $f\geqq 0$.

(G_α.2) $\alpha G_\alpha 1\leqq 1$.

(G_α.3) $(\alpha-\beta)G_\alpha G_\beta f(x)+G_\alpha f(x)-G_\beta f(x)=0$ (Resolvent Equation).

From (G_α.3), we can see that the range of $\mathfrak{B}(X)$ by the Green operator G_α is independent of α. Its range is denoted by $\mathfrak{R}$.

Since the topology of X is discrete, we may regard the Green operator G_α as mapping from any continuous function to another continuous function. Therefore noting (W.3), we have the following *strong Markov property*.

(P.4) For any Markov time σ, $x\in X$ and $B\in\mathfrak{B}_W$, we have

$$P_x(w\,; w^+_\sigma\in B/\mathfrak{B}_{\sigma+})=P_{x_\sigma}(B) \quad \text{with } P_x\text{-probability 1.} \tag{1.7}$$

We shall define the quantity q_x and the kernel $\Pi(x,y)$ $(x,y\in X^*)$ as follows:

$$q_x^{-1} = E_x(\tau_1), \tag{1.8}$$

$$\begin{aligned} \Pi(x, y) &= P_x(x_{\tau_1} = y) \qquad &\text{if} \quad q_x \neq 0, \\ &= \delta_{x,y} \qquad &\text{if} \quad q_x = 0. \end{aligned} \tag{1.9}$$

Then from the right continuity of path functions, we have

$$0 \leqq q_x < \infty \qquad \text{if} \quad x \in X, \qquad q_\infty = 0. \tag{1.10}$$

$$\begin{aligned} &\Pi(x, y) \geqq 0, \qquad \sum_{y \in X^*} \Pi(x, y) = 1, \\ &\Pi(x, x) = 0 \qquad \text{if} \quad q_x \neq 0. \end{aligned} \tag{1.11}$$

LEMMA 1.2. *x_{τ_1} and τ_1 are independent with respect to P_x-probability.*

PROOF. Let A be a subset of X^*. We have

$$\begin{aligned} P_x(\tau_1 > t, x_{\tau_1} \in A) &= P_x(\tau_1 > t, x_{\tau_1}(w_t^+) \in A) \\ &= E_x(P_{x_t}(x_{\tau_1} \in A); \tau_1 > t) = P_x(x_{\tau_1} \in A) \cdot P_x(\tau_1 > t). \end{aligned} \tag{1.12}$$

The kernel $\Pi_\alpha(x, y)$ is defined by

$$\Pi_\alpha(x, y) = E_x(e^{-\alpha\tau_1}; x_{\tau_1} = y). \tag{1.13}$$

Using Lemma 1.2 and the well known relation $P_x(\tau_1 \geqq t) = e^{-q_x t}$, we have

$$\begin{aligned} \Pi_\alpha(x, y) &= E_x(e^{-\alpha\tau_1}) \cdot P_x(x_{\tau_1} = y) \\ &= \left(\int_0^\infty e^{-\alpha t} q_x e^{-q_x t} dt\right) \Pi(x, y) = \frac{q_x}{\alpha + q_x} \Pi(x, y). \end{aligned} \tag{1.14}$$

By the strong Markov property, the function $G_\alpha f$ is changed into

$$\begin{aligned} G_\alpha f(x) &= E_x\left(\int_0^{\tau_1} e^{-\alpha t} f(x_t) dt\right) + E_x\left(e^{-\alpha t_1} E_{x_{\tau_1}}\left(\int_0^\infty e^{-\alpha t} f(x_t) dt\right)\right) \\ &= \frac{1}{\alpha + q_x} f(x) + \sum_{y \in X} \Pi_\alpha(x, y) G_\alpha f(y). \end{aligned} \tag{1.15}$$

Writing $q_x \sum_{y \in X} \Pi(x, y) G_\alpha f(y) = q\Pi G_\alpha f(x)$ and using (1.14) and (1.15), we have

$$\alpha G_\alpha f - f = q\Pi G_\alpha f - q G_\alpha f.$$

Put $G_\alpha f = 0$ in (1.15) then we get $f = 0$. Therefore for $f \in \mathfrak{R}$, inverse operator G_α^{-1} can be defined. We shall define the *generator* $\mathfrak{G}$ as follows:

$$\mathfrak{G} f \equiv \alpha f - G_\alpha^{-1} f = q\Pi f - qf. \tag{1.16}$$

The *domain of generator* $\mathfrak{G}$ is denoted by $\mathfrak{D}(\mathfrak{G})$, i. e. $\mathfrak{D}(\mathfrak{G}) = \mathfrak{R}$. Then a Markov process is uniquely determined by the generator $\mathfrak{G}$ (see K. Itô [7]). The operator $q(\Pi - I)$ is called *Dynkin generator* and denoted by $\tilde{\mathfrak{G}}$.

A Markov process $x(t)$ is called a *k-th instantaneous return process* if it satisfies

$$P_x(\sigma_\infty(w) \leqq \tau_{\omega(k+1)}(w)) = 1 \tag{1.17}$$

for every $x \in X$. Particularly, when $k = 0$, $x(t)$ is called a *minimal process.*

Let $\boldsymbol{M}=(X^*, W, \mathfrak{B}_W, P_x\,;\, x\in X^*)$ be a Markov process and let $x^k(w)$ be a mapping from W into itself defined by

$$(1.18)\qquad \begin{aligned} x^k(t, w) &= x(t, w) &&\text{if } t<\tau_{\omega(k+1)}\,,\\ &=\infty &&\text{if } t\geqq\tau_{\omega(k+1)}\,. \end{aligned}$$

Then a stochastic process $\boldsymbol{M}^k=(X^*, W, \mathfrak{B}_W, P_x^k\,;\, x\in X^*)$ (or briefly $x^k(t)$) defined by

$$(1.19)\qquad P_x^k(B)=P_x(w\,;\, x^k(w)\in B)\,,\qquad B\in\mathfrak{B}_W$$

becomes a k-th instantaneous return process. It is clear that the *pair* $\{q, \Pi\}$ of the process $x(t)$ and that of the process $x^k(t)$, i. e. the quantities $\{q_x, \Pi(x, y)\,;\, x, y\in X^*\}$ for the processes $x(t)$ and $x^k(t)$ defined by (1.8) and (1.9), coincide completely. Moreover the Green function $G_\alpha^k f$ for the process $x^k(t)$ becomes

$$(1.20)\qquad G_\alpha^k f(x)=E_x^k\Big(\int_0^\infty e^{-\alpha t}f(x_t)dt\Big)=E_x\Big(\int_0^{\tau_{\omega(k+1)}} e^{-\alpha t}f(x_t)dt\Big)\,.$$

So we shall call $x^k(t)$ the *k-th instantaneous return process induced by $x(t)$.*

Let $x(t)$ be a Markov process with the pair $\{q, \Pi\}$ and let $x_\alpha^0(t)$ be a minimal process with the pair $\{\alpha+q, \Pi_\alpha\}$ where Π_α is defined by (1.13). Such $x_\alpha^0(t)$ is called the *α-order minimal process.* The Dynkin generator $\tilde{\mathfrak{G}}_\alpha$ of $x_\alpha^0(t)$ becomes

$$\tilde{\mathfrak{G}}_\alpha=(q+\alpha)(\Pi_\alpha-I)=q(\Pi-I)-\alpha=\tilde{\mathfrak{G}}-\alpha\,.$$

The *first passage time for* $A\subset X$ (relative to the minimal process $x^0(t)$ induced by $x(t)$) is defined by

$$(1.21)\qquad \begin{aligned} \sigma_A(w) &= \inf\{t\,;\, x_t^0(w)\in A\} &&\text{if } x_t^0(w)\in A \text{ for some } t\geqq 0\,,\\ &=\infty \quad\text{otherwise.} \end{aligned}$$

Then $\sigma_A(w)$ is also a Markov time, because

$$\begin{aligned} \{\sigma_A\geqq t\} &= \{w\,;\, x_r^0(w)\notin A \text{ for every rational } r<t\}\\ &=\bigcap_{r<t}[\{w\,;\, x_r(w)\notin A, r<\tau_\omega\}\cup\{r>\tau_\omega\}]\in\mathfrak{B}_t\,. \end{aligned}$$

A state x is called *recurrent* if it satisfies

$$(1.22)\qquad P_x(w\,;\, \sigma_{\{x\}}(w_{\tau_1}^+)<+\infty\,|\,\tau_1(w)<\infty)=1\,,$$

and a subset R of X an *indecomposable recurrent set* if R contains a recurrent state x satisfying

$$(1.23)\qquad \begin{aligned} &P_x(\sigma_y<\infty)>0 \qquad\text{for any } y\in R \text{ and,}\\ &P_x(w\,;\, \sigma_{R^c}<+\infty)=0\,. \end{aligned}$$

We know that X is uniquely decomposed into $X=\bigcup_i R_i+N$, where each R_i is an indecomposable recurrent set and N is the non recurrent part of X.

2. $x^0(t)$-superharmonic functions and Martin boundaries.

Since our arguments in the sequel are essentially based on the Martin boundary theory, we shall sketch it following [**11**, **12**].

Let $\mathfrak{F}(X)$ be the family of finite valued functions on X^* taking the value 0 at ∞. If $u \in \mathfrak{F}(X)$ satisfies

$$\tilde{\mathfrak{G}}u \leqq 0 \qquad (\tilde{\mathfrak{G}}u \geqq 0), \tag{2.1}$$

u is called an $x^0(t)$-*superharmonic* ($x^0(t)$-*subharmonic*) function. Especially if u is both $x^0(t)$-superharmonic and $x^0(t)$-subharmonic, u is called an $x^0(t)$-*harmonic function.* A non-negative $x^0(t)$-harmonic function u is called *minimal* if any nonnegative $x^0(t)$-harmonic function v which does not exceed u is a constant multiple of u. A function $u \in \mathfrak{F}(X)$ is called the $x^0(t)$-*potential* of f, if u can be written in the form

$$u = G^0 f = E_x^0\Big(\int_0^\infty f(x_t)dt\Big). \tag{2.2}$$

THEOREM 2.1.[1] (Analogue of Riesz decomposition theorem). *An $x^0(t)$-superharmonic function u is decomposed by means of $x^0(t)$-potential $\sum_{y \in N} G^0(x, y)[-\tilde{\mathfrak{G}}u(y)]$ and $x^0(t)$-harmonic function $\lim_{n\to\infty} \Pi^n u$ into the form*

$$u(x) = \sum_{y\in N} G^0(x, y)[-\tilde{\mathfrak{G}}u(y)] + \lim_{n\to\infty} \Pi^n u, \tag{2.3}$$

if and only if there exists an $x^0(t)$-harmonic function which does not exceed u.

THEOREM 2.2.[2] (1) *If $u \in \mathfrak{F}(X)$ is the $x^0(t)$-potential of f, the set $\{x; f(x) \neq 0\}$ is contained in the transient part N.* (2) *If u is the $x^0(t)$-potential of positive function f, we have*

$$\lim_{t \uparrow \tau_\omega} u(x_t) = 0 \tag{2.4}$$

with P_x—1.[3]

From now on, we shall consider the Markov process satisfying the following condition:

(P.5)[4] There exists at least one state c such that $P_c(\sigma_{\{y\}} < +\infty) > 0$ for any $y \in X$. Such state c is called *center*.

In the sequel we shall fix a center e. Define

$$K(c, x, y) = \frac{P_x(\sigma_{\{y\}} < +\infty)}{P_c(\sigma_{\{y\}} < +\infty)}, \tag{2.5}$$

1) See [**12**, Theorem 2.8].

2) See [**3**, Theorem 3.1], or [**12**, Theorem 2.6].

3) By this notation, we mean P_x-probability 1 for every $x \in X$.

4) We assume this condition only for the simplicity. According to [**6**], we can establish the Martin boundary theory by introducing a reference measure if the process does not satisfy this condition.

then $K(c, x, y)$ becomes an $x^0(t)$-superharmonic function of x for any fixed y, and $K(c, x, y)=K(c, x, y')$ (for every x) induces that $y=y'$ or that both y and y' belong to the same indecomposable recurrent set. Let us denote by $\hat{X}$, the union of N and r_i from R_i. We can metrize $\hat{X}$ by

$$\rho(y, z)=\int_X \frac{|K(c, x, y)-K(c, x, z)|}{1+|K(c, x, y)-K(c, x, z)|} m(dx), \tag{2.6}$$

where m is a totally finite measure which is positive on any state $x \in X$. The completion of $\hat{X}$ by ρ-metric is denoted by M, and called the *canonical Martin space*. Moreover $\partial X \equiv M-N$ is called the *Martin boundary*, and especially $\{\cup r_i\}$ $(\subset \partial X)$, the *degenerate boundary points*. The element of ∂X is denoted by b. Let b be a non-degenerate boundary point and $\{y_n\}$ a sequence converging to b in ρ-metric, then $\lim_{n\to\infty} K(c, x, y_n)$ exists and is denoted by $K(c, x, b)$. The set of all b such that $K(c, x, b)$ is minimal harmonic is called *minimal part* of ∂X and denoted by $(\partial X)_1$. $(\partial X)_1$ is a Borel set of X. The natural mapping from X onto $\hat{X}$ is denoted by θ.

Let D be a closed subset of M and $\mathfrak{U}(D)$ be the family of all open sets containing D. The *réduite* $u_D(x)$ or $u(x, D)$ of a nonnegative $x^0(t)$-superharmonic function u is

$$u_D(x)=\inf_{G\in\mathfrak{U}(D)} E_x(u(x_{\sigma_{[G]}}))=\lim_{n\to\infty} E_x(u(x_{\sigma_{[G_n]}})), \tag{2.7}$$

where $[G]=\theta^{-1}(G \cap \hat{X})$, and $\{G_n\}$ is any sequence in $\mathfrak{U}(D)$ such that $G_n\downarrow$ and $\bar{G}_n \downarrow D$. For any Borel set B of ∂X, we can define the réduite $u_B(x)$ or $u(x, B)$, which is the extension of the réduite defined above.

THEOREM 2.3. (1)[5] *Let u be a nonnegative $x^0(t)$-superharmonic function, and B a Borel set of ∂X, then $u(x, B)$ can be uniquely represented by means of a Radon measure μ over ∂X whose total mass is concentrated in $(\partial X)_1$, in the form*

$$u(x, B)=\int_B K(c, x, b)\mu(db). \tag{2.8}$$

The measure μ is independent of B and is characterized by

$$\mu(B)=u(c, B). \tag{2.9}$$

(2)[6] *If u is an $x^0(t)$-harmonic function satisfying $\lim_{n\to\infty} \Pi^n|u|(c)<+\infty$, then*

$$u(x)=\int_{(\partial X)_1} K(c, x, b)\mu(db). \tag{2.10}$$

The réduite $\chi_X(x, B)$ of the indicator function of the state space X is denoted by $h(x, B)$ and is called the *harmonic measure to the Martin boundary*.

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5) [**12**, Lemma 4.8 and Main Theorem].
6) [**12**, Theorem 4.3].

THEOREM 2.4. (1)[7] *$\lim_{n\to\infty} \theta(x_{\tau_n})$ exists with P_x-1, and is a random variable over $\partial X\cup\{\infty\}$. Moreover, if we set $x_{\tau_{\omega}-}(w)=\lim_{n\to\infty}\theta(x_{\tau_n}(w))$, we have*

$$h(x, B)=P_x(x_{\tau_{\omega}-}(w)\in B). \tag{2.11}$$

(2) *If u is a bounded $x^0(t)$-harmonic function, there exists a bounded measurable function $\tilde{u}(b)$ over $\partial X\cup\{\infty\}$ such that $\tilde{u}(\infty)=0$ and*

$$\begin{aligned} u(x) &= \int_{(\partial X)_1} K(c, x, b)\tilde{u}(b)h(c, db)=\int_{(\partial X)_1}\tilde{u}(b)h(x, db) \\ &= E_x(\tilde{u}(x_{\tau_{\omega}-})), \end{aligned} \tag{2.12}$$

where $\tilde{u}$ is uniquely determined without $h(c, \cdot)$-measure 0. $\tilde{u}$ is called the boundary function of u.

REMARK. Since $x_{\tau_{\omega}-}(w)$ is measurable in the smallest Borel field containing $\bigcup_i \mathfrak{B}_{\tau_n+}$, applying martingale theory to (2.12) we have with P_x-1

$$\lim_{n\to\infty} u(x_{\tau_n}(w))=\tilde{u}(x_{\tau_{\omega}-}(w)). \tag{2.13}$$

3. $x_\alpha^0(t)$-harmonic functions and the classification of boundary points.

In this section, we shall establish some relations between $x^0(t)$-harmonic functions and $x_\alpha^0(t)$-harmonic functions, which are applied to classify the boundary points.

First we shall define several families of functions:

$\mathfrak{H}=\{u\,;u$ is $x^0(t)$-harmonic and satisfies $\lim_{n\to\infty}\Pi^n|u|(c)<\infty\}$,

$\mathfrak{H}^+=\{u\,;u$ is nonnegative and $x^0(t)$-harmonic$\}$,

$\mathfrak{H}_\alpha=\{u_\alpha\,;u_\alpha$ is $x_\alpha^0(t)$-harmonic and satisfies $\lim_{n\to\infty}\Pi_\alpha^n|u|(c)<\infty\}$,

$\mathfrak{H}_\alpha^+=\{u_\alpha\,;u_\alpha$ is a nonnegative $x_\alpha^0(t)$-harmonic function of $\mathfrak{H}_\alpha\}$.

If u is an element of $\mathfrak{H}^+$, then

$$\tilde{\mathfrak{G}}_\alpha u=(\tilde{\mathfrak{G}}-\alpha)u=-\alpha u\leqq 0 \tag{3.1}$$

holds, which shows that u is an $x_\alpha^0(t)$-superharmonic function. Hence by Theorem 2.1, we have

$$\begin{aligned} u(x) &= \sum_{y\in N} G_\alpha^0(x, y)[-\tilde{\mathfrak{G}}_\alpha u(y)]+\lim_{n\to\infty}\Pi_\alpha^n u(x) \\ &= \alpha G_\alpha^0 u(x)+\lim_{n\to\infty}\Pi_\alpha^n u(x). \end{aligned} \tag{3.2}$$

Setting

$$u_\alpha(x)=\lim_{n\to\infty}\Pi_\alpha^n u(x), \tag{3.3}$$

the above formula (3.2) can be rewritten as

7) [**11**].

(3.4) $$u(x)=\alpha G_\alpha^0 u(x)+u_\alpha(x)\,.$$

Since any element of $\mathfrak{H}$ can be expressed by the difference of nonnegative $x^0(t)$-harmonic functions,[8] the formula (3.4) holds for any u of $\mathfrak{H}$.

LEMMA 3.1 (W. Feller). *The functions* u_α *and* u_β *defined by* (3.3) *satisfy*

(3.5) $$u_\alpha-u_\beta=(\beta-\alpha)G_\alpha^0 u_\beta=(\beta-\alpha)G_\beta^0 u_\alpha\,.$$

PROOF. Since $G_\alpha^0 u$ coincides with the Green function of minimal process $x^0(t)$ induced by $x(t)$, $G_\alpha^0 u$ satisfies the resolvent equation. Hence using (3.4), we get

(3.6)
$$\begin{aligned}(\beta-\alpha)G_\alpha^0 u_\beta&=(\beta-\alpha)G_\alpha^0 u-\beta(\beta-\alpha)G_\alpha^0 G_\beta^0 u\\&=(\beta-\alpha)G_\alpha^0 u-\beta(G_\alpha^0 u-G_\beta^0 u)\\&=\beta G_\beta^0 u-\alpha G_\alpha^0 u=u_\alpha-u_\beta\,.\end{aligned}$$

The formula $u_\alpha-u_\beta=(\beta-\alpha)G_\beta^0 u_\alpha$ can be also derived by the same method.

Conversely if u_α is an element of $\mathfrak{H}_\alpha$, we have

(3.7) $$\tilde{\mathfrak{G}}u_\alpha=\alpha u_\alpha\,.$$

Hence if u_α is nonnegative, $-u_\alpha$ is $x(t)$-superharmonic. But generally $-\lim_{n\to\infty}\Pi^n u_\alpha$ may be $-\infty$. So we shall restrict $\mathfrak{H}_\alpha$ to $\bar{\mathfrak{H}}_\alpha$ such that $\bar{\mathfrak{H}}_\alpha$ is the family of functions u_α of $\mathfrak{H}_\alpha$ satisfying $\overline{\lim_{n\to\infty}}\,\Pi^n|u_\alpha|<+\infty$. If u_α is a nonnegative function of $\bar{\mathfrak{H}}_\alpha$, $-u_\alpha$ is Riesz decomposable and satisfies

(3.8)
$$\begin{aligned}-u_\alpha(x)&=\sum_{y\in N}G^0(x,y)[-\tilde{\mathfrak{G}}(-u_\alpha(y)]+\lim_{n\to\infty}\Pi^n[-u_\alpha(x)]\\&=\alpha\sum_{y\in N}G^0(x,y)u_\alpha(y)-\lim_{n\to\infty}\Pi^n u_\alpha(x)\,.\end{aligned}$$

Writing

(3.9) $$u_0(x)=\lim_{n\to\infty}\Pi^n u_\alpha(x)\,,$$

we get

(3.10) $$u_0(x)=\alpha G^0 u_\alpha(x)+u_\alpha(x)\,.$$

The above formula also holds for any $u_\alpha\in\bar{\mathfrak{H}}_\alpha$.

REMARK. (i) In the case that $u\in\mathfrak{H}\cap\mathfrak{B}(x)$ and $u_\alpha\in\mathfrak{H}_\alpha\cap\mathfrak{B}(x)$, the formulas (3.4), (3.10) and Lemma 3.1 have been obtained in [**5**] by somewhat different method.

(ii) We can easily show that $\bar{\mathfrak{H}}_\alpha$ coincides with the range of $\lim_{n\to\infty}\Pi_\alpha^n u$ where $u\in\mathfrak{H}$.

DEFINITION 3.1. The mapping from $\bar{\mathfrak{H}}_\alpha$ into $\mathfrak{H}$ defined by (3.9) is called the *canonical mapping*.

Now, if $b\in(\partial X)_1$, $K(c,x,b)$ belongs to $\mathfrak{H}^+$. We shall denote $\lim_{n\to\infty}\sum_{y\in X}\Pi_\alpha^n(x,y)$

8) See [**12**].

$\times K(c, y, b)$ by $K_\alpha(c, x, b)$. Since $K_\alpha(c, x, b)$ and $K_\beta(c, x, b)$ statisfy (3.5), $K_\alpha(c, x, b) \equiv 0$ leads to $K_\beta(c, x, b) \equiv 0$, and vice versa, i.e. the set b such that $K_\alpha(c, x, b) \equiv 0$ is independent of α.

DEFINITION 3.2.[9] If $b \in (\partial X)_1$ and $K_\alpha(c, x, b) \not\equiv 0$, b is called the *exit boundary point*. If $b \in (\partial X)_1$ and $K_\alpha(c, x, b) \equiv 0$ or $b \in (\partial X) - (\partial X)_1$, b is called the *passive boundary point*. The set of all exist boundary points is denoted by $(\partial X)_e$ and that of passive ones, by $(\partial X)_p$.

LEMMA 3.2. *The set $(\partial X)_e$ is measurable.*

PROOF. Since $K(c, x, b)$ is (x, b)-measurable, $K_\alpha(c, x, b)$ is also (x, b)-measurable from the definition of K_α. Therefore the set

$$(\partial X)_e = \bigcup_{x \in X} \{b\,;\, K_\alpha(c, x, b) > 0\}$$

is measurable.

LEMMA 3.3. *If $b \in (\partial X)_e$, the canonical image of $K_\alpha(c, x, b)$ coincides with $K(c, x, b)$.*

PROOF. Evidently $K(c, x, b) \geqq K_\alpha(c, x, b)$ holds. Operating Π^n and letting $n \to \infty$, we get

$$\begin{aligned} \text{(3.11)} \qquad & \lim_{n\to\infty} \Pi^n K_\alpha(c, x, b) (\equiv \lim_{n\to\infty} \sum_{y\in X} \Pi^n(x, y) K_\alpha(c, y, b)) \\ & \leqq \lim_{n\to\infty} \Pi^n K(c, x, b) = K(c, x, b). \end{aligned}$$

Since $K(c, x, b)$ is minimal, there exists a constant k $(0 < k \leqq 1)$ such that $\lim_{n\to\infty} \Pi^n K_\alpha(c, x, b) = kK(c, x, b)$. Therefore $K_\alpha(c, x, b) \leqq kK(c, x, b)$. Operating Π_α^n and letting $n \to \infty$, we get $K_\alpha(c, x, b) \leqq kK_\alpha(c, x, b)$, which shows $k \geqq 1$. Therefore k must be one.

Now, the set of all elements $u \in \mathfrak{H}$ such that $u(x) = u(x, (\partial X)_e)$ is denoted by $\bar{\mathfrak{H}}$ and the set of all elements $u \in \mathfrak{H}$ such that $u(x) = u(x, (\partial X)_p)$, by $\mathfrak{H}^{(0)}$. Let u be a function of $\mathfrak{H}$, and (2.10) be its representation formula. Then operating Π_α^n to (2.10) and letting $n \to \infty$, $u_\alpha(x)$ defined by (3.3) becomes

$$\text{(3.12)} \qquad u_\alpha(x) = \int_{(\partial X)_1} K_\alpha(c, x, b) \mu(db) = \int_{(\partial X)_e} K_\alpha(c, x, b) \mu(db).$$

By the similar argument, applying Lemma 3.3, $u_0(x)$ defined by (3.9) becomes

$$\text{(3.13)} \qquad u_0(x) = \int_{(\partial X)_e} K(c, x, b) \mu(db).$$

9) Our definition of the exit boundary points is that in the sense of [**5**] and is different from the definitions in [**3**] and [**6**]. Their exit boundary points in [**3**] and [**6**] are nothing but our Martin boundary points (∂X). Our classification is significant because our Markov process is time-continuous. The probabilistic meaning of exit and passive boundaries is shown in Theorem 3.2. There exists a more probabilistic definition of exit and passive boundaries, which is equivalent to ours. Such definition will appear some time or other.

THEOREM 3.1. *Let $u \in \mathfrak{H}$ and, u_α and u_0 be the functions defined by* (3.3) *and* (3.9).

(1) *$\bar{\mathfrak{H}}$ coincides with the range of the canonical image of $\bar{\mathfrak{H}}_\alpha$.*

(2) *u belongs to $\bar{\mathfrak{H}}$ if and only if $u = u_0$.*

(3) *u belongs to $\mathfrak{H}^{(0)}$ if and only if $u_0 = 0$.*

(4) $\lim_{n\to\infty} \Pi^n_\alpha u_0 = u_\alpha$.

PROOF. Since any function belonging to the range of the canonical image is given by (3.13), (i) will be clear. If $u(x)$ is a function of $\bar{\mathfrak{H}}$, it is written in

$$u(x) = u(x, (\partial X)_e) = \int_{(\partial X)_e} K(c, x, b)\mu(db)\,. \tag{3.14}$$

Hence $u = u_0$, and vice versa. Further, if $u(x)$ belongs to $\mathfrak{H}^{(0)}$, it is written in

$$u(x) = u(x, (\partial X)_p) = \int_{(\partial X)_p} K(c, x, b)\mu(db)\,. \tag{3.15}$$

Therefore, by (3.13) u_0 becomes

$$u_0(x) = \int_{(\partial X)_p \cap (\partial X)_e} K(c, x, b)\mu(db) = 0\,. \tag{3.16}$$

Conversely if $u_0 = 0$, we have $\mu((\partial X)_e) = 0$ and u satisfies (3.15). To prove (4), operate Π^n_α to (3.13) and let $n \to \infty$, and we get $\lim_{n\to\infty} \Pi^n_\alpha u_0 = u_\alpha$ immediately.

THEOREM 3.2. (1) $P_x(\tau_\omega < \infty \mid x_{\tau_\omega -} \in (\partial X)_e) = 1$.

(2) $P_x(\tau_\omega = \infty \mid x_{\tau_\omega -} \in (\partial X)_p) = 1$.

PROOF. If we put $u(x) = h(x, (\partial X)_e) = P_x(x_{\tau_\omega -} \in (\partial X)_e)$, it belongs to $\bar{\mathfrak{H}}$. If we operate Π^n_α to $u(x)$, we have

$$\Pi^n_\alpha u(x) = E_x(e^{-\alpha\tau_n} P_{x_{\tau_n}}(x_{\tau_\omega -} \in (\partial X)_e)) = E_x(e^{-\alpha\tau_n}\,;\, x_{\tau_\omega -} \in (\partial X)_e)\,. \tag{3.17}$$

Therefore

$$u_\alpha(x) = \lim_{n\to\infty} \Pi^n_\alpha u(x) = E_x(e^{-\alpha\tau_\omega}\,;\, x_{\tau_\omega -}(w) \in (\partial X)_e)\,. \tag{3.18}$$

Conversely operating Π^n to the above formula, we have

$$\begin{aligned} \Pi^n u_\alpha(x) &= E_x(E_{x_{\tau_n}}(e^{-\alpha\tau_\omega}\,;\, \tau_\omega < \infty, x_{\tau_\omega -} \in (\partial X)_e)) \\ &= E_x(e^{-\alpha\tau_\omega(w^+_{\tau_n})}\,;\, \tau_\omega(w^+_{\tau_n}) < \infty, x_{\tau_\omega -}(w^+_{\tau_n}) \in (\partial X)_e) \\ &= E_x(e^{-\alpha(\tau_\omega - \tau_n)}\,;\, \tau_\omega < \infty, x_{\tau_\omega -} \in (\partial X)_e)\,. \end{aligned} \tag{3.19}$$

Letting $n \to \infty$, the canonical image of $u_\alpha(x)$ becomes

$$u_0(x) = P_x(\tau_\omega < \infty \mid x_{\tau_\omega -} \in (\partial X)_e)\,. \tag{3.20}$$

By Theorem 3.1 (2) $u = u_0$ holds, which shows (1). Next put $u(x) = h(x, (\partial X)_p)$, and it belongs to $\mathfrak{H}^{(0)}$. Hence by the similar calculation as (3.17),

$$\lim_{n\to\infty} \Pi^n_\alpha u(x) = E_x(e^{-\alpha\tau_\omega}\,;\, x_{\tau_\omega -} \in (\partial X)_p) = 0\,. \tag{3.21}$$

Therefore

(3.22) $$P_x(\tau_\omega < \infty \mid x_{\tau_\omega -} \in (\partial X)_p) = 0\,,$$

that is,

(3.23) $$P_x(x_{\tau_\omega -} \in (\partial X)_p) = P_x(\tau_\omega = \infty \mid x_{\tau_\omega -} \in (\partial X)_p)\,,$$

which shows (2).

THEOREM 3.3. *The exit boundary $(\partial X)_e$ does not contain the degenerate boundary points.*

PROOF. If b is a degenerate boundary point, there exists $y \in \bigcup R_i$ such that

$$K(c, x, b) = \frac{P_x(\sigma_{\{y\}} < \infty)}{P_c(\sigma_{\{y\}} < \infty)}\,.$$

Therefore it is enough to prove $\lim_{n\to\infty} \sum_{z\in X} \Pi_\alpha^n(x, z) P_z(\sigma_{\{y\}} < \infty) \equiv 0$. We know that if $x_t(w)$ once reaches a recurrent point y then it passes through y infinitely often (with P_x—1). Hence we have $\{w\,;\, \sigma_{\{y\}} < \infty\} = \{w\,;\, \sigma_{\{y\}}(w_{\tau_n}^+) < \infty\}$ with P_x—1. Moreover, since $P_x(\tau_\omega = \infty) = 1$ for $x \in \bigcup_i R_i$, we get

$$\begin{aligned}
&\sum_{z\in X} \Pi_\alpha^n(x, z) P_z(\sigma_{\{y\}} < \infty) = E_x(e^{-\alpha\tau_n} P_{x_{\tau_n}}(\sigma_{\{y\}} < \infty)) = E_x(e^{-\alpha\tau_n}\,;\, \sigma_{\{y\}} < \infty) \\
&\underset{n\to\infty}{\longrightarrow} E_x(e^{-\alpha\tau_\omega}\,;\, \sigma_{\{y\}} < \infty) = E_x(e^{-\alpha\sigma_{\{y\}}} e^{-\alpha\tau_\omega(w_{\sigma_{\{y\}}}^+)}\,;\, \sigma_{\{y\}} < \infty) \\
&= E_x(e^{-\alpha\sigma_{\{y\}}} E_{x_{\sigma_{\{y\}}}}(e^{-\alpha\tau_\omega})) = 0\,.
\end{aligned}$$

4. The representation of bounded $x_\alpha^0(t)$-harmonic functions.

Let u_α be a bounded $x_\alpha^0(t)$-harmonic function and u_0, the canonical image of u_α and $\tilde{u}_0$, the boundary function of u_0. Then it follows that

(4.1) $$\begin{aligned} u_\alpha(x) &= \int_{(\partial X)_e} K_\alpha(c, x, b)\tilde{u}_0(b) h(c, db) \\ &= \int_{(\partial X)_e} \tilde{u}_0(b) h_\alpha(x, db)\,, \end{aligned}$$

where $h_\alpha(x, db) = K_\alpha(c, x, b) h(c, db)$. But according to the argument of the previous section, we have

(4.2) $$\begin{aligned} h_\alpha(x, B) &= \lim_{n\to\infty} \sum_{y\in X} \Pi_\alpha^n(x, y) h(y, B) \\ &= E_x(e^{-\alpha\tau_\omega}\,;\, x_{\tau_\omega -}(w) \in B)\,. \end{aligned}$$

Therefore the formula (4.1) can be rewritten in

(4.3) $$u_\alpha(x) = E_x(e^{-\alpha\tau_\omega} \tilde{u}_0(x_{\tau_\omega -}))\,,$$

which proves the first part of the following

THEOREM 4.1. (1) *Any bounded $x_\alpha^0(t)$-harmonic function u_α is expressible in* *the form* (4.1) *or* (4.3), *using the boundary function* $\tilde{u}_0$ *of the canonical image of* u_α.

(2) $\lim_{n\to\infty} u_\alpha(x_{\tau_n}(w)) = \tilde{u}_0(x_{\tau_\omega-}(w))$ *holds with* P_x-1.

PROOF OF (2). In (3.10), since $\lim_{n\to\infty} u_0(x_{\tau_n}(w)) = u_0(x_{\tau_\omega-}(w))$ and $\lim_{n\to\infty} G^0 u(x_{\tau_n}) = 0$ with P_x-1 (Theorem 2.2), we get (2) immediately.

REMARK. Conversely if $\tilde{u}$ is a bounded Borel function on $\partial X \cup \{\infty\}$ such that $u(\infty)=0$, the function

$$u_\alpha(x) = E_x(e^{-\alpha\tau_\omega}\tilde{u}(x_{\tau_\omega-})) \tag{4.4}$$

becomes a bounded $x_\alpha^0(t)$-harmonic function. But generally the formula

$$\lim_{n\to\infty} u_\alpha(x_{\tau_n}(w)) = \tilde{u}(x_{\tau_\omega-}(w)) \tag{4.5}$$

does not hold. In fact if $u_0(x)$ is the canonical image of u_α and $\tilde{u}_0$, its boundary function,

$$\begin{aligned} \tilde{u}_0(b) &= \tilde{u}(b) \qquad \text{if} \quad b\in(\partial X)_e\,, \\ &= 0 \qquad\quad \text{if} \quad b\in(\partial X)_p \end{aligned} \tag{4.6}$$

holds except for the set of $h(c,\cdot)$-measure 0. Therefore we may call $(\partial X)_e$ the *PWB resolutive boundary points* for u_α defined by (4.4) and $(\partial X)_p$, the *PWB non-resolutive ones*.

From Theorem 4.1 and the above Remark, we get

COROLLARY 1. *If* B *is a Borel subset of* $(\partial X)_e$, *we have with* P_x-1,

$$\begin{aligned} \lim_{n\to\infty} E_{x_{\tau_n}(w)}(e^{-\alpha\tau_\omega}\,;\, x_{\tau_\omega-}\in B) &= 1 \qquad \textit{if} \quad x_{\tau_\omega-}(w)\in B\,, \\ &= 0 \qquad \textit{if} \quad x_{\tau_\omega-}(w)\notin B\,. \end{aligned} \tag{4.7}$$

COROLLARY 2. *With* P_x-1, *we have*

$$\begin{aligned} \lim_{n\to\infty} E_{x_{\tau_n}(w)}(e^{-\alpha\tau_\omega}) &= 1 \qquad \textit{if} \quad x_{\tau_\omega-}(w)\in(\partial X)_e\,, \\ &= 0 \qquad \textit{if} \quad x_{\tau_\omega-}(w)\in(\partial X)_p\,. \end{aligned} \tag{4.8}$$

THEOREM 4.2. *Every function* f *such that both* f *and* $\tilde{\mathfrak{G}}f\in\mathfrak{B}(X)$ *can be expressed by the difference of* $x_\alpha^0(t)$*-superharmonic function* (*i.e.* $x_\alpha^0(t)$*-Riesz decomposable*), *and both* $x_\alpha^0(t)$*-potential part and* $x_\alpha^0(t)$*-harmonic part of the* $x_\alpha^0(t)$*-Riesz decomposition of* f *are bounded. Moreover the boundary function of its* $x_\alpha^0(t)$*-harmonic part is independent of* α.

PROOF. Writing $g=(\alpha-\tilde{\mathfrak{G}})f$ and $u_\alpha = f - G_\alpha^0 g$, u_α is an $x_\alpha^0(t)$-harmonic function. Therefore $f=u_\alpha+G_\alpha^0 g$ is nothing but the formula of $x_\alpha^0(t)$-Riesz decomposition. Since $\tilde{\mathfrak{G}}f$ is bounded, clearly $G_\alpha^0 g$ is bounded. Hence u_α is also bounded. Since

$$|G_\alpha^0 g(x)| \leqq \|g\| E_x\Big(\int_0^{\tau_\omega} e^{-\alpha t}dt\Big) = \frac{1}{\alpha}\|g\|(1-E_x(e^{-\alpha\tau_\omega}))\,, \tag{4.9}$$

from Corollary of Theorem 4.1, we get with P_x—1

(4.10) $$\lim_{n\to\infty} G_\alpha^0 g(x_{\tau_n})=0 \qquad \text{if} \quad x_{\tau_\omega -}\in(\partial X)_e\,.$$

Therefore

(4.11) $$\lim_{n\to\infty} f(x_{\tau_n})=\lim_{n\to\infty} u_\alpha(x_{\tau_n}) \qquad \text{for} \quad x_{\tau_\omega -}\in(\partial X)_e$$

holds with P_x—1, which shows the last statement of this theorem.

Now, we shall introduce the several notations. The family of all bounded Borel functions $\tilde{u}$ on $(\partial X)_1\cup\{\infty\}$ such that $\tilde{u}(\infty)=0$ is denoted by $\mathfrak{B}(\partial X)$. The norm of $\tilde{u}\in\mathfrak{B}(\partial X)$ is defined by

(4.12) $$\|\tilde{u}\|=\text{ess sup}\,|\tilde{u}(b)|$$

where the essential superior limit is taken with respect to $h(c,\cdot)$-measure. If $\|\tilde{u}_1-\tilde{u}_2\|=0$, we shall regard $\tilde{u}_1$ and $\tilde{u}_2$ as the same element. Then $\mathfrak{B}(\partial X)$ becomes a Banach space. The mapping h from the Banach space $\mathfrak{B}(\partial X)$ to the Banach space $\mathfrak{H}\cap\mathfrak{B}(X)$ defined by

(4.13) $$h\tilde{u}(x)=\int_{(\partial X)_1}\tilde{u}(b)h(x,db)$$

is one to one and onto, because any element of $\mathfrak{H}\cap\mathfrak{B}(X)$ is represented in the form $\int_{(\partial X)_1}\tilde{u}(b)h(x,db)$ and $\tilde{u}$ is uniquely determined except $h(c,\cdot)$-measure 0. Moreover h is an isometric linear operator. Let $\mathfrak{B}((\partial X)_e)$ be the subspace of $\mathfrak{B}(\partial X)$ consisting of elements of $\mathfrak{B}(\partial X)$ which vanish on $(\partial X)_p$. Then $\bar{\mathfrak{H}}\cap\mathfrak{B}(X)$ coincides with the image of $\mathfrak{B}((\partial X)_e)$ by the operator h. The mapping h_α from $\mathfrak{B}((\partial X)_e)$ to $\mathfrak{H}_\alpha\cap\mathfrak{B}(X)$ defined by

(4.14) $$h_\alpha u(x)=\int_{(\partial X)_e}\tilde{u}(b)h_\alpha(x,db)$$

becomes also an one to one isometric linear operator. In the sequel we shall consider any bounded function on $(\partial X)\cup\{\infty\}$ (taking the value 0 at ∞) as an element of Banach space $\mathfrak{B}(\partial X)$ or $\mathfrak{B}((\partial X)_e)$.

DEFINITION 4.1. All the functions f of $\mathfrak{B}(X)$ such that $\tilde{\mathfrak{G}}f\in\mathfrak{B}(X)$ are denoted by $\mathfrak{B}_{(1)}(X)$. The boundary function of the $x_\alpha^0(t)$-harmonic function obtained by $x_\alpha^0(t)$-Riesz decomposition of $f\in\mathfrak{B}_{(1)}(X)$ is called the *boundary value of f*.

Finally, we shall give the generator's domain of the minimal process, which is the reformulation of [**5**, Theorem 6.2] from our point of view.

THEOREM 4.3. *The generator's domain $\mathfrak{D}(\mathfrak{G})$ of the minimal process is*

(4.15) $$\mathfrak{D}(\mathfrak{G})=\{u\,;(\alpha)\,u,\tilde{\mathfrak{G}}u\in\mathfrak{B}(X) \text{ and } (\beta)\lim_{n\to\infty}\Pi_\alpha^n u=0\}$$

or equivalently

(4.16) $\mathfrak{D}(\mathfrak{G})=\{u\,;(\alpha)\ u, \tilde{\mathfrak{G}}u\in\mathfrak{B}(X)\ \text{and}\ (\beta')\ \tilde{u}=0,\ \text{where}\ \tilde{u}\ \text{is the boundary value of}\ u\}\,.$

PROOF. Denote the right side of (4.15) by $\tilde{\mathfrak{D}}(\mathfrak{G})$. In Section 1, we have already shown that $G_\alpha f$ satisfies (α). To prove that $G_\alpha f$ satisfies (β), we shall rewrite $G_\alpha f$ as

$$G_\alpha f(x)=E_x\Big(\int_0^{\tau_n}e^{-\alpha t}f(x_t)dt\Big)+E_x(e^{-\alpha\tau_n}G_\alpha f(x_{\tau_n})) \tag{4.17}$$

$$=E_x\Big(\int_0^{\tau_n}e^{-\alpha t}f(x_t)dt\Big)+\Pi_\alpha^n G_\alpha f(x)\,.$$

Letting $n\to\infty$, $E_x\Big(\int_0^{\tau_n}e^{-\alpha t}f(x_t)dt\Big)$ converges to $G_\alpha f(x)$. Hence $\lim_{n\to\infty}\Pi_\alpha^n G_\alpha f(x)=0$. Conversely take any u from $\tilde{\mathfrak{D}}(\mathfrak{G})$, and put $g=(\alpha-\tilde{\mathfrak{G}})u$ and $v=u-G_\alpha g$. Then v is an $x_\alpha^0(t)$-harmonic function because $g=(\alpha-\tilde{\mathfrak{G}})G_\alpha g$. Since u and $G_\alpha g$ satisfy (β), v also satisfies (β). Concequently

$$v=\Pi_\alpha v=\cdots=\Pi_\alpha^n v\xrightarrow[n\to\infty]{}0\,. \tag{4.18}$$

Therefore $v=0$, i.e. $u=G_\alpha f$. Thus we have proved (4.15). Since $\lim_{n\to\infty}\Pi_\alpha^n u=0$ shows that u is an $x_\alpha^0(t)$-potential, the equivalence of (β) and (β') will be clear.

The process being completely characterized by the generator together with its domain, the minimal process is characterized by the pair $\{q,\Pi\}$. But, as is well known, the non minimal processes can not be characterized only by $\{q,\Pi\}$. In Sections 5 and 6, we shall determine all the factors characterizing the instantaneous return process, using the results obtained hitherto.

5. The boundary conditions for the first instantaneous return processes.

We shall now introduce the new kernels ${}_{(2)}\Pi(x,A)$ and ${}_{(2)}\Pi_\alpha(x,A)$ as

$${}_{(2)}\Pi(x,A)=P_x(x_{\tau_\omega}\in A,\tau_\omega<\infty)\,, \tag{5.1}$$

$${}_{(2)}\Pi_\alpha(x,A)=E_x(e^{-\alpha\tau_\omega}\,;\,x_{\tau_\omega}\in A)\,. \tag{5.2}$$

If we consider the kernel ${}_{(2)}\Pi_\alpha(x,A)$ as a function of x, it is a bounded $x_\alpha^0(t)$-harminic function, because

$${}_{(2)}\Pi_\alpha(x,A)=E_x(e^{-\alpha\tau_1}e^{-\alpha\tau_\omega(w_{\tau_1}^+)}) \tag{5.3}$$

$$=E_x(e^{-\alpha\tau_1}E_{x_{\tau_1}}(e^{-\alpha\tau_\omega}\,;\,x_{\tau_\omega}\in A))=\Pi_\alpha\,{}_{(2)}\Pi_\alpha(x,A)\,.$$

Moreover ${}_{(2)}\Pi(x,A)$ is the canonical image of ${}_{(2)}\Pi_\alpha(x,A)$, since

$$\lim_{n\to\infty}\Pi^n\,{}_{(2)}\Pi_\alpha(x,A)=\lim_{n\to\infty}E_x(e^{-\alpha(\tau_\omega-\tau_n)}\,;\,x_{\tau_\omega}\in A,\tau_\omega<\infty) \tag{5.4}$$

$$=P_x(x_{\tau_\omega}\in A,\tau_\omega<\infty)\,.$$

Therefore ${}_{(2)}\Pi(x, A)$ and ${}_{(2)}\Pi_\alpha(x, A)$ are represented by means of $\tilde{\Pi}(b, A) \in \mathfrak{B}((\partial X)_e)$ as follows;

$$
{}_{(2)}\Pi(x, A) = \int_{(\partial X)_e} \tilde{\Pi}(b, A) h(x, db) = h\tilde{\Pi}(x, A), \tag{5.5}
$$

$$
{}_{(2)}\Pi_\alpha(x, A) = \int_{(\partial X)_e} \tilde{\Pi}(b, A) h_\alpha(x, db) = h_\alpha\tilde{\Pi}(x, A). \tag{5.6}
$$

Noting the relation

$$
\sum_{y \in X^*} {}_{(2)}\Pi(x, y) = P_x(\tau_\omega < \infty) = h(x, (\partial X)_e), \tag{5.7}
$$

we get

$$
\begin{aligned}
&\tilde{\Pi}(b, y) \geqq 0, \\
&\sum_{y \in X^*} \tilde{\Pi}(b, y) = 1 \qquad \text{if } b \in (\partial X)_e, \\
&\qquad\qquad\quad = 0 \qquad \text{if } b \in (\partial X)_p.
\end{aligned} \tag{5.8}
$$

REMARK. Because of the uniqueness of the Laplace transform, the formula (5.6) is equivalent to

$$
P_x(x_{\tau_\omega} \in A, \tau_\omega \leqq t) = \int_{(\partial X)_e} \tilde{\Pi}(b, A) P_x(x_{\tau_\omega -} \in db, \tau_\omega \leqq t), \tag{5.9}
$$

which may be proved by the direct calculation. Probabilistically speaking, the formulas (5.5) and (5.9) interprete the following circumstances. Consider the particle whose motion yields to the given Markov process $x(t)$ with right continuous paths. Such particle will move in accordance with the minimal process $x^0(t)$ having the same $\{q, \Pi\}$ as $x(t)$, until it converges to some boundary point b. If b is a passive boundary point, the particle has to take infinite time before it reaches b, and if b is an exit boundary point, it reaches b after some finite time. In the latter case, as soon as the particle reaches b, it returns to the interior X^* with the probability distribution $\tilde{\Pi}(b, \cdot)$ not depending on the past movement and starts from scratch in accordance with the minimal process.

Let L be the operator of the space $\mathfrak{B}_{(1)}(X)$ into the space $\mathfrak{B}((\partial X)_e)$ defined for $u \in \mathfrak{B}_{(1)}(X)$ by

$$
Lu(b) = \tilde{u}(b) - \sum_{y \in X} \tilde{\Pi}(b, y) u(y), \tag{5.10}
$$

where $\tilde{u}$ is the boundary value of u. To inquire a boundary condition for $G_\alpha f$ to satisfy, we shall rewrite $G_\alpha f$ as follows;

$$
\begin{aligned}
G_\alpha f(x) &= G_\alpha^0 f(x) + E_x(e^{-\alpha\tau_\omega} G_\alpha f(x_{\tau_\omega})) \\
&= G_\alpha^0 f(x) + {}_{(2)}\Pi_\alpha G_\alpha f(x) = G_\alpha^0 f(x) + h_\alpha \tilde{\Pi} G_\alpha f(x).
\end{aligned} \tag{5.11}
$$

Since $G_\alpha^0 f(x)$ and $h_\alpha \tilde{\Pi} G_\alpha f(x)$ are an $x_\alpha^0(t)$-potential and an $x_\alpha^0(t)$-harmonic function respectively, the above formula is nothing but the Riesz decomposition of $G_\alpha f$. Therefore the boundary value $\widetilde{G_\alpha f}$ of $G_\alpha f$ satisfies $\widetilde{G_\alpha f}=\tilde{\Pi}G_\alpha f$ i.e. $LG_\alpha f=0$.

To get all the conditions which determine the generator's domain $\mathfrak{D}(\mathfrak{G})$ completely, it is convenient to discuss the first instantaneous return process and the higher order ones separately. So at the remainder of this section we shall treat this problem in the case of the first instantaneous return process.

THEOREM 5.1. *The generator's domain* $\mathfrak{D}(\mathfrak{G})$ *of the first instantaneous return process is given by*

$$\mathfrak{D}(\mathfrak{G})=\{u\,;(\alpha)\,u \text{ and } \tilde{\mathfrak{G}}u\in\mathfrak{B}(X),(\beta)\,Lu=0 \text{ and } (\gamma)\lim_{n\to\infty}{}_{(2)}\Pi_\alpha^n u=0\}\,. \tag{5.12}$$

PROOF. Denote the right side of (5.12) by $\tilde{\mathfrak{D}}(\mathfrak{G})$. We have already shown that $u=G_\alpha f$ satisfies (α) and (β). Let us rewrite $G_\alpha f$ as

$$\begin{aligned} G_\alpha f(x)&=E_x\Big(\int_0^{\tau_{\omega 2}}e^{-\alpha t}f(x_t)dt\Big)\\ &=E_x\Big(\int_0^{\tau_{\omega n}}e^{-\alpha t}f(x_t)dt\Big)+E_x(e^{-\alpha\tau_{\omega n}}G_\alpha f(x_{\tau_{\omega n}}))\\ &=E_x\Big(\int_0^{\tau_{\omega n}}e^{-\alpha t}f(x_t)dt\Big)+{}_{(2)}\Pi_\alpha^n G_\alpha f(x)\,, \end{aligned} \tag{5.13}$$

and letting $n\to\infty$, we get $\lim_{n\to\infty}{}_{(2)}\Pi_\alpha^n G_\alpha f(x)=0$. Therefore $G_\alpha f\in\tilde{\mathfrak{D}}(\mathfrak{G})$. Conversely take any element u from $\tilde{\mathfrak{D}}(\mathfrak{G})$ and put $f=(\alpha-\tilde{\mathfrak{G}})u$ and $v=u-G_\alpha f$. Since $(\alpha-\tilde{\mathfrak{G}})G_\alpha f=f$, v satisfies $(\alpha-\tilde{\mathfrak{G}})v=0$, which shows that v is an $x_\alpha^0(t)$-harmonic function. Moreover v is in $\tilde{\mathfrak{D}}(\mathfrak{G})$ because both u and $G_\alpha f$ are in $\tilde{\mathfrak{D}}(\mathfrak{G})$. Therefore

$$v=h_\alpha\tilde{\Pi}v={}_{(2)}\Pi_\alpha v=\cdots={}_{(2)}\Pi_\alpha^n v\xrightarrow[n\to\infty]{}0$$

which shows $u=G_\alpha f$. Thus we have proved $\tilde{\mathfrak{D}}(\mathfrak{G})\subseteqq\mathfrak{D}(\mathfrak{G})$.

COROLLARY. *The first instantaneous return process is completely characterized by the following factors;*

$$\{q_x,\Pi(x,y),\tilde{\Pi}(b,y)\,;\,x,y\in X^*,b\in(\partial X)_e\}\,. \tag{5.14}$$

6. The boundary conditions for the k-th instantaneous return processes.

In the previous section, we have obtained the three factors $\{q,\Pi,\tilde{\Pi}\}$ which determine the first instantaneous return process completely. To obtain all the factors determining the k-th instantaneous return process, it is necessary to define the boundaries of higher order besides the boundary which has been already introduced. First we shall discuss the second boundary and the second instantaneous return process in detail, for our procedure is applicable to the k-th instantaneous return process by induction.

Let $x(t)$ be a second instantaneous return process and ${}_{(2)}\Pi(x, A)=P_x(x_{\tau_\omega}\in A, \tau_\omega<\infty)$, ${}_{(2)}\Pi_\alpha(x, A)=E_x(e^{-\alpha\tau_\omega}; x_{\tau_\omega}\in A)$ $(\alpha>0)$ as was defined in the previous section. We now consider a new time discrete Markov process[10] ${}_{(2)}x(t)$ having ${}_{(2)}\Pi$ as the one step transition probability. In the same way, the time discrete Markov process induced by ${}_{(2)}\Pi_\alpha$ is denoted by ${}_{(2)}x_\alpha(t)$. The Dynkin generator ${}_{(2)}\tilde{\mathfrak{G}}$ $({}_{(2)}\tilde{\mathfrak{G}}_\alpha)$ of the process ${}_{(2)}x(t)$ $({}_{(2)}x_\alpha(t))$ is defined by

$$(6.1)\qquad {}_{(2)}\tilde{\mathfrak{G}}={}_{(2)}\Pi-I \qquad ({}_{(2)}\tilde{\mathfrak{G}}_\alpha={}_{(2)}\Pi_\alpha-I).$$

If u is a nonnegative ${}_{(2)}x(t)$-harmonic function, i.e. ${}_{(2)}\tilde{\mathfrak{G}}u=0$, it is ${}_{(2)}x_\alpha(t)$-superharmonic function, because ${}_{(2)}\tilde{\mathfrak{G}}_\alpha u\leqq{}_{(2)}\tilde{\mathfrak{G}}u=0$. Therefore by the ${}_{(2)}x_\alpha(t)$-Riesz decomposition, we get

$$(6.2)\qquad u=\lim_{n\to\infty}{}_{(2)}\Pi_\alpha^n u+\sum_{n\geqq0}{}_{(2)}\Pi_\alpha^n(-{}_{(2)}\tilde{\mathfrak{G}}_\alpha u).$$

Noting that $u={}_{(2)}\Pi u=h\tilde{\Pi}u$, it is plain to see that u is $x^0(t)$-harmonic. Further u is the canonical image of ${}_{(2)}\Pi_\alpha u=h_\alpha\tilde{\Pi}u$. Hence recalling the formula (3.4), we get

$$(6.3)\qquad -{}_{(2)}\tilde{\mathfrak{G}}_\alpha u=u-{}_{(2)}\Pi_\alpha u=\alpha G_\alpha^0 u.$$

Therefore (6.2) becomes

$$(6.4)\qquad \begin{aligned} u&=\lim_{n\to\infty}{}_{(2)}\Pi_\alpha^n u+\alpha\sum_{n\geqq0}{}_{(2)}\Pi_\alpha^n G_\alpha^0 u\\ &=\lim_{n\to\infty}{}_{(2)}\Pi_\alpha^n u+\alpha\sum_{n\geqq0}E.\left(\int_{\tau_{\omega n}}^{\tau_{\omega(n+1)}}e^{-\alpha t}u(x_t)dt\right)\\ &=\lim_{n\to\infty}{}_{(2)}\Pi_\alpha^n u+\alpha G_\alpha^1 u, \end{aligned}$$

where

$$G_\alpha^1 u(x)=E_x\left(\int_0^{\tau_{\omega 2}}e^{-\alpha t}u(x_t)dt\right).$$

Noting that $G_\alpha^1 f$ satisfies the resolvent equation and using the relation (6.4), we can easily prove the following lemma by the similar argument as Lemma 3.1.

LEMMA 6.1. *Suppose u a nonnegative ${}_{(2)}x(t)$-harmonic function and put ${}_{(2)}u_\alpha=\lim_{n\to\infty}{}_{(2)}\Pi_\alpha^n u$. Then we have*

$$(6.6)\qquad {}_{(2)}u_\alpha-{}_{(2)}u_\beta=(\beta-\alpha)G_\alpha^1\,{}_{(2)}u_\beta=(\beta-\alpha)G_\beta^1\,{}_{(2)}u_\alpha.$$

10) In the definition of the Markov process in Section 1, if we take the discrete time parameter $T=\{0, 1, 2, \cdots, +\infty\}$ for the continuous one, we get the time discrete Markov process. For detail, refer to [**12**].

Let us denote by ${}_{(2)}X$ the union of center c and y such that ${}_{(2)}\Pi(c,y)>0$. Then ${}_{(2)}\Pi(x,y)=0$ for every $x\in X$ and $y\notin {}_{(2)}X$, because

$$0={}_{(2)}\Pi(c,y)\geqq P_c(\sigma_x<\infty\,|\,x_{\tau_\omega}(w^+_{\sigma_x})=y) \\ =P_c(\sigma_x<\infty)\,{}_{(2)}\Pi(x,y)\,. \tag{6.7}$$

Therefore we can confine the Markov process ${}_{(2)}x(t)$ on the state ${}_{(2)}X$, where the state c is also a center of the confined process ${}_{(2)}x(t)$. Hence we can construct the Martin boundary $\partial_{(2)}X$, which is called the *second boundary.* The K-function corresponding to $b\in\partial_{(2)}X$ is denoted by ${}_{(2)}K(c,x,b)$ and the minimal part of $\partial_{(2)}X$ by $(\partial_{(2)}X)_1$. The ${}_{(2)}x(t)$-harmonic measure for a Borel set $B\subset\partial_{(2)}X$ is denoted by ${}_{(2)}h(x,B)$ where $x\in{}_{(2)}X$. Moreover for $x\notin{}_{(2)}X$, we shall define the ${}_{(2)}x(t)$-harmonic measure as follows;

$${}_{(2)}h(x,B)=\sum_{y\in{}_{(2)}X}{}_{(2)}\Pi(x,y)\,{}_{(2)}h(y,B)\,. \tag{6.8}$$

Then it is clear that any bounded ${}_{(2)}x(t)$-harmonic function over X can be represented as

$$u(x)=\int_{(\partial_{(2)}X)_1}{}_{(2)}\tilde u(b)\,{}_{(2)}h(x,db)={}_{(2)}h\,{}_{(2)}\tilde u(x)\,, \tag{6.9}$$

where ${}_{(2)}\tilde u$ is uniquely determined except for ${}_{(2)}h(c,\cdot)$-measure 0. If ${}_{(2)}K(c,x,b)$ is a minimal ${}_{(2)}x(t)$-harmonic function and

$${}_{(2)}K_\alpha(c,x,b)\equiv\lim_{n\to\infty}\sum_{y\in{}_{(2)}X}{}_{(2)}\Pi^n_\alpha(x,y)\,{}_{(2)}K(c,y,b)\neq 0\,, \tag{6.10}$$

b is called the *second exit boundary point.* The set $(\partial_{(2)}X)_e$ of all second exit boundary points is independent of α, according to Lemma 6.1. $(\partial_{(2)}X)_p=(\partial_{(2)}X)-(\partial_{(2)}X)_e$ is called the *second passive boundary.*

It is easy to show that all the analogous results of Sections 3 and 4 hold without any essential change. For example, denoting the limit of $x_{\tau_{\omega n}}(w)$ in the topology of the second Martin space by $x_{\tau_{\omega^2}-}(w)$, we get

$$P_x(\tau_{\omega 2}<\infty\,|\,x_{\tau_{\omega^2}-}\in(\partial_{(2)}X)_e)=1\,, \\ P_x(\tau_{\omega 2}=\infty\,|\,x_{\tau_{\omega^2}-}\in(\partial_{(2)}X)_p)=1\,. \tag{6.11}$$

And

$$\lim_{n\to\infty}E_{x_{\tau_{\omega n}}(w)}(e^{-\alpha\tau_{\omega^2}})=1 \qquad \text{if}\quad x_{\tau_{\omega^2}-}(w)\in(\partial_{(2)}X)_e\,, \\ =0 \qquad \text{if}\quad x_{\tau_{\omega^2}-}(w)\in(\partial_{(2)}X)_p \tag{6.12}$$

holds with P_x—1. Moreover any bounded ${}_{(2)}x_\alpha(t)$-harmonic function u can be represented as

$$u(x)=\int_{(\partial_{(2)}X)_e}{}_{(2)}\tilde u_0(b)\,{}_{(2)}h_\alpha(x,db)={}_{(2)}h_\alpha\,{}_{(2)}\tilde u_0(x)\,, \tag{6.13}$$

where ${}_{(2)}\tilde u_0$ is the boundary function of ${}_{(2)}u_0=\lim_{n\to\infty}{}_{(2)}\Pi^n u$ and ${}_{(2)}h_\alpha(x,B)=\lim_{n\to\infty}\sum_{y\in X}{}_{(2)}\Pi^n_\alpha(x,y)\,{}_{(2)}h(y,B)$. The analogue of Theorem 4.2 is the following

THEOREM 6.1. *Let $\mathfrak{B}_{(2)}(X)$ be the family of functions u such that* (α) $u, \tilde{\mathfrak{G}}u \in \mathfrak{B}(X)$ *and* (β) $Lu=0$. *Every $u \in \mathfrak{B}_{(2)}(X)$ is ${}_{(2)}x_\alpha(t)$-Riesz decomposable and the potential part of ${}_{(2)}x_\alpha(t)$-Riesz decomposition of u is written in the form $G_\alpha^1 g$, where g is an element of $\mathfrak{B}(X)$. Moreover, the boundary function of ${}_{(2)}x_\alpha(t)$-harmonic part is independent of α.*

RROOF. Take any element u from $\mathfrak{B}_{(2)}(X)$ and put $g=(\alpha-\tilde{\mathfrak{G}})u$ and $v=u-G_\alpha^1 g$. Then v is a bounded $x_\alpha^0(t)$-harmonic function because $G_\alpha^1 g$ satisfies $g=(\alpha-\tilde{\mathfrak{G}})G_\alpha^1 g$. Since u and $G_\alpha^1 g$ belong to $\mathfrak{B}_{(2)}(X)$, v belongs also to $\mathfrak{B}_{(2)}(X)$. Therefore if $\tilde{v}$ is the boundary value of v, we have

$$v=h_\alpha \tilde{v}=h_\alpha \tilde{\Pi} v={}_{(2)}\Pi_\alpha v\,, \tag{6.14}$$

which means that v is ${}_{(2)}x_\alpha(t)$-harmonic. Hence $u=G_\alpha^1 g+v$ is the formula of ${}_{(2)}x_\alpha(t)$-Riesz decomposition. Next noting the relation

$$|\alpha G_\alpha^1 g(x)| \leqq \|g\| \{1-E_x(e^{-\alpha\tau_{\omega^2}})\} \tag{6.15}$$

and (6.12), we can easily prove the second statement of our theorem.

Next we denote the set of all bounded Borel functions vanishing except on $(\partial_{(2)}X)_e$ by $\mathfrak{B}((\partial_{(2)}X)_e)$. Then such function family constitute a Banach space with the norm $\|\tilde{f}\|={}_{(2)}h(c,\cdot)$-ess sup $|\tilde{f}(b)|$ $(b \in (\partial_{(2)}X)_e)$.

DEFINITION 6.1. The boundary function ${}_{(2)}\tilde{u} \in \mathfrak{B}((\partial_{(2)}X)_e)$ of Theorem 6.1 is called the *second boundary value of u.*

Now, if we define the kernels ${}_{(3)}\Pi(x, A)$ and ${}_{(3)}\Pi_\alpha(x, A)$ by

$${}_{(3)}\Pi(x, A)=P_x(x_{\tau_{\omega^2}} \in A,\, \tau_{\omega^2}<\infty)\,, \tag{6.16}$$

$${}_{(3)}\Pi_\alpha(x, A)=E_x(e^{-\alpha\tau_{\omega^2}};\, x_{\tau_{\omega^2}} \in A)\,, \tag{6.17}$$

then they are represented by means of ${}_{(2)}\tilde{\Pi}(b, A) \in \mathfrak{B}((\partial_{(2)}X)_e)$ as

$${}_{(3)}\Pi(x, A)=\int_{(\partial_{(2)}X)_e} {}_{(2)}\tilde{\Pi}(b, A)\,{}_{(2)}h(x, db)={}_{(2)}h\,{}_{(2)}\tilde{\Pi}(x, A)\,, \tag{6.18}$$

$${}_{(3)}\Pi_\alpha(x, A)=\int_{(\partial_{(2)}X)_e} {}_{(2)}\tilde{\Pi}(b, A)\,{}_{(2)}h_\alpha(x, db)={}_{(2)}h_\alpha\,{}_{(2)}\tilde{\Pi}(x, A)\,. \tag{6.19}$$

${}_{(2)}\tilde{\Pi}(b,\cdot)$ is a probability measure over X^* for any fixed $b \in (\partial_{(2)}X)_e$. Let ${}_{(2)}L$ be the operator of $\mathfrak{B}_{(2)}(X)$ into $\mathfrak{B}((\partial_{(2)}X)_e)$ defined by

$${}_{(2)}Lu={}_{(2)}\tilde{u}-{}_{(2)}\tilde{\Pi} u\,, \tag{6.20}$$

where ${}_{(2)}\tilde{u}$ is the boundary value of $u \in \mathfrak{B}_{(2)}(X)$. Then we have

THEOREM 6.2. *The generator's domain $\mathfrak{D}(\mathfrak{G})$ of the second instantaneous return process is given by*

$$\mathfrak{D}(\mathfrak{G})=\left\{\begin{array}{l} u\,;\ (\alpha)\ u, \tilde{\mathfrak{G}}u \in \mathfrak{B}(X),\ (\beta_1)\ Lu=0,\ (\beta_2)\ {}_{(2)}Lu=0 \\ \text{and } (\gamma)\ \lim_{n\to\infty} {}_{(3)}\Pi_\alpha^n u=0\,. \end{array}\right\}. \tag{6.21}$$

PROOF. Denote the right side of (6.21) by $\tilde{\mathfrak{D}}(\mathfrak{G})$. It has been shown in the previons section that every $G_\alpha f$ satisfies (α) and (β_1). To prove that $G_\alpha f$ satisfies (β_2), we shall rewrite $G_\alpha f$ as follows:

$$\begin{aligned}(6.22)\qquad G_\alpha f(x) &= G_\alpha^1 f(x) + E_x(e^{-\alpha\tau_{\omega^2}} G_\alpha f(x_{\tau_{\omega^2}})) \\ &= G_\alpha^1 f(x) + {}_{(3)}\Pi_\alpha G_\alpha f(x) \\ &= G_\alpha^1 g(x) + {}_{(2)}h_{\alpha\,(2)}\tilde{\Pi} G_\alpha f(x)\,.\end{aligned}$$

We may consider that the above formula is ${}_{(2)}x_\alpha(t)$-Riesz decomposition of $G_\alpha f$ and that ${}_{(2)}h_{\alpha\,(2)}\tilde{\Pi} G_\alpha f$ is its ${}_{(2)}x_\alpha(t)$-harmonic part. Therefore ${}_{(2)}\widetilde{G_\alpha f} = {}_{(2)}\tilde{\Pi} G_\alpha f$, where ${}_{(2)}\widetilde{G_\alpha f}$ is the second boundary value of $G_\alpha f$. Thus we have proved (β_2). To prove that $G_\alpha f$ satisfies (γ), we shall rewrite $G_\alpha f$ as follows;

$$\begin{aligned}(6.23)\qquad G_\alpha f(x) &= E_x\Big(\int_0^{\tau_{\omega^{2n}}} e^{-\alpha t} f(x_t)dt\Big) + E_x(e^{-\alpha\tau_{\omega^{2n}}} G_\alpha f(x_{\tau_{\omega^{2n}}})) \\ &= E_x\Big(\int_0^{\tau_{\omega^{2n}}} e^{-\alpha t} f(x_t)dt\Big) + {}_{(3)}\Pi_\alpha^n G_\alpha f(x)\,.\end{aligned}$$

Letting $n \to \infty$ in the above formula, we get $\lim_{n\to\infty} {}_{(3)}\Pi_\alpha^n G_\alpha f = 0$, because

$$(6.24)\qquad \lim_{n\to\infty} E_x\Big(\int_0^{\tau_{\omega^{2n}}} e^{-\alpha t} f(x_t)dt\Big) = E_x\Big(\int_0^{\tau_{\omega^3}} e^{-\alpha t} f(x_t)dt\Big) = G_\alpha f(x)$$

by virtue of the definition of the second instantaneous return process. Conversely for any given u from $\tilde{\mathfrak{D}}(\mathfrak{G})$, put $f = (\alpha - \tilde{\mathfrak{G}})u$ and $v = u - G_\alpha f$, then v is $x_\alpha^0(t)$-harmonic and belongs to $\tilde{\mathfrak{D}}(\mathfrak{G})$. Therefore

$$(6.25)\qquad v = h_\alpha \tilde{v} = h_\alpha \tilde{\Pi} v = {}_{(2)}\Pi_\alpha v\,.$$

Hence v can be rewritten in

$$(2.26)\qquad v = {}_{(2)}h_\alpha \tilde{v} = {}_{(2)}h_{\alpha\,(2)}\tilde{\Pi} v = {}_{(3)}\Pi_\alpha v\,.$$

Therefore

$$(6.27)\qquad v = {}_{(3)}\Pi_\alpha^n v \xrightarrow[n\to\infty]{} 0\,,$$

which shows $\tilde{\mathfrak{D}}(\mathfrak{G}) \subseteq \mathfrak{D}(\mathfrak{G})$. Thus we have accomplished the proof of Theorem 6.2.

COROLLARY. *The second instantaneous return process is completely characterized by the factors*

$$(2.28)\qquad \{q_x, \Pi(x,y), \tilde{\Pi}(b,y), {}_{(2)}\tilde{\Pi}(b',y)\,;\, x,y \in X^*, b \in (\partial X)_e \text{ and } b' \in (\partial_{(2)}X)_e\}\,.$$

REMARK. In the previous section, although we have given the generetor's domain $\mathfrak{D}(\mathfrak{G})$ of the first instantaneous return process, we shall here give its another form using the second boundary, i.e.

$$(6.29)\qquad \mathfrak{D}(\mathfrak{G}) = \{u\,;\, (\alpha)\, u, \tilde{\mathfrak{G}}u \in \mathfrak{B}(X), (\beta_1)\, Lu = 0 \text{ and } (\beta_2)\, {}_{(2)}\tilde{u} = 0\}\,.$$

For, it is easily shown that the above (β_2) is equivalent to the third condition (γ) of (5.12).

Quite in the same way we can introduce the *i-th boundary* $(\partial_{(i)}X)$ and the *i-th exit boundary* $(\partial_{(i)}X)_e$. The kernel ${}_{(i+1)}\Pi(x, A)=P_x(x_{\tau_{\omega^i}}\in A, \tau_{\omega^i}<\infty)$ and ${}_{(i+1)}\Pi_\alpha(x, A)=E_x(e^{-\alpha\tau_{\omega^i}}; x_{\tau_{\omega^i}}\in A)$ can be represented by means of the i-th exit boundary as

$$(6.30)\qquad {}_{(i+1)}\Pi(x, A)=\int_{(\partial_{(i)}X)_e}{}_{(i)}\tilde{\Pi}(b, A)\,{}_{(i)}h(x, db)\,,$$

$$(6.31)\qquad {}_{(i+1)}\Pi_\alpha(x, A)=\int_{(\partial_{(i)}X)_e}{}_{(i)}\tilde{\Pi}(b, A)\,{}_{(i)}h_\alpha(x, db)\,.$$

Let $\mathfrak{B}_{(k)}(X)=\{u\,;(\alpha)\,u, \tilde{\mathfrak{G}}u\in\mathfrak{B}(X),$ and $(\beta_i)\,{}_{(i)}Lu=0\ (i=1, 2, \cdots, k-1)\}$ where ${}_{(i)}Lu(b)={}_{(i)}\tilde{u}(b)-\sum_{y\in X}{}_{(i)}\tilde{\Pi}(b, y)u(y)\ (i=1, 2, \cdots, k-1)$ and ${}_{(i)}\tilde{u}$ is the i-th boundary value of u.[11] Then we can define the k-th boundary value ${}_{(k)}\tilde{u}$ of u for every $u\in\mathfrak{B}_{(k)}(X)$ by induction. If ${}_{(k)}L$ is the operator defined for $u\in\mathfrak{B}_{(k)}(X)$ as ${}_{(k)}Lu(b)={}_{(k)}\tilde{u}-\sum_{y\in X}{}_{(k)}\tilde{\Pi}(b, y)u(y)$, then the following theorem will hold.

THEOREM 6.3. *The generator's domain* $\mathfrak{D}(\mathfrak{G})$ *of the k-th instantaneous return process is given by*

$$(6.32)\qquad \mathfrak{D}(\mathfrak{G})=\left\{\begin{array}{l} u\,;(\alpha)\,u, \tilde{\mathfrak{G}}u\in\mathfrak{B}(X), (\beta_i)\,{}_{(i)}Lu=0\ (i=1, 2, \cdots, k) \\ \textit{and}\ (\gamma)\lim_{n\to\infty}{}_{(k+1)}\Pi_\alpha^n u=0 \end{array}\right\}.$$

COROLLARY. *The k-th instantaneous return process is completely characterized by the following factors*

$$(6.33)\qquad \{q_x, \Pi(x, \cdot), \tilde{\Pi}(b, \cdot), \cdots, {}_{(i)}\tilde{\Pi}(b_i, \cdot), \cdots, {}_{(k)}\tilde{\Pi}(b_k, \cdot)\,;\ x\in X^*\ \textit{and}\ b_i\in(\partial_{(i)}X)_e\}\,.$$

7. The construction of instantaneous return processes.

From Section 1 to 6, we have studied the Markov processes with right continuous paths when they exist, and shown that the generator's domain of the k-th instantaneous return process is uniquely determined by (6.33). In this section we shall solve the construction problem of instantaneous return processes which is formulated as follows:

THEOREM 7.1.[12] *Let* $x^k(t)$ *be the k-th instantaneous return process whose generator's domain is given by* (6.32) *and* $(\partial_{(k+1)}X)_e$, *the set of all the* $(k+1)$*-th exit boundary points and* ${}_{(k+1)}h_\alpha(x, B)$, *the* α*-order harmonic measure for the* $(k+1)$*-th exit boundary. Take an arbitrary measurable function* ${}_{(k+1)}\tilde{\Pi}(b, y)$ *defined*

11) The first boundary value ${}_{(1)}\tilde{u}$ is the boundary value u defined in Section 4, and ${}_{(1)}L$ and ${}_{(1)}\tilde{\Pi}$ denote L and Π respectively.

on $(\partial_{(k+1)}X)_e \times X^*$ *satisfying*

$$(7.1) \qquad {}_{(k+1)}\tilde{\Pi}(b, y) \geqq 0, \qquad \sum_{y \in X^*} {}_{(k+1)}\tilde{\Pi}(b, y) = 1 .$$

Then we can construct the $(k+1)$*-th instantaneous return process whose generator's domain is*

$$(7.2) \qquad \mathfrak{D}(\mathfrak{G}) = \left\{ \begin{array}{l} u; (\alpha)\, u, \tilde{\mathfrak{G}}u \in \mathfrak{B}(X), (\beta_i)\, {}_{(i)}Lu = 0 \ (i = 1, 2, \cdots, k), (\beta_{k+1}) \\ {}_{(k+1)}Lu = {}_{(k+1)}\tilde{u} - {}_{(k+1)}\tilde{\Pi}u = 0 \ \textit{and} \ (\gamma) \lim_{n\to\infty} {}_{(k+2)}\Pi^n_\alpha u = 0 \end{array} \right\},$$

where ${}_{(k+1)}\tilde{u}$ *is the* $k+1$*-th boundary value of* u *and*

$${}_{(k+2)}\Pi_\alpha(x, y) = \int_{(\partial_{(k+1)}X)_e} {}_{(k+1)}\tilde{\Pi}(b, y)\, {}_{(k+1)}h_\alpha(x, db) .$$

For simplicity we shall discuss the case $k = 0$; the proof goes with no essential change for every k.

CONSTRUCTION. Let $\boldsymbol{M}^0 = \{X^*, W, \mathfrak{B}_W, P^0_x; x \in X^*\}$ be the minimal process with the pair $\{q, \Pi\}$ and satisfy the condition (P.5), and let (∂X), $(\partial X)_e$ and $(\partial X)_p$ be the Martin boundary, its exit part and its passive part respectively. $\tilde{\Pi}(b, y)$ is an arbitrary measurable function on $(\partial X)_e \times X^*$ satisfying (5.8). Next we consider an abstruct probability field $(\tilde{W}, \mathfrak{B}_{\tilde{W}}, \tilde{P})$ over which the following family of stochastic processes and random variables are defined. (A) ${}_iY^{(x)}_t(\tilde{w})$ $(x \in X^*, t \in [0, +\infty]$ and $i = 0, 1, 2, \cdots)$ is right continuous with respect to t for any fixed $\tilde{w}, x$ and i, and is subject to the following probability measure (not depending on i)

$$(7.3) \qquad \tilde{P}(\tilde{w}\,; {}_iY^{(x)}(\tilde{w}) \in B) = P^0_x(B) \qquad \text{for any} \quad B \in \mathfrak{B}_W .$$

(B) ${}_iZ^{(b)}(\tilde{w})$ $(b \in (\partial X) \cup \{\infty\}, i = 1, 2, \cdots)$ are random variables which are measurable with respect to $(b, \tilde{w})$, and satisfy

$$(7.4) \qquad \begin{aligned} \tilde{P}({}_iZ^{(b)}(\tilde{w}) \in A) &= \tilde{\Pi}(b, A) && \text{if} \quad b \in (\partial X)_e , \\ &= \delta(\infty, A) && \text{if} \quad b \in (\partial X)_p \cup \{\infty\} , \end{aligned}$$

where $\delta(\infty, \cdot)$ is the unit measure at ∞. (C) ${}_iY^{(x)}_t(\tilde{w})$ and ${}_jZ^{(b)}(\tilde{w})$ $(x \in X^*, b \in (\partial X) \cup \{\infty\}$, and $i = 0, 1, \cdots, j = 1, 2, \cdots)$ are mutually independent.

We shall define the jumping times for the process ${}_iY^{(x)}_t(\tilde{w})$ as

---

12) It is well known (e.g. [1]) that for any pair $\{q, \Pi\}$ satisfying (1.10) and (1.11) we can construct the minimal process $x^0(t)$ whose generator's domain is given by

$$\mathfrak{D}(\mathfrak{G}) = \{u\,; (\alpha) u, \tilde{\mathfrak{G}}u = q(\Pi - I)u \in \mathfrak{B}(X) \text{ and } (\beta) \lim_{n\to\infty} \Pi_\alpha{}^n u = 0\},$$

where $\Pi_\alpha(x, \cdot) = q_x/(\alpha + q_x) \cdot \Pi(x, \cdot)$. Further the condition (P.5) is equivalent to $\sum_{n \geqq 0} \Pi^n(c, y) q_y{}^{-1} > 0$ for any y. In this theorem, we assume that the given instantaneous return process $x^k(t)$ satisfies this condition.

(7.5)
$$
\begin{aligned}
&{}_i\tau_1^{(x)}(\tilde{w}) = \inf\{t\,;\,{}_iY_t^{(x)}(\tilde{w}) \neq {}_iY_0^{(x)}(\tilde{w})\}\,,\\
&{}_i\tau_{n+1}^{(x)}(\tilde{w}) = \inf\{t\,;\,{}_iY^{(x)}_{{}_i\tau_n^{(x)}+t}(\tilde{w}) \neq {}_iY^{(x)}_{{}_i\tau_n^{(x)}}(\tilde{w})\} + {}_i\tau_n^{(x)}(\tilde{w}) \qquad (n>1),\\
&{}_i\tau_\omega^{(x)}(\tilde{w}) = \lim_{n\to\infty} {}_i\tau_n^{(x)}(\tilde{w})\,.
\end{aligned}
$$

A new stochastic process $X_t^{(x)}(\tilde{w})$ is defined as

(7.6)
$$
\begin{aligned}
X_t^{(x)} &= {}_0Y_t^{(x)} \quad \text{if} \quad 0 \leqq t < \tau_\omega^{(x)} \equiv {}_0\tau_\omega^{(x)}\,,\\
&\qquad Z_1^{(x)} = {}_1Z^{\left(\tilde{X}_1^{(x)}\right)} \quad \text{where} \quad \tilde{X}_1^{(x)} \equiv \lim_{t\uparrow\tau_\omega^{(x)}} X_t^{(x)}\,,\\
&= {}_nY^{\left(z_n^{(x)}\right)}_{t-\tau_{\omega n}^{(x)}} \quad \text{if} \quad \tau_{\omega n}^{(x)} \leqq t < \tau_{\omega(n+1)}^{(x)} \equiv \tau_{\omega n}^{(x)} + {}_n\tau_\omega^{\left(z_n^{(x)}\right)}\\
&\qquad Z_{n+1}^{(x)} = {}_{n+1}Z^{\left(\tilde{X}_{n+1}^{(x)}\right)} \quad \text{where} \quad \tilde{X}_{n+1}^{(x)} \equiv \lim_{t\uparrow\tau_{\omega(n+1)}^{(x)}} X_t^{(x)}\\
&= \infty \quad \text{if} \quad t \geqq \tau_{\omega 2}^{(x)} \equiv \lim_{n\to\infty} \tau_{\omega n}^{(x)}\,.
\end{aligned}
$$

Then $X^{(x)}(\tilde{w}) = (X_t^{(x)}(\tilde{w})\,;\, t \in [0, +\infty])$ is also a right continuous function. Let W and $\mathfrak{B}_W$ be the space of paths and its Borel field defined in Section 1. Put

(7.7)
$$
P_x(B) = \tilde{P}(\tilde{w}\,;\, X^{(x)}(\tilde{w}) \in B) \qquad \text{for} \quad B \in \mathfrak{B}_W\,,
$$

then $\boldsymbol{M} = (X, W, \mathfrak{B}_W, P_x\,;\, x \in X)$ is the first instantaneous return process whose generator's domain is given by

(7.8)
$$
\mathfrak{D}(\mathfrak{G}) = \left\{ \begin{array}{l} u\,;\, (\alpha)\ u, \tilde{\mathfrak{G}}u \in \mathfrak{B}(X),\ (\beta)\ \tilde{u} - \sum\limits_{y \geqq X} \tilde{\Pi}(b,y)u(y) = 0 \ \text{and} \\ (\gamma)\ \lim\limits_{n\to\infty} {}_{(2)}\Pi_\alpha^n u = 0 \end{array} \right\},
$$

where $\tilde{u}$ is the boundary value of u and ${}_{(2)}\Pi_\alpha(x,y) = \int_{(\partial X)_e} \tilde{\Pi}(b,y)h_\alpha(x,db)$.

It is clear that $\boldsymbol{M}$ satisfies the conditions (P.1), (P.2) and $P_x(\tau_{\omega 2} \geqq \sigma_\infty) = 1$ for every $x \in X$. Moreover, since ${}_iY_t^{(x)}$ and ${}_iZ^{(b)}$ are independent, we have

(7.9)
$$
\begin{aligned}
P_x(x_{\tau_\omega} \in A, \tau_\omega < \infty) &= \tilde{P}({}_iZ^{(x)} \in A, \tau_\omega^{(x)} < \infty)\\
&= \int_{(\partial X)_e} \tilde{P}({}_1Z^{(b)} \in A)\tilde{P}(\tilde{X}_1^{(x)} \in db) = \int_{(\partial X)_e} \tilde{\Pi}(b,A)h(x,db) = h\tilde{\Pi}(x,A)\,.
\end{aligned}
$$

Therefore if we prove that $\boldsymbol{M}$ satisfies (P.3) i.e. the Markov property, our process $\boldsymbol{M}$ is the first instantaneous return (Markov) process having $\mathfrak{D}(\mathfrak{G})$ of (7.8) as its generator's domain. To prove that our process $\boldsymbol{M}$ satisfies the Markov property, it is enough to show for every $0 \leqq t_1 < t_2 < \cdots < t_n, y_1, y_2, \cdots, y_n \in X^*$

(7.10)
$$
P_x(\Lambda_n) = P_x(x_{t_1} = y_1)P_{y_1}(x_{t_2-t_1} = y_2) \cdots P_{y_{n-1}}(x_{t_n - t_{n-1}} = y_n)\,,
$$

where $\Lambda_n = \{w\,;\, x_{t_1} = y_1, \cdots, x_{t_n} = y_n\}$ (see K. Itô [7]). We shall prove the above relation following the technique due to K. L. Chung [**0**].

LEMMA 7.1. *The following formulae hold.*

$$(7.11)\quad P_x(\Lambda_n\,;\,\tau_{\omega l}\leqq t_{n-1}<\tau_{\omega(l+1)},\,\tau_{\omega m}\leqq t_n<\tau_{\omega(m+1)})$$
$$=P_x(\Lambda_{n-1}\,;\,\tau_{\omega l}\leqq t_{n-1}<\tau_{\omega(l+1)})$$
$$\times P_{y_{n-1}}(x_{t_n-t_{n-1}}=y_n\,;\,\tau_{\omega(m-l)}\leqq t_n-t_{n-1}<\tau_{\omega(m-l+1)})\quad (l<m)\,,$$

$$(7.12)\quad P_x(\Lambda_n\,;\,\tau_{\omega m}\leqq t_{n-1}<t_n<\tau_{\omega(m+1)})$$
$$=P_x(\Lambda_{n-1}\,;\,\tau_{\omega m}\leqq t_{n-1}<\tau_{\omega(m+1)})$$
$$\times P_{y_{n-1}}(x_{t_n-t_{n-1}}=y_n\,;\,\tau_\omega>t_n-t_{n-1})\,.$$

PROOF. We shall only prove (7.11). The proof of (7.12) is similar and simpler than that of (7.11). First we shall show the following formula.

$$(7.13)\quad P_x(\Lambda_{n-1}\,;\,\tau_{\omega l}\leqq t_{n-1}<\tau_{\omega(l+1)},\,x_{\tau_{\omega(l+1)}}=y,\,\tau_{\omega(l+1)}\leqq u)$$
$$=P_x(\Lambda_{n-1}\,;\,\tau_{\omega l}\leqq t_{n-1}<\tau_{\omega(l+1)})P_{y_{n-1}}(\tau_\omega\leqq u-t_{n-1},\,x_{\tau_\omega}=y)\,.$$

Put $\Lambda_n^{(x)}=\{\tilde{w}\,;\,X_{t_1}^{(x)}(\tilde{w})=y_1,\cdots,X_{t_n}^{(x)}(\tilde{w})=y_n\}$ and ${}_l\tilde{Y}^{(x)}=\lim_{t\uparrow{}_l\tau_\omega^{(x)}}{}_lY_t^{(x)}$, then the left member of (7.13) is equal to

$$\sum_{i=1}^{n-3}\tilde{P}(\Lambda_{n-1}^{(x)}\,;\,t_i<\tau_{\omega l}^{(x)}\leqq t_{i+1}<t_{n-1}<\tau_{\omega(l+1)}^{(x)},\,X_{\tau_{\omega(l+1)}^{(x)}}^{(x)}=y,\,\tau_{\omega(l+1)}^{(x)}\leqq u)$$
$$=\sum_{i=1}^{n-3}\sum_{z\in X^*}\int_0^{t_{1+i}}\tilde{P}(\{\Lambda_i^{(x)}\,;\,t_i<\tau_{\omega l}^{(x)},\,X_{\tau_{\omega l}^{(x)}}^{(x)}=z,\,\tau_{\omega l}^{(x)}\in ds\}$$
$$\cap\{{}_lY_{t_j-s}^{(z)}=y_j\ (i+1\leqq j<n)\ {}_{l+1}Z^{({}_{l+1}\tilde{Y}^{(z)})}=y,\,{}_l\tau_\omega^{(z)}\leqq u-s\})$$
$$=\sum_{i=1}^{n-3}\sum_{z\in X^*}\int_0^{t_{i+1}}\tilde{P}(\Lambda_i^{(x)}\,;\,t_i<\tau_{\omega l}^{(x)},\,X_{\tau_{\omega l}^{(x)}}^{(x)}=z,\,\tau_{\omega l}^{(x)}\in ds)$$
$$\times\tilde{P}({}_lY_{t_j-s}^{(z)}=y_j\ (i+1\leqq j<n)\ {}_{l+1}Z^{({}_{l+1}\tilde{Y}^{(z)})}=y,\,{}_l\tau_\omega^{(z)}\leqq u-s)$$
$$=\sum_{i=1}^{n-3}\sum_{z\in X^*}\int_0^{t_{i+1}}\tilde{P}(\Lambda_i^{(x)}\,;\,t_i<\tau_{\omega l}^{(x)},\,X_{\tau_{\omega l}^{(x)}}^{(x)}=z,\,\tau_{\omega l}^{(x)}\in ds)$$
$$\times\int_{(\partial X)_e}\tilde{P}({}_lY_{t_j-s}^{(z)}=y_j\,(i+1\leqq j<n)\,;\,{}_l\tau_\omega^{(z)}\leqq u-s,\,{}_l\tilde{Y}^{(z)}\in db)\tilde{\Pi}(b,y)\,.$$

Using the Markov property of ${}_lY_t^{(z)}$, the above member is

$$\sum_{i=1}^{n-3}\sum_{z\in X^*}\int_0^{t_{i+1}}\tilde{P}(\Lambda_i^{(x)}\,;\,t_i<\tau_{\omega l}^{(x)},\,X_{\tau_{\omega l}^{(x)}}^{(x)}=z,\,\tau_{\omega l}^{(x)}\in ds)\tilde{P}({}_lY_{t_j-s}^{(z)}=y_j\ (i+1\leqq j<n))$$
$$\times\int_{(\partial X)_e}\tilde{P}({}_l\tau_\omega^{(y_{n-1})}\leqq u-t_{n-1},\,{}_l\tilde{Y}^{(y_{n-1})}\in db)\tilde{\Pi}(b,y)\,.$$

Noting

$$(7.14)\quad \int_{(\partial X)_e}\tilde{P}({}_l\tau_\omega^{(y_{n-1})}\leqq u-t_{n-1},\,{}_l\tilde{Y}^{(y_{n-1})}\in db)\tilde{\Pi}(b,y)$$
$$=P_{y_{n-1}}(\tau_\omega\leqq u-t_{n-1},\,x_{\tau_\omega}=y)$$

and retracing our step, we have the formula (7.13).

Here we shall define the stochastic process ${}_iX_t^{(x)}(\tilde{w})$ having the same probability law as $X_t^{(x)}(\tilde{w})$ by

$$
\begin{aligned}
(7.15)\qquad {}_iX_t^{(x)} &= {}_iY_t^{(x)} \quad \text{if} \quad 0 \leqq t < {}_i\tau_\omega^{(x)} \\
{}_iZ_1^{(x)} &= {}_iZ^{({}_i\tilde{X}_1^{(x)})} \quad \text{where} \quad {}_i\tilde{X}_1^{(x)} \equiv \lim_{t \uparrow {}_i\tau_\omega^{(x)}} {}_iX_t^{(x)} \\
&= {}_{i+n}Y_{t-{}_i\tau_{\omega n}^{(x)}}^{({}_iZ_n^{(x)})} \quad \text{if} \quad {}_i\tau_{\omega n}^{(x)} \leqq t < {}_i\tau_{\omega(n+1)}^{(x)} \equiv {}_i\tau_{\omega n}^{(x)} + {}_{n+i}\tau_\omega^{({}_iZ_n^{(x)})} \\
{}_iZ_{n+1}^{(x)} &= {}_{n+i}Z^{({}_i\tilde{X}_{n+1}^{(x)})} \quad \text{where} \quad {}_i\tilde{X}_{n+1}^{(x)} \equiv \lim_{t \uparrow {}_i\tau_{\omega(n+1)}^{(x)}} {}_iX_t^{(x)} \\
&= \infty \quad \text{if} \quad t > {}_i\tau_{\omega 2}^{(x)} \equiv \lim {}_i\tau_{\omega n}^{(x)} .
\end{aligned}
$$

Then the left member of (7.11) is equal to

$$
\begin{aligned}
&\sum_{y \in X^*} \tilde{P}(\Lambda_{n-1}^{(x)} ; \tau_{\omega l}^{(x)} \leqq t_{n-1} < \tau_{\omega(l+1)}^{(x)}, X_{\tau_{\omega(l+1)}^{(x)}}^{(x)} = y, X_{t_n}^{(x)} = y_n, \tau_{\omega m}^{(x)} \leqq t_n < \tau_{\omega(m+1)}^{(x)}) \\
&= \sum_{y \in X^*} \int_{t_{n-1}}^{t_n} \tilde{P}(\{\Lambda_{n-1}^{(x)} ; \tau_{\omega l}^{(x)} \leqq t_{n-1} < \tau_{\omega(l+1)}^{(x)}, X_{\tau_{\omega(l+1)}^{(x)}}^{(x)} = y, \tau_{\omega(l+1)}^{(x)} \in du\} \\
&\qquad\qquad \cap \{{}_{l+1}X_{t_n-u}^{(y)} = y_n, {}_{l+1}\tau_{(m-l-1)}^{(y)} \leqq t_n - u < {}_{l+1}\tau_{(m-l)}^{(y)}\}) \\
&= \sum_{y \in X^*} \int_{t_{n-1}}^{t_n} \tilde{P}(\{\Lambda_{n-1}^{(x)} ; \tau_{\omega l}^{(x)} \leqq t_{n-1} < \tau_{\omega(l+1)}^{(x)}, X_{\tau_{\omega(l+1)}^{(x)}}^{(x)} = y, \tau_{\omega(l+1)}^{(x)} \in du\}) \\
&\qquad\qquad \times \tilde{P}({}_{l+1}X_{t_n-u}^{(y)} = y_n, {}_{l+1}\tau_{\omega(m-l-1)}^{(y)} \leqq t_n - u < {}_{l+1}\tau_{\omega(m-l)}^{(y)}) .
\end{aligned}
$$

Substituting (7.13) to the last member, it becomes

$$
\begin{aligned}
&P_x(\Lambda_{n-1} ; \tau_{\omega l} \leqq t_{n-1} < \tau_{\omega(l+1)}) \cdot \sum_{y \in X^*} \int_{t_{n-1}}^{t_n} \tilde{P}({}_l\tau_\omega^{(y_{n-1})} + t_{n-1} \in du, {}_lX_{{}_l\tau_\omega^{(y_{n-1})}}^{(y_{n-1})} = y) \\
&\qquad\qquad \times \tilde{P}({}_{l+1}X_{t_n-u}^{(y)} = y_n, {}_{l+1}\tau_{(m-l-1)}^{(y)} \leqq t_n - u < {}_{l+1}\tau_{\omega(m-l)}^{(y)}) \\
&= P_x(\Lambda_{n-1} ; \tau_{\omega l} \leqq t_{n-1} < \tau_{\omega(l+1)}) \\
&\quad \times \sum_{y \in X^*} \tilde{P}({}_lX_{{}_l\tau_\omega^{(y_{n-1})}}^{(y_{n-1})} = y, {}_{l+1}X_{t_n-t_{n-1}-{}_l\tau_\omega^{(y_{n-1})}}^{(y)} = y_n, {}_l\tau_\omega^{(y_{n-1})} + {}_{l+1}\tau_{\omega(m-l-1)}^{(y)} \\
&\qquad\qquad\qquad\qquad = t_n - t_{n-1} < {}_l\tau_\omega^{(y_{n-1})} + {}_{l+1}\tau_{\omega(m-l)}^{(y)}) .
\end{aligned}
$$

If we put ${}_lZ_1^{(y_{n-1})} = {}_lX_{{}_l\tau_\omega^{(y_{n-1})}}^{(y_{n-1})}$, ${}_{l+1}X_{t_n-t_{n-1}-{}_l\tau_\omega^{(y_{n-1})}}^{({}_lZ_1^{(y_{n-1})})} = {}_lX_{t_n-t_{n-1}}^{(y_{n-1})}$ and ${}_l\tau_\omega^{(y_{n-1})} + {}_{l+1}\tau_{\omega(m-l-1)}^{({}_lZ_1^{(y_{n-1})})} = {}_l\tau_{\omega(m-l)}^{(y_{n-1})}$ hold by definition. Hence the last member is

$$
P_x(\Lambda_{n-1} ; \tau_{\omega l} \leqq t_{n-1} < \tau_{\omega(l+1)}) \tilde{P}({}_lX_{t_n-t_{n-1}}^{(y_{n-1})} = y_n, {}_l\tau_{\omega(m-l)}^{(y_{n-1})} \leqq t_n - t_{n-1} < \tau_{\omega(m-l+1)}^{(y_{n-1})}) .
$$

Thus we have proved the formula (7.11).

We now return to the proof of the Markov property. Using Lemma 7.1, we have

$$(7.16)\quad P_x(A_n)=\sum_{m\geqq n\geqq 0}P_x(A_n\,;\,\tau_{\omega l}\leqq t_{n-1}<\tau_{\omega(l+1)},\tau_{\omega m}\leqq t_n<\tau_{\omega(m+1)})$$

$$=\sum_{m=l\geqq 0}P_x(A_{n-1}\,;\,\tau_{\omega m}\leqq t_{n-1}<\tau_{\omega(m+1)})\cdot P_{y_{n-1}}(x_{t_n-t_{n-1}}=y_n,\,t_n-t_{n-1}<\tau_\omega)$$

$$+\sum_{m>l\geqq 0}P_x(A_{n-1}\,;\,\tau_{\omega l}\leqq t_{n-1}<\tau_{\omega(l+1)})P_{y_{n-1}}(x_{t_n-t_{n-1}}=y_n,\,\tau_{\omega(m-l)}\leqq t_n-t_{n-1}<\tau_{\omega(m-l+1)})$$

$$=\sum_{l=0}^{\infty}P_x(A_{n-1}\,;\,\tau_{\omega l}\leqq t_{n-1}<\tau_{\omega(l+1)})P_{y_{n-1}}(x_{t_n-t_{n-1}}=y_n)$$

$$=P_x(A_{n-1})P_{y_{n-1}}(x_{t_n-t_{n-1}}=y_n)\,.$$

Repeating our argument, we get (7.10).

Thus we have proved Theorem 7.1 for the case $k=0$.

8. Examples

I. Let $X=\{0,1,2,3,\cdots\}$ and $x^0(t)$ be the minimal process over X satisfying

$$(8.1)\qquad \Pi(0,1)=\Pi(0,2)=\frac{1}{2}\,,\qquad \Pi(2n,2n+2)=1\,,$$

$$\Pi(2n-1,2n)=\Pi(2n-1,2n+1)=\frac{1}{2}\qquad (n=1,2,\cdots)\,.$$

$$(8.2)\qquad q_0=1\,,\qquad q_{2n-1}=2^{-n},\qquad q_{2n}=2^{+n}\qquad (n=1,2,\cdots)\,.$$

In this process, 0 is the unique center and K-function $K(0,x,y)$ is given by

$$(8.3)\qquad K(0,2m,2n-1)=0\qquad \text{if}\quad m\geqq 1\,.$$

$$K(0,2m,2n)=\begin{cases}0 & \text{if}\quad n<m\,,\\ \dfrac{1}{1-2^{-(n+1)}} & \text{if}\quad n\geqq m\geqq 1\,.\end{cases}$$

$$K(0,2m-1,2n)=\begin{cases}0 & \text{if}\quad n<m\ (m\geqq 1)\,,\\ \dfrac{1-2^{m-(n+1)}}{1-2^{-(n+1)}} & \text{otherwise}\,.\end{cases}$$

$$K(0,2m-1,2n-1)=\begin{cases}0 & \text{if}\quad n<m\,,\\ 2^m & \text{if}\quad n\geqq m\,.\end{cases}$$

Obviously $\{2n\}$ and $\{2n-1\}$ are two fundamental sequences which are not equivalent to each other. Denote the boundary points determined by $\{2n\}$ and $\{2n-1\}$ by b_1 and b_2. Then

$$(8.4)\qquad K(0,x,b_1)=1\,.$$

$$K(0,x,b_2)=\begin{cases}0 & \text{if}\quad x \text{ is even}\,,\\ 1 & \text{if}\quad x \text{ is } 0\,,\\ 2^m & \text{if}\quad x=2m-1\,.\end{cases}$$

From the definition of réduite, we get $h(x,b_1)=1$ and $h(x,b_2)=0$. Now, operating Π_α^n to $K(0, x, b_1)$, we get

$$\Pi_\alpha^n K(0, 2m, b_1) = \frac{2^m}{\alpha+2^m} \cdots \frac{2^{m+n-1}}{\alpha+2^{m+n-1}} = \frac{1}{\prod_{k=0}^{n-1}\left(1+\frac{\alpha}{2^{m+k}}\right)}. \tag{8.5}$$

Since $\sum_{k=0}^{\infty} \frac{\alpha}{2^{m+k}} < \infty$ implies that $\prod_{k=0}^{\infty}\left(1+\frac{\alpha}{2^{m+k}}\right) < +\infty$, $K_\alpha(0, 2m, b_1) > 0$ and therefore b_1 is an exit boundary point. Next operating Π_α^n to $K(0, x, b_2)$, we get

$$\Pi_\alpha^n K(0, 0, b_2) = \left(\frac{1}{2}\right)^n \prod_{k=0}^{n-1} \frac{2^{-k}}{\alpha+2^{-k}} \cdot 2^n = \frac{1}{\prod_{k=0}^{n-1}(1+\alpha 2^k)} \xrightarrow[n\to\infty]{} 0, \tag{8.6}$$

so that $K_\alpha(0, 0, b_2) = 0$, namely, b_2 is a passive boundary point. Since $h(x, b_1) = 1$, we get $P_x(\tau_\omega < \infty) = h(x, b_1) = 1$.

REMARK. A function $s_A(x)$ defined by

$$s_A(x) = P_x(\varliminf_{n\to\infty} (x_{\tau_n} \in A)) \tag{8.7}$$

is called a sojourn solution unless it is identically zero (see W. Feller [4]). In the above example, constant 1 is the only one sojourn solution because every bounded $x^0(t)$-harmonic function is constant. Therefore the Feller boundary consists of one point. According to the notation of [5], the canonical image of $X_\alpha(x) = \lim_{n\to\infty} \Pi_\alpha^n 1(x)$ is denoted by $\bar{X}(x)$. By the similar calculation as in the proof of Theorem 3.2, we get $X_\alpha(x) = E_x(e^{-\alpha\tau_\omega}) > 0$ for some x. Hence the boundary point b_1 is exit in the sense of [5].

II. A simple example of the first instantaneous return process is this: Let $x^0(t)$ be an arbitrary minimal Markov process with $\{q, \Pi\}$ satisfying (P.5) and $(\partial X)_e$ be its exit boundary points. $\tilde{\Pi}(b, y)$ is a probability measure on X^* which does not depend on any $b \in (\partial X)_e$. Then as in the previous section we can construct the first instantaneous return porcess $x^0(t)$ by giving $\{q, \Pi, \tilde{\Pi}\}$. Since this process satisfies $P_x(\tau_{\omega 2} = \infty) = 1$ (see Appendix), we need not consider any instantaneous return process of higher than first order. This process was treated in [1].

III. Let $x^0(t)$ be an arbitrary minimal Markov process with $\{q, \Pi\}$ satisfying (P.5), and $(\partial X)_e$ be its exit boundary. Assume that $\tilde{\Pi}(b, \cdot)$ coincides with each another for every $b \in B_1$ and $b \in (\partial X)_e - B_1$, and $\tilde{\Pi}(b, \infty) = 1$ for $b \in (\partial X)_e - B_1$. Moreover we change notations as follows: $\tilde{\Pi}(\cdot) = \tilde{\Pi}(b, \cdot)$ for $b \in B_1$, $\tilde{\Pi}(X) = \frac{1}{m+1}$, $\alpha(y) = (m+1)\tilde{\Pi}(y)$, $X_\alpha(x) = h_\alpha(x, B_1)$ and $A_\alpha(x) = \sum_{y\in X} \alpha(y) G_\alpha^0 \chi_{\{y\}}(y) = \sum_{y\in X} \alpha(y) G_\alpha^0(y, x)$. Then from Theorem 2 of Appendix the Green kernel of the first instantaneous return process having the factors $\{q, \Pi, \tilde{\Pi}\}$ is

(8.8)
$$G_\alpha\chi_{\{y\}}(x)=G_\alpha^0(x,y)+X_\alpha(x)\frac{1}{1-\sum_{z\in X}\tilde{\Pi}(b,z)X_\alpha(z)}\times\sum_{z'\in X}\tilde{\Pi}(b,z')G_\alpha^0(z',y)$$
$$=G_\alpha^0(x,y)+X_\alpha(x)\frac{A_\alpha(y)}{m+\sum_{y\in X}\alpha(z)(1-X_\alpha(z))}\,.$$

This process also satisfies the condition $P_x(\tau_{\omega 2}=\infty)=1$. The above formula (8.8) has been obtained in [5] analytically. But the definition of the instantaneous return process of [5] is a little wider than ours. The case $\sum_{y\in X}\alpha(y)=\infty$ is not contained in our definition.

Although Examples II and III satisfy the condition $P_x(\tau_{\omega 2}=\infty)=1$, there exists surely a process such that $P_x(\tau_{\omega 2}<\infty)>0$ holds. In fact.

IV. *A modification of dyadic branching scheme.* Let $X=\{c,0,1,00,01,10,11,\cdots,\delta,\delta 0,\delta 1,\cdots$ where $\delta=a_1\cdots a_k$ $(a_i=0$ or $1)\}$, and $x^0(t)$ be the minimal process satisfying $\Pi(\delta,\delta 0)=\Pi(\delta,\delta 1)=\frac{1}{2}$, $q_c=1$ and $q_\delta=2^k$ (if $\delta=a_1\cdots a_k$). According to [12], its Martin boundary (∂X) is the countable infinite direct product of the compact Hausdorff space consisting of two points 0 and 1. Every point of ∂X is denoted by $b=b_1b_2\cdots(b_i=0$ or $1)$. We shall make such b coorrespond to the real value $\left(\frac{b_1}{2}+\cdots+\frac{b_n}{2^n}+\cdots\right)$. But, since $b=b_1\cdots b_n1000\cdots$ and $b'=b_1\cdots b_n0111\cdots$ are different points in (∂X), we shall in such case make b correspond to $\left(\frac{b_1}{2}+\cdots+\frac{b_n}{2^n}+\frac{1}{2^{n+1}}\right)+$ and b' to $\left(\frac{b_1}{2}+\cdots+\frac{b_n}{2^n}+\frac{1}{2^{n+1}}\right)-$. If $b=b_1\cdots b_n\cdots$, $K(c,x,b)$ is

(8.10)
$$K(c,x,b)=2^k\qquad\text{if}\quad x=b_1\cdots b_k,$$
$$=0\qquad\text{otherwise.}$$

Moreover, by the similar calculation as Example I, we get

(8.11)
$$K_\alpha(c,c,b)=\frac{1}{\lim_{n\to\infty}\prod_{k=0}^{n-1}\left(1+\frac{\alpha}{2^k}\right)}=\frac{1}{\prod_{k=0}^{\infty}\left(1+\frac{\alpha}{2^k}\right)}>0\,.$$

Therefore $(\partial X)=(\partial X)_e$. Next, we shall define $\tilde{\Pi}(b,y)$ as follows;

(8.12)
$$\Pi(b,a_1\cdots a_n0)=1\qquad\text{if}\quad b\in\left[\frac{2^n-2}{2^n}+,\frac{2^n-1}{2^n}-\right]$$
$$\text{where}\qquad a_i=1\qquad\text{for}\qquad 1\leqq i\leqq n\,.$$

Let $x^1(t)$ denote the first instantaneous return process corresponding to the above $\{q,\Pi,\tilde{\Pi}\}$. We have

(8.13) $h\left(c,\left[\frac{2^n-2}{2^n}+,\frac{2^n-1}{2^n}-\right]\right)=\frac{1}{2^n}$,

$h\left(a_1\cdots a_{n-1}0,\left[\frac{2^n-2}{2^n}+,\frac{2^n-1}{2^n}-\right]\right)=1$ if $a_i=1$ for $1\leqq i\leqq n-1$,

and

(8.14) $h_\alpha\left(x,\left[\frac{2^n-2}{2^n}+,\frac{2^n-1}{2^n}-\right]\right)=\frac{1}{\prod_{k=n}^{\infty}\left(1+\frac{\alpha}{2^k}\right)}>0$, if $x=a_1\cdots a_{n-1}0$ $(a_i=1)$,

$h_\alpha\left(x,\left[\frac{2^n-2}{2^n}+,\frac{2^n-1}{2^n}-\right]^c\right)=0$, if $x=a_1\cdots a_{n-1}0$ $(a_i=1)$.

The kernels ${}_{(2)}\Pi$ and ${}_{(2)}\Pi_\alpha$ become

(8.15) ${}_{(2)}\Pi(c,y)=2^{-2n}$ if $y=a_1\cdots a_{n-1}0$ $(a_i=1)$,

$=0$ otherwise.

${}_{(2)}\Pi(x,y)=1$ if $x=a_1\cdots a_{n-1}0$, $y=a_1\cdots a_n0$ $(a_1=1)$.

(8.16) ${}_{(2)}\Pi_\alpha(x,y)=\frac{1}{\prod_{k=n}^{\infty}\left(1+\frac{\alpha}{2^k}\right)}$ if $x=a_1\cdots a_{n-1}0$ and $y=a_1\cdots a_n0$ $(a_i=1)$,

$=0$ otherwise.

Therefore the set ${}_{(2)}X$ which is covered with the center c with respect to ${}_{(2)}\Pi$ is $\{c, a_1\cdots a_n0;\ n=1,2,\cdots,$ and $a_i=1\}$. The corresponding ${}_{(2)}K$-function is

(8.17) ${}_{(2)}K(c,x,y)=1$ if $x\leqq y$,

$=0$ if $x>y$.

Therefore $\{a_1\cdots a_n0\}$ $(n=1,2,\cdots)$ is the unique fundamental sequence and the corresponding second boundary is one point ${}_{(2)}b$. Moreover,

(8.18) ${}_{(2)}K(c,x,b)=1$ for all $x\in{}_{(2)}X$.

Operating ${}_{(2)}\Pi_\alpha^m$ to ${}_{(2)}K$, we get

(8.19) $$\sum_{y\in_{(2)}X}{}_{(2)}\Pi_\alpha^m(10,y)K(c,y,{}_{(2)}b)=\prod_{n=1}^{m}\frac{1}{\prod_{k=n+1}^{\infty}\left(1+\frac{\alpha}{2^k}\right)}$$

$$=\frac{1}{\prod_{k=1}^{m}\left(1+\frac{\alpha}{2^k}\right)^{k-1}\prod_{k'=m+1}^{\infty}\left(1+\frac{\alpha}{2^{k'}}\right)^m}\xrightarrow[m\to\infty]{}\frac{1}{\prod_{k=1}^{\infty}\left(1+\frac{\alpha}{2^k}\right)^k}>0,$$

because $\sum_{k=1}^{\infty}\frac{\alpha k}{2^k}<\infty$. Therefore $K_\alpha(c,10,{}_{(2)}b)>0$, that is, ${}_{(2)}b$ is the second exit boundary point. Further in this case

(8.18) $P_x(\tau_{\omega 2}<\infty)={}_{(2)}h(x,(\partial_{(2)}X)_e)={}_{(2)}h(x,{}_{(2)}b)=1$.

Hence we can construct the second instantaneous return process by giving an arbitrary measure ${}_{(2)}\tilde{\Pi}({}_{(2)}b,\cdot)$ on X^*. But since the second exit boundary is one point, all the second instantaneous return processes $x^2(t)$ satisfy $P_x(\tau_{\omega 3}=\infty)=1$ for any choice of ${}_{(2)}\tilde{\Pi}({}_{(2)}b,\cdot)$[13]. But if we choose $\tilde{\Pi}(b,\cdot)$ suitably, the second exit boundary has infinitely many points and moreover if the ${}_{(2)}\tilde{\Pi}(b,\cdot)$'s are suitably chosen, the third exit boundary points will be not void. Thus, in such case, we shall be able to construct the third instantaneous return processes actually.

Appendix. On the processes satisfying the condition $P_x(\tau_{\omega k+1}=\infty)=1$.

Let $x^k(t)$ be the k-th instantaneous return process with the factors $\{q,\Pi,\tilde{\Pi},\cdots,{}_{(k)}\tilde{\Pi}\}$. In general, since $P_x(\tau_{\omega k+1}=\infty)<1$ or equivalently since ${}_{(k+1)}h(c,(\partial_{(k+1)}X)_e)>0$,[14] we can construct infinitely many $(k+1)$-th instantaneous return processes with the same $\{q,\Pi,\tilde{\Pi},\cdots,{}_{(k)}\tilde{\Pi}\}$. In other words, $\{q,\Pi,\tilde{\Pi},\cdots,{}_{(k)}\tilde{\Pi}\}$ do not determine the Markov process with right continuous paths, though these factor determine the k-th instantaneous return process uniquely. But if the relation $P_x(\tau_{\omega k+1}=\infty)=1$ is satisfied, our process $x^k(t)$ is maximal in the sense that if $x(t)$ is a Markov process with right continuous paths having the same $\{q,\Pi,\tilde{\Pi},\cdots,{}_{(k)}\tilde{\Pi}\}$, $x(t)$ is nothing but $x^k(t)$. In this section we shall give some useful conditions under which $P_x(\tau_{\omega k+1}=\infty)=1$ holds.

Let $\mathfrak{B}((\partial_{(k)}X)_e)$ be the family of all the bounded Borel functions over $\partial_{(k)}X\cup\{\infty\}$ vanishing except for $(\partial_{(k)}X)_e$. We shall define the operator ${}_{(k)}V_\alpha$ from $\mathfrak{B}((\partial_{(k)}X)_e)$ into itself as

(1) $$ {}_{(k)}V_\alpha\tilde{f}(b)=\sum_{y\in X}{}_{(k)}\tilde{\Pi}(b,y)\,{}_{(k)}h_\alpha\tilde{f}(y), \qquad \text{for } \tilde{f}\in\mathfrak{B}((\partial_{(k)}X)_e). $$

Then

(2) $$ \begin{aligned} {}_{(k+1)}\Pi_\alpha^n(x,y) &= \int_{(\partial_{(k)}X)_e}{}_{(k)}h_\alpha(x,db)\,{}_{(k)}V_\alpha^{n-1}\,{}_{(k)}\tilde{\Pi}(b,y) \\ &= {}_{(k)}h_\alpha\,{}_{(k)}V_\alpha^{n-1}\,{}_{(k)}\tilde{\Pi}(x,y), \end{aligned} $$

where ${}_{(k)}V_\alpha^{n-1}$ is the iterated operator of ${}_{(k)}V_\alpha$. Particularly, $(\partial_{(1)}X)_e$, ${}_{(1)}\tilde{\Pi}$, ${}_{(1)}h_\alpha$ and ${}_{(1)}V_\alpha$ express $(\partial X)_e$, $\tilde{\Pi}$, h_α and V_α respectively.

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13) See Appendix.

14) It is easy to show that analogously to Theorem 3.2 or (6.13), the relations

$$ P_x(\tau_{\omega k+1}<\infty\,|\,x_{\tau_{\omega k+1}-}\in(\partial_{(k+1)}X)_e)=1, $$
$$ P_x(\tau_{\omega k+1}=\infty\,|\,x_{\tau_{\omega k+1}-}\in(\partial_{(k+1)}X)_p)=1 $$

hold for any k, where $x_{\tau_{\omega k+1}-}(w)$ is $\lim_{n\to\infty}x_{\tau_{\omega k_n}}(w)$ with respect to the canonical Martin space of k-th order. Therefore

$$ P_x(\tau_{\omega k+1}<\infty)={}_{(k+1)}h(x,(\partial_{(k+1)}X)_e)=\int_{(\partial_{(k+1)}X)_e}{}_{(k+1)}K(c,x,b)\,{}_{(k+1)}h(c,db) $$

which shows the equivalence of $P_x(\tau_{\omega k+1}=\infty)=1$ (for all $x\in X$) and $h(c,(\partial_{(k+1)}X)_e)=0$.

THEOREM 1. *The following conditions are equivalent.*

(i) $P_x(\tau_{\omega^{k+1}}=\infty)=1$ *for every* $x \in X$.

(ii) $\lim_{n\to\infty} {}_{(k+1)}\Pi_\alpha^n(x, X^*)=0$,

(iii) $\lim_{n\to\infty} {}_{(k)}V_\alpha^n 1(b)=0$.

(iv) $(I-{}_{(k)}V_\alpha)\tilde{f}=\tilde{g}$ has at most one solution $\tilde{f}$ for any $\tilde{g}\in \mathfrak{B}((\partial_{(k)}X)_e)$.

PROOF. When $P_x(\tau_{\omega^k}=\infty)=1$ for all $x\in X$, this theorem is trivial. So we assume $P_x(\tau_{\omega^k}<\infty)>0$ for some x, which is equivalent to ${}_{(k+1)}\Pi_\alpha(c, X^*)>0$. Since $E_x(e^{-\alpha\tau_{\omega^{k+1}}})=\lim_{n\to\infty} {}_{(k+1)}\Pi_\alpha^n(x, X^*)$, the equivalence of (i) and (ii) is clear. (ii)$\leftrightarrow$(iii). From (2), we have

$$\lim_{n\to\infty} {}_{(k+1)}\Pi_\alpha^n(x, X^*)=\lim_{n\to\infty} {}_{(k)}h_\alpha {}_{(k)}V_\alpha^{n-1}1(x)={}_{(k)}h_\alpha[\lim_{n\to\infty} {}_{(k)}V_\alpha^{n-1}1](x)\,. \tag{3}$$

Therefore $\lim_{n\to\infty} {}_{(k)}V_\alpha^{n-1}1(b)=0$ is equivalent to $\lim_{n\to\infty} {}_{(k+1)}\Pi_\alpha^n(x, X^*)=0$. (ii)$\to$(iv). If $(I-{}_{(k)}V_\alpha)\tilde{f}=\tilde{g}$ has two solutions $\tilde{f}_1$ and $\tilde{f}_2$ for some $\tilde{g}\in\mathfrak{B}((\partial_{(k)}X)_e)$, $\tilde{f}=\tilde{f}_1-\tilde{f}_2$ satisfies $(I-{}_{(k)}V_\alpha)\tilde{f}=0$, i.e.,

$$|\tilde{f}|=|{}_{(k)}V_\alpha\tilde{f}|=\cdots=|{}_{(k)}V_\alpha^n\tilde{f}|\leqq\|\tilde{f}\|\,{}_{(k)}V_\alpha^n 1\xrightarrow[n\to\infty]{}0\,. \tag{4}$$

(iv)$\to$(iii). Since $\lim_{n\to\infty} {}_{(k)}V_\alpha^n 1(b)$ satisfies $(I-{}_{(k)}V_\alpha)\lim_{n\to\infty} {}_{(k)}V_\alpha^n 1=0$, $\lim_{n\to\infty} {}_{(k)}V_\alpha^n 1=0$.

COROLLARY. *If the norm of* ${}_{(k)}V_\alpha$ *is smaller than* 1, $P_x(\tau_{\omega^{k+1}}=\infty)=1$ *holds for every* $x\in X^*$.

PROOF. By the general theory of linear operator, we have ${}_{(k)}V_\alpha^n 1(b)\leqq \|{}_{(k)}V_\alpha^n\|\leqq\|{}_{(k)}V_\alpha\|^n$. Letting $n\to\infty$, we get $\lim_{n\to\infty} {}_{(k)}V_\alpha^n 1=0$.

According to the above theorem, we may define the inverse operator $(I-{}_{(k)}V_\alpha)^{-1}$ of $(I-{}_{(k)}V_\alpha)$ for the range of $(I-{}_{(k)}V_\alpha)$, if the process satisfies $P_x(\tau_{\omega^{k+1}}=\infty)=1$. Further, if we recall the boundary operator ${}_{(k)}L$, we have

$$({}_{(k)}L\,{}_{(k)}h_\alpha)\tilde{f}\equiv{}_{(k)}L({}_{(k)}h_\alpha\tilde{f})=(I-{}_{(k)}V_\alpha)\tilde{f}\,. \tag{5}$$

Therefore the inverse operator $({}_{(k)}L\,{}_{(k)}h_\alpha)^{-1}$ of ${}_{(k)}L\,{}_{(k)}h_\alpha$ is $(I-{}_{(k)}V_\alpha)^{-1}$.

THEOREM 2. *When* $P_x(\tau_{\omega^{k+1}}=\infty)=1$ *holds,* $G_\alpha f$ *is given by*

$$G_\alpha f(x)=G_\alpha^{k-1}f(x)+{}_{(k)}h_\alpha(I-{}_{(k)}V_\alpha)^{-1}\,{}_{(k)}\tilde{\Pi}G_\alpha^{k-1}f(x) \tag{6}$$

or

$$G_\alpha f(x)=G_\alpha^{k-1}f(x)-{}_{(k)}h_\alpha({}_{(k)}L\,{}_{(k)}h_\alpha)^{-1}\,{}_{(k)}LG_\alpha^{k-1}f(x)\,,[15] \tag{7}$$

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15) Particularly if $P_x(\tau_{\omega^2}=\infty)=1$ holds, this formula is

$$G_\alpha f(x)=G_\alpha{}^0 f(x)-h_\alpha(Lh_\alpha)^{-1}LG_\alpha{}^0 f\,.$$

We have an interest in expressing $G_\alpha f$ in this form. The Green function of the other type of countable Markov processes, e.g. the reflecting barrier process is expected to be denoted in the above form by introducing a suitable boundary operator L. In connection with the multidmensional diffusion process, see T. Ueno [**10**].

where

$$G_\alpha^{k-1}f(x)=E_x\Big(\int_0^{\tau_{\omega^k}}e^{-\alpha t}f(x_t)dt\Big). \tag{8}$$

Moreover, the generator's domain $\mathfrak{D}(\mathfrak{G})$ *is*

$$\mathfrak{D}(\mathfrak{G})=\{u\,;(\alpha)\,u,\tilde{\mathfrak{G}}u\in\mathfrak{B}(X)\ \textit{and}\ (\beta_i)\,{}_{(i)}Lu=0\ (i=1,2,\cdots,k)\}. \tag{9}$$

PROOF. Using the strong Markov property, we have

$$\begin{aligned}G_\alpha f(x)&=G_\alpha^{k-1}f(x)+\sum_{n\geqq 1}E_x\Big(\int_{\tau_{\omega^k n}}^{\tau_{\omega^k(n+1)}}e^{-\alpha t}f(x_t)dt\Big)\\ &=G_\alpha^{k-1}f(x)+\sum_{n\geqq 1}{}_{(k+1)}\Pi_\alpha^nG_\alpha^{k-1}f(x)\\ &=G_\alpha^{k-1}f(x)+{}_{(k)}h_\alpha\{\sum_{n\geqq 0}{}_{(k)}V_\alpha^n\}{}_{(k)}\tilde{\Pi}G_\alpha^{k-1}f(x).\end{aligned} \tag{10}$$

Since $\sum_{n\geqq 0}{}_{(k)}V_\alpha^n{}_{(k)}\tilde{\Pi}G_\alpha^{k-1}f$ is one and only one solution of $(I-{}_{(k)}V_\alpha)\tilde{g}={}_{(k)}\tilde{\Pi}G_\alpha^{k-1}f$, we have $\sum_{n\geqq 0}{}_{(k)}V_\alpha^n\tilde{\Pi}G_\alpha^{k-1}f=(I-{}_{(k)}V_\alpha)^{-1}{}_{(k)}\tilde{\Pi}G_\alpha^{k-1}f$. Hence we get the formula (6). To obtain (7), it is enough to show $-{}_{(k)}LG_\alpha^{k-1}f={}_{(k)}\tilde{\Pi}G_\alpha^{k-1}f$. Since $G_\alpha^{k-1}f$ is an $x_\alpha^{k-1}(t)$-potential, its boundary value is zero. Therefore

$${}_{(k)}LG_\alpha^{k-1}f(b)=0-\sum_{y\in X}{}_{(k)}\tilde{\Pi}(b,y)G_\alpha^{k-1}f(y)=-{}_{(k)}\tilde{\Pi}G_\alpha^{k-1}f(b). \tag{11}$$

Thus we have proved the first statement of this theorem. In Theorem 6.3, we have already given the generator's domain of the k-th instantaneous return process. But in this case, since $\lim_{n\to\infty}{}_{(k+1)}\Pi_\alpha^n(x,X^*)=0$, every function $u\in\mathfrak{B}(X)$ satisfies

$$\lim_{n\to\infty}|{}_{(k+1)}\Pi_\alpha^nu|\leqq\lim_{n\to\infty}\|u\|{}_{(k+1)}\Pi_\alpha^n(x,X^*)=0. \tag{12}$$

Therefore we need not the condition (γ) in (6.32). Thus we have accomplished the proof of this theorem.

Here we shall list several sufficient conditions for $P_x(\tau_{\omega^{k+1}}=\infty)=1$ to hold, which we can easily check up.

THEOREM 3. *If any one of the following conditions holds, then* $P_x(\tau_{\omega^{k+1}}=\infty)=1$ *for every* $x\in X$.

(i) *The k-th exit boundary* $(\partial_{(k)}X)_e$ *are divided into finitely many disjoint Borel sets* $B_1,\cdots,B_n$ *such that* ${}_{(k)}\tilde{\Pi}(b,A)$ *is independent of* b *so long as* b *runs over each* B_k.

(ii) $\inf_{b\in(\partial_{(k)}X)_e}{}_{(k)}\tilde{\Pi}(b,\infty)>0.$

(iii) *There exists a finite set* A *such that* $P_c(x_{\tau_{\omega^k}}\notin A)=0$.

PROOF. (i) By the definition of operator ${}_{(k)}V_\alpha$, we have

$${}_{(k)}V_\alpha 1(b)=\sum_{y\in X}{}_{(k)}\tilde{\Pi}(b,y){}_{(k)}h_\alpha 1(y)=\sum_{y\in X}{}_{(k)}\tilde{\Pi}(b,y)E_y(e^{-\alpha\tau_{\omega^k}}). \tag{13}$$

Therefore $V_\alpha 1(b) < 1$ for every fixed b. Since ${}_{(k)}V_\alpha 1(b)$ is a finitely valued function, we get $\| {}_{(k)}V_\alpha \| = \max_{b \in (\partial_{(k)}X)_e} {}_{(k)}V_\alpha 1(b) < 1$. Therefore by the Corollary of Theorem 1, $P_x(\tau_{\omega^{k+1}} = \infty) = 1$. (ii) Since

$$\| \sum_{y \in X} {}_{(k)}\tilde{\Pi}(b, y) \| = 1 - \inf_{b \in (\partial_{(k)}X)_e} {}_{(k)}\tilde{\Pi}(b, \infty) < 1 \tag{14}$$

we get $\|{}_{(k)}V_\alpha\| = \sum_{y \in X} {}_{(k)}\tilde{\Pi}(b, y) h_\alpha 1(y)\| \leqq \| \sum_{y \in X} {}_{(k)}\tilde{\Pi}(c, y)\| < 1$. (iii) Noting $P_c(x_{\tau_{\omega^k}} \in A^c) = {}_{(k)}h_{(k)}\tilde{\Pi}(c, A^c)$, ${}_{(k)}\tilde{\Pi}(b, A^c) = 0$ for every $b \in (\partial_{(k)}X)_e$. Put $K = \max_{y \in A} E_y(e^{-\alpha\tau_{\omega^k}})$, then $K < 1$ and

$$\| {}_{(k)}V_\alpha \| \leqq K \| \sum_{y \in X} \tilde{\Pi}(b, y) \| \leqq K < 1\,. \tag{15}$$

REMARK. Probabilistically speacking, ${}_{(k)}\tilde{\Pi}(b, \infty)$ is the probability that the particle is absorbed to ∞ as soon as it reached the k-th exit boundary point b. The condition (ii) of the above theorem shows that the probability of being absorbed is uniformly positive with respect to b. If we take $k=1$ in Theorem 3 (i), it will be clear that Examples II and III of Section 8 satisfy $P_x(\tau_{\omega^2} = \infty) = 1$. Similarly, if we consider $k=2$, we see easily that Example IV of previous section satisfies $P_x(\tau_{\omega^3} = \infty) = 1$.

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