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Note on meromorphic mappings in complex spaces

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1. If $\tau: X \to Y$ is a meromorphic mapping in the sense of Remmert with (irremovable) singularities M, then there exists a proper modification $(\tilde{X}, \tilde{M}, \pi, M, X)$ such that there exists a holomorphic mapping $\lambda: \tilde{X} \to Y$ for which $\lambda = \tau \circ \pi$ on $\tilde{X} - \tilde{M}$. (See Remmert [**4**] and Iwahashi [**3**].) In this paper we consider the following converse problem: Let M be a nowhere dense closed subset of a normal complex space X, and τ a holomorphic mapping of $X - M$ into a complex space Y. Assume there exist a connected normal complex space $\tilde{X}$, a holomorphic surjective mapping $\pi: \tilde{X} \to X$ and a holomorphic mapping $\lambda: \tilde{X} \to Y$ for which $\lambda = \tau \circ \pi$ on $\pi^{-1}(X - M)$. Then we may ask: what character does the mapping τ have? We give the following two results on this problem: (1) If the space $\tilde{X}$ has a countable basis, the set of irremovable singularities of τ is closed and almost thin of order 2. (2) If π is proper, the mapping τ is meromorphic in the sense of Remmert.

At the end of this paper we give a remark on removable singularities of a holomorphic (or meromorphic) function in normal complex spaces.

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2. I adopt the notion of *complex spaces* (= *complex β-spaces*) and the related notions such as *holomorphic mapping, analytic set, normalization of an analytic set* and so on as given in Grauert-Remmert [**2**]. The number $n = \dim_P X$ is the *complex* dimension of a complex space X at $P \in X$.

Let $\tau: X \to Y$ be a holomorphic mapping of a complex space X into a complex space Y. The number $r_\tau(P) = \dim_P X - \dim_P \tau^{-1}\tau(P)$ is called the *rank of the mapping* τ *at a point* $P \in X$. $r_\tau = \sup_{P \in X} r_\tau(P)$ is called the *global rank of the mapping* τ *on* X. The set

$$\{P \in X \mid r_\tau(P) < \limsup_{P' \to P} r_\tau(P')\}$$

is called the *set of degeneration of the mapping* τ. When the space X is irreducible, the set $\{P \in X \mid r_\tau(P) < r_\tau\}$ coincides with the set of degeneration of τ.

A subset M of a complex space X is said to be *thin of order* q if for every point $P \in M$ there exists an open neighborhood U of P in X in which an analytic set M^0 is given such that $M \cap U \subset M^0$, $\dim_P M^0 \leqq \dim_P X - q$. A subset M of X is said to be *almost thin of order q* if $M = \bigcup_{i=1}^{\infty} M_i$, where each M_i is thin of order q. First we remark the following fact:

THEOREM 1. *Let $\tau: X \to Y$ be a holomorphic mapping of a normal complex space X into a normal complex space Y, and N be the set of degeneration of τ. If the space X has a countable basis, then the set $\tau(N)$ is almost thin of order 2 in Y.*

PROOF. The space X has at most countable number of components. Therefore we may assume that X and Y are connected and pure-dimensional. If $\dim Y$ is 0 or 1, N is empty. So we assume that $\dim X = m$, $\dim Y = n$ and $n \geqq 2$. The set $\{P \in X \mid r_\tau(P) \leqq n-2\}$ is analytic in X and its image under τ is almost thin of order 2 in Y (Stoll [**6**, Hilfssatz 6.3][1]). Hence, for the proof it is sufficient to consider the case of $r_\tau = n$, $r_\tau(P) = n-1$ at every point $P \in N$ and $m \geqq n$. The normalization of the analytic set N is denoted by (N^*, μ). The space N^* is a normal complex space and has also a countable basis. Let $\bigcup_{i=1}^{\infty} N_i^*$ be a decomposition into connected components of N^* and μ_i be the restriction of the mapping μ to N_i^*. $N_i = \mu(N_i^*)$ is an irreducible component of N and $N = \bigcup_{i=1}^{\infty} N_i$. For every point $P^* \in N_i^*$ ($i = 1, 2, \cdots$), there exists $P^{*\prime} \in N_i^*$ in an arbitrary neighborhood of P^* such that $P' = \mu(P^{*\prime})$ is an ordinary point of N_i and there exists a neighborhood U of P' in X for which $N \cap U = N_i \cap U$. In a sufficiently small neighborhood of P' there is at least one irreducible $m-n+1$ dimensional component of $\tau^{-1}\tau(P')$ contained in N, accordingly in N_i. Therefore, $\dim_{P'} \tau^{-1}\tau(P') \cap N_i \geqq m-n+1$. Since the set of degeneration N is at most $m-1$ dimensional and μ_i is the mapping of the normalization, we have

$$
\begin{aligned}
r_{\tau \circ \mu_i}(P^{*\prime}) &= \dim N_i^* - \dim_{P^{*\prime}} (\tau \circ \mu_i)^{-1}(\tau \circ \mu_i)(P^{*\prime}) \\
&= \dim N_i - \dim_{P'} \tau^{-1}\tau(P') \cap N_i \leqq (m-1)-(m-n+1) = n-2.
\end{aligned}
$$

By the lower semi-continuity of the rank of a holomorphic mapping (Remmert [**4**, Satz 15]), the global rank of $\tau \circ \mu_i$ on N_i^* is at most $n-2$. Hence, $\tau \circ \mu_i(N_i^*) = \tau(N_i)$ is almost thin of order 2 and $\tau(N) = \bigcup_{i=1}^{\infty} \tau(N_i)$ is so. q. e. d.

3. Now we state;

THEOREM 2. *Let M be a subset of an n-dimensional connected normal complex space X such that $X-M$ is dense in X and let τ be a mapping (it need not be continuous) of $X-M$ into a complex space Y. Suppose that there exist a normal complex space $\tilde{X}$ and a holomorphic mapping $\pi: \tilde{X} \to X$ such that the*

1) Every complex space in Stoll [**6**] is normal and has a countable basis. But in this lemma there is no need for the image space to have a countable basis.

global rank of π *on each component of* $\tilde{X}$ *is* n $(=\dim X)$ *and there exists a holomorphic mapping* $\lambda: \tilde{X}\rightarrow Y$ *for which* $\lambda=\tau\circ\pi$ *on* $\pi^{-1}(X-M)$. *The set of degeneration of* π *is denoted by* $\tilde{N}$.

Then $\pi(\tilde{X}-\tilde{N})$ *is an open set in* X *and there exists a holomorphic mapping* $\tau^{*}: \pi(\tilde{X}-\tilde{N})\rightarrow Y$ *such that* $\tau^{*}=\tau$ *on* $(X-M)\cap\pi(\tilde{X}-\tilde{N})$ *and* $\lambda=\tau^{*}\circ\pi$ *on* $\tilde{X}-\tilde{N}$.

PROOF. Under our assumptions, the restriction of π to $\tilde{X}-\tilde{N}$ is an open mapping (Remmert [4, Sats 28]). Hence $\pi(\tilde{X}-\tilde{N})$ is an open set in X.

For every point $P\in\pi(\tilde{X}-\tilde{N})$, we take two arbitrary points $\tilde{P}^{1}$ and $\tilde{P}^{2}$ of $\pi^{-1}(P)\cap(\tilde{X}-\tilde{N})$. If $P\notin M$, then $\lambda(\tilde{P}^{1})=\lambda(\tilde{P}^{2})$. Suppose that $P\in M$. Since M contains no interior points, $\tilde{M}=\pi^{-1}(M)$ can not contain interior points in $\tilde{X}-\tilde{N}$. Therefore there exist $\tilde{P}_{\nu}{}^{1}\in\tilde{X}-(\tilde{N}\cup\tilde{M})$ $(\nu=1,2,\cdots)$ such that $\lim_{\nu\rightarrow\infty}\tilde{P}_{\nu}{}^{1}=\tilde{P}^{1}$. Define $P_{\nu}=\pi(\tilde{P}_{\nu}{}^{1})$. Choose a neighborhood $\tilde{U}$ of $\tilde{P}^{2}$ such that $\tilde{U}\cap\tilde{N}=\phi$ and there exists a metric σ in $\tilde{U}$. Let $\tilde{U}_{\mu}$ be a neighborhood of $\tilde{P}^{2}$ such that

$$\tilde{U}_{\mu}\subset\{\tilde{P}\in\tilde{U}\mid\sigma(\tilde{P}^{2},\tilde{P})<1/\mu\}\qquad(\mu=1,2,\cdots).$$

$\pi(\tilde{U}_{\mu})=U_{\mu}$ is a neighborhood of P. Since $\lim_{\nu\rightarrow\infty}P_{\nu}=P$, the sequence $\{P_{\nu}\}$ contains a subsequence $\{P_{\nu_{\mu}}\}$ for which $P_{\nu_{\mu}}\in U_{\mu}$, $(\mu=1,2,\cdots)$. We can choose a point $\tilde{P}_{\nu_{\mu}}^{2}\in\tilde{U}_{\mu}\cap\pi^{-1}(P_{\nu_{\mu}})$ for every ν_{μ}. Since $\lim_{\mu\rightarrow\infty}\tilde{P}_{\nu_{\mu}}^{2}=\tilde{P}^{2}$, $\lambda(\tilde{P}^{1})=\lim_{\mu\rightarrow\infty}\lambda(\tilde{P}_{\nu_{\mu}}^{1})=\lim_{\mu\rightarrow\infty}\tau\circ\pi(\tilde{P}_{\nu_{\mu}}^{1})=\lim_{\mu\rightarrow\infty}\tau\circ\pi(\tilde{P}_{\nu_{\mu}}^{2})=\lim_{\mu\rightarrow\infty}\lambda(\tilde{P}_{\nu_{\mu}}^{2})=\lambda(\tilde{P}^{2})$. Therefore for every point $P\in\pi(\tilde{X}-\tilde{N})$ the mapping λ is constant on $(\tilde{X}-\tilde{N})\cap\pi^{-1}(P)$.

Suppose that P is an arbitrary point of $\pi(\tilde{X}-\tilde{N})$. Let $\Sigma_{\tau}(P)$ be the set of all points $Q\in Y$ such that there exist $P_{\nu}\in X-M$ $(\nu=1,2,\cdots)$ for which $\lim_{\nu\rightarrow\infty}P_{\nu}=P$ and $\lim_{\nu\rightarrow\infty}\tau(P_{\nu})=Q$. It is trivial that $\Sigma_{\tau}(P)\neq\phi$, because there exists at least one point $\tilde{P}\in(\tilde{X}-\tilde{N})\cap\pi^{-1}(P)$. We can easily prove in the same way that $Q=\lambda(\tilde{P})$ for an arbitrary point $Q\in\Sigma_{\tau}(P)$. Hence $\Sigma_{\tau}(P)$ contains one and only one point for every $P\in\pi(\tilde{X}-\tilde{N})$.

Define $\tau^{*}(P)=\Sigma_{\tau}(P)$ for every $P\in\pi(\tilde{X}-\tilde{N})$. By a Stein's theorem ([5, Staz 2, Zusatz][2)]), τ^{*} is a holomorphic mapping from $\pi(\tilde{X}-\tilde{N})$ into Y. Thus our assertion is proved. q. e. d.

A point $P\in X$ is called a *regular point of* τ^{*} if there exist an open neighborhood U of P and a holomorphic mapping τ^{**} from $\pi(\tilde{X}-\tilde{N})\cup U$ into Y such that $\tau^{**}=\tau^{*}$ on $\pi(\tilde{X}-\tilde{N})$. The point $P\in X$ is called a *(irremovable) singular point of* τ^{*} if it is not a regular point of τ^{*}. We have,

COROLLARY 1. *In addition to the same assumptions as in Theorem* 2, *we assume that* $\tilde{X}$ *has a countable basis and* π *is surjective. Then the set* S *of all singular points of* τ^{*} *is closed and almost thin of order* 2.

2) Every complex space in Stein [5] is normal. But in this theorem there is no need for the image space to be normal.

PROOF. It is trivial that S is closed. By Theorem 1, $\pi(\tilde{N})$ is almost thin of order 2. Since π is surjective, $S\subset X-\pi(\tilde{X}-\tilde{N})\subset\pi(\tilde{N})$. Hence S is almost thin of order 2. q. e. d.

I adopt the definition of a meromorphic mapping in the sense of Remmert ($=SR$-meromorphic mapping) as given in Stoll [6]; $\tau: X\rightarrow Y$ is called a *meromorphic mapping (in the sense of Remmert) of a complex space X into a complex space Y* if it has the following properties: (a) There exists a closed thin set M of order 1 in X, and τ is a holomorphic mapping of $X-M$ into Y. (b) We denote by T the graph of τ. The closure $\bar{T}$ of T in $X\times Y$ is analytic in $X\times Y$. (c) For every point $P\in X$, the set $(\{P\}\times Y)\cap\bar{T}$ is not empty and compact. Then we have the following fact:

COROLLARY 2. *In addition to the same assumptions as in Theorem 2, we assume that the mapping π is proper and surjective. Then τ^{*} is a meromorphic mapping from X into Y.*

PROOF. Since π is proper, $N=\pi(\tilde{N})$ is an analytic set of at most $n-2$ dimension and consequently $X-\pi(\tilde{X}-\tilde{N})$ is closed thin of order 1. We denote by T the set

$$\{(P, \tau^{*}(P))\mid P\in\pi(\tilde{X}-\tilde{N})\},$$

and by G the set

$$\{(\pi(\tilde{P}), \lambda(\tilde{P}))\mid\tilde{P}\in\tilde{X}\}.$$

The closure of T in $X\times Y$ is denoted by $\bar{T}$. It is trivial that

$$T=\{(\pi(\tilde{P}), \lambda(\tilde{P}))\mid\tilde{P}\in\tilde{X}-\tilde{N}\}$$

and $G\subset\bar{T}$. Conversely, suppose that $(P, Q)\in\bar{T}$. Then there exist $P_{\nu}\in\pi(\tilde{X}-\tilde{N})$ such that $\lim_{\nu\rightarrow\infty}(P_{\nu}, \tau^{*}(P_{\nu}))=(P, Q)$. Since π is proper, we can select a subsequence $\{P_{\nu_{\mu}}\}$ of the sequence $\{P_{\nu}\}$ such that there exist $\tilde{P}_{\nu_{\mu}}\in(\tilde{X}-\tilde{N})\cap\pi^{-1}(P_{\nu_{\mu}})$ and $\tilde{P}\in\tilde{X}$ for which $\lim_{\mu\rightarrow\infty}\tilde{P}_{\nu_{\mu}}=\tilde{P}$. Since

$$(P, Q)=\lim_{\nu\rightarrow\infty}(P_{\nu}, \tau^{*}(P_{\nu}))=\lim_{\mu\rightarrow\infty}(\pi(\tilde{P}_{\nu_{\mu}}), \lambda(\tilde{P}_{\nu_{\mu}}))=(\pi(\tilde{P}), \lambda(\tilde{P})),$$

$\bar{T}$ is contained in G. Thus we have $\bar{T}=G$.

The set $H=\{(\tilde{P}, \pi(\tilde{P}), \lambda(\tilde{P}))\mid\tilde{P}\in\tilde{X}\}$ is analytic in $\tilde{X}\times X\times Y$. We denote by α the natural projection

$$H\rightarrow X\times Y \quad \text{such that} \quad (\tilde{P}, \pi(\tilde{P}), \lambda(\tilde{P}))\rightarrow(\pi(\tilde{P}), \lambda(\tilde{P})),$$

and similarly by β

$$X\times Y\rightarrow X \quad \text{such that} \quad (P, Q)\in X\times Y\rightarrow P\in X.$$

Let K be a compact set in $X\times Y$. $\pi^{-1}\beta(K)$ is compact in X. Since $\pi^{-1}\beta(K)\times K$ is compact in $\tilde{X}\times X\times Y$ and H is closed in $\tilde{X}\times X\times Y$, $\alpha^{-1}(K)=(\pi^{-1}\beta(K)\times K)\cap H$ is a compact subset of H. Hence the mapping α is proper and $\alpha(H)=G=\bar{T}$

is analytic in $X \times Y$. For every point $P \in X$, the set $(\{P\} \times Y) \cap \bar{T}$ coincides with $\{P\} \times \lambda\pi^{-1}(P)$, and consequently it is compact and not empty. q. e. d.

4. *Let X be a connected normal complex space, M be a closed almost thin set of order 2 and f be a holomorphic (or meromorphic) function in $X-M$. We would like to show that f can be extended to the whole space X by a holomorphic (or meromorphic) function.*

When M is a closed almost thin set of order 2 and furthermore thin of order 1, this fact is used as a well-known result in Stoll's paper ([**6**]), but I don't know his proof. When M is a closed thin set of order 2, this fact is well-known. (See Grauert-Remmert [**2**, p. 270].) Therefore, it is sufficient to prove the following Theorem 3.

A subset M of a connected normal complex space X is said to be *of F-type* with respect to holomorphic (or meromorphic) functions if for every point $P \in M$ there exists an open neighborhood U of P in X in which a nowhere dense relatively-closed subset M^0 in U is given such that $M \cap U \subset M^0$ and each function holomorphic (or meromorphic) in $U-M^0$ can be extended to the whole U by a holomorphic (or meromorphic) function. Now, we have the following theorem:

THEOREM 3. *Let X be a connected normal complex space and M be a closed subset. If $M = \bigcup_{i=1}^{\infty} M_i$ and each M_i is of F-type with respect to holomorphic (or meromorphic) functions, then M itself is also of F-type with respect to holomorphic (or meromorphic) functions.*

PROOF. For every point $P \in M$, there exists an open neighborhood V which has a metric σ and we choose an open relatively-compact neighborhood U of P such that $U \subset \bar{U} \subset V$. Let f be a holomorphic (or meromorphic) function in $U-M$ and S be the set of all (irremovable) singularities of f in U. Suppose S were not empty. S is relatively-closed in U and $S \subset M$. $S_i = S \cap M_i$ is also of F-type and $S = \bigcup_{i=1}^{\infty} S_i$. We denote $\bar{S} \cap (\bar{U}-U)$ by $\bar{S}_0$. Then we have easily $\bar{S} = \bigcup_{i=0}^{\infty} \bar{S}_i$.

With respect to the relative topology induced from X to $\bar{S}$, $\bar{S}$ is a complete metric space and each $\bar{S}_i$ is its closed subset. By the Baire's theorem, there exists a non-negative integer i_0 such that there exist a point $P_0 \in \bar{S}_{i_0}$ and a neighborhood U_1 of P_0 in X in which $\bar{S}_{i_0} \cap U_1 = \bar{S} \cap U_1$.

We distinguish three cases as follows:

(1) If $i_0 = 0$, i. e. $P_0 \in \bar{S}_0$, then $S \cap U_1$ would be empty, because $\bar{S}_0 \subset \bar{U}-U$, $S \subset U$ and $\bar{S}_0 \cap U_1 = \bar{S} \cap U_1$. But this contradicts the facts that $P_0 \in \bar{S}$ and U_1 is a neighborhood of P_0.

(2) If $i_0 \neq 0$ and $P_0 \in S_{i_0}$, then there would exist an open neighborhood U_2 of P_0 in X and a subset $S_{i_0}^0$ of U_2 which would satisfy the conditions of F-type. As $P_0 \in S_{i_0} \subset U$, we may assume $U_2 \subset U \cap U_1$. Since $S_{i_0} \cap U_2 \subset S_{i_0}^0$, $\bar{S}_{i_0} \cap U_1 = \bar{S} \cap U_1$ and $S_{i_0}^0$ is relatively-closed in U_2, we have $\bar{S} \cap U_2 \subset S_{i_0}^0$. The function f is holomorphic (or meromorphic) in $U_2 - S$, so it is holomorphic (or meromorphic) in $U_2 - S_{i_0}^0$, and consequently in U_2. This contradicts the fact $P_0 \in S$
(3) If $i_0 \neq 0$ and $P_0 \in \bar{S}_{i_0} - S_{i_0}$, then we could choose a point $P_0' \in S_{i_0}$ and a neighborhood U_1' of P_0' contained in U_1. In the same way as in the case (2), this leads to a contradiction.

These contradictions prove our theorem. q. e. d.

The field of all meromorphic functions on a connected normal complex space X is denoted by $K(X)$. We have,

COROLLARY 1. *We assume the hypothesis of Corollary* 1 *of Theorem* 2 *and furthermore Y to be connected normal. Suppose $\tau^* \circ f$ is a meromorphic function in $X-S$ for every $f \in K(Y)$. Then there exists a homomorphism $\eta: K(Y) \to K(X)$ which satisfies the relation $\eta(f) = f \circ \tau^*$ on a dense open subset of X for every $f \in K(Y)$.*

From Theorem 3, we have the following fact: (For the proof see Andreotti-Stoll [**1**, p. 316].)

COROLLARY 2. *We assume the hypothesis of Corollary* 1 *of Theorem* 2 *and furthermore Y to be K-complete. Then the set of all singular points of τ^* is empty, i.e. τ^* can be extended to a holomorphic mapping of X into Y.*

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