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8-manifolds admitting no differentiable structure

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J. Milnor [**5**], [**6**] and R. Thom [**14**] have given examples of compact unbounded triangulated topological 8-manifolds admitting no differentiable structure compatible with the given triangulations. In this paper we shall prove that some of these examples do not admit any differentiable structure, compatible or not with the given triangulations, as in the case of Kervaire's 10-manifold [**4**]. The well-known result of Milnor [**5**] on the existence of non-canonical differentiable structures on the 7-sphere is responsible for this situation. An analogous result holds also for the 15-sphere (Shimada [**9**], Tamura [**13**]), whence follows the existence of 16-manifolds admitting no differentiable structure. This will be shown at the same time.

1. 3-sphere bundles over the 4-sphere.

We recall here some results about 3-sphere bundles over the 4-sphere (resp. 7-pshere bundles over the 8-sphere). For the proofs of them, see Milnor [**5**], Shimada [**9**], Tamura [**12**], [**13**].

Let $\rho,\ \sigma : S^3 \to SO(4)$ (resp. $\rho',\ \sigma' : S^7 \to SO(8)$) be maps defined by

$$\rho(u)v = uvu^{-1}, \qquad \sigma(u)v = uv,$$

$$(\text{resp. } \rho'(x)y = xyx^{-1}, \qquad \sigma'(x)y = xy,)$$

where u and v denote quaternions with norm 1 (resp. x and y denote Cayley numbers with norm 1). Then the homotopy classes $\{\rho\}, \{\sigma\}$ (resp. $\{\rho'\}, \{\sigma'\}$) are generators of $\pi_3(SO(4)) \approx Z+Z$ (resp. $\pi_7(SO(8)) \approx Z+Z$). Let

$$\xi_{m,n} = (B^7_{m,n},\ S^4,\ S^3,\ \pi_{m,n})$$

$$(\text{resp. } \xi'_{m,n} = (B^{15}_{m,n},\ S^8,\ S^7,\ \pi'_{m,n}))$$

be the S^3 bundle over S^4 (resp. S^7 bundle over S^8) with the characteristic map $m\{\rho\}+n\{\sigma\}$ (resp. $m\{\rho'\}+n\{\sigma'\}$). Moreover let

$$\bar{\xi}_{m,n} = (\bar{B}^8_{m,n},\ S^4,\ D^4,\ \bar{\pi}_{m,n})$$

$$(\text{resp. } \bar{\xi}'_{m,n} = (\bar{B}^{16}_{m,n},\ S^8,\ D^8,\ \bar{\pi}'_{m,n}))$$

be the 4-cell bundle over S^4 (resp. 8-cell bundle over S^8) associated with $\xi_{m,n}$

(resp. $\xi'_{m,n}$). $B^7_{m,n}$ and $\bar{B}^8_{m,n}$ (resp. $B^{15}_{m,n}$ and $\bar{B}^{16}_{m,n}$) have (C^∞) differentiable structures naturally defined by bundle structures. Thus $B^7_{m,n}$ (resp. $B^{15}_{m,n}$) is a compact unbounded differentiable 7-(resp. 15-) manifold and $\bar{B}^8_{m,n}$ (resp. $\bar{B}^{16}_{m,n}$) is a compact differentiable 8-(resp. 16-) manifold with the boundary $\partial\bar{B}^8_{m,n}=B^7_{m,n}$ (resp. $\partial\bar{B}^{16}_{m,n}=B^{15}_{m,n}$). The homology groups of $B^7_{m,n}$ (resp. $B^{15}_{m,n}$) are as follows:

$$H_0(B^7_{m,n};Z)\approx H_7(B^7_{m,n};Z)\approx Z,\qquad H_i(B^7_{m,n};Z)=0\qquad i\neq 0,3,4,7,$$

$$H_3(B^7_{m,n};Z)\approx Z_n,\qquad H_4(B^7_{m,n};Z)=\begin{cases}0 & n\neq 0\\ Z & n=0\end{cases}$$

$$\text{(resp. } H_0(B^{15}_{m,n};Z)\approx H_{15}(B^{15}_{m,n};Z)\approx Z,\qquad H_i(B^{15}_{m,n};Z)=0\qquad i\neq 0,7,8,15,$$

$$H_7(B^{15}_{m,n};Z)\approx Z_n,\qquad H_8(B^{15}_{m,n};Z)=\begin{cases}0 & n\neq 0\\ Z & n=0\end{cases}\text{)}.$$

$H_3(B^7_{m,n};Z)$ (resp. $H_7(B^{15}_{m,n};Z)$) is generated by a cycle $\pi^{-1}_{m,n}(x_0)$ ($x_0\in S^4$) (resp. $\pi'^{-1}_{m,n}(x_0)$ ($x_0\in S^8$)). $B^7_{m,1}$ (resp. $B^{15}_{m,1}$) is homeomorphic to S^7 (resp. S^{15}) and $B^7_{0,1}$ (resp. $B^{15}_{0,1}$) is diffeomorphic to the standard S^7 (resp. S^{15}). $\bar{B}^8_{m,n}$ (resp. $\bar{B}^{16}_{m,n}$) has the homotopy type of S^4 (resp. S^8).

The first (resp. the second) Pontrjagin class of $\bar{B}^8_{m,n}$ (resp. $\bar{B}^{16}_{m,n}$) is given by

$$p_1(\bar{B}^8_{m,n})=\pm 2(2m+n)\alpha_4\qquad\text{(resp. } p_2(\bar{B}^{16}_{m,n})=\pm 6(2m+n)\alpha_8\text{)},$$

where α_4 is a generator of $H^4(\bar{B}^8_{m,n};Z)\approx Z$ (resp. α_8 is a generator of $H^8(\bar{B}^{16}_{m,n};Z)\approx Z$).

As is well-known, we have

$$\pi_7(S^4)\approx Z+Z_{12}\qquad\text{(resp. } \pi_{15}(S^8)\approx Z+Z_{120}\text{)}.$$

The homotopy class $\{\nu_4\}$ (resp. $\{\nu_8\}$) represented by the Hopf map $\nu_4=J(\sigma)$: $S^7\to S^4$ (resp. $\nu_8=J(\sigma'):S^{15}\to S^8$) generates the infinite cyclic direct summand Z of $\pi_7(S^4)$ (resp. $\pi_{15}(S^8)$) and the homotopy class $\{\gamma_4\}$ (resp. $\{\gamma_8\}$) represented by $\gamma_4=J(\rho):S^7\to S^4$ (resp. $\gamma_8=J(\rho'):S^{15}\to S^8$) generates the finite cyclic direct summand Z_{12} of $\pi_7(S^4)$ (resp. Z_{120} of $\pi_{15}(S^8)$), where $J:\pi_n(SO(r))\to\pi_{n+r}(S^r)$ is the J-homomorphism. Then, choosing the orientation of $B^7_{m,1}$ (resp. $B^{15}_{m,1}$) properly, the homotopy class of the map $\pi_{m,1}:B^7_{m,1}\to S^4$ (resp. $\pi'_{m,1}:B^{15}_{m,1}\to S^8$) is given as follows:

$$\{\pi_{m,1}\}=\{\nu_4\}+m\{\gamma_4\}\qquad\text{(resp. } \{\pi'_{m,1}\}=\{\nu_8\}+m\{\gamma_8\}\text{)}.$$

2. 3-connected compact unbounded differentiable 8-manifold with the 4th Betti number 1.

In this section we consider a 3-connected compact unbounded differentiable 8-manifold M^8 such that $H_4(M^8;Z)\approx Z$. The notation D^n will be used for the closed disk in Euclidean space R^n bounded by the unit sphere S^{n-1}.

Let $i: D^8 \to M^8$ be a differentiable imbedding. Then the compact differentiable 8-manifold $V^8 = M^8 - i(\text{Int } D^8)$ with the boundary $i(\partial D^8)$ is a handlebody, an element of $\mathcal{H}(8, 1, 4)$ by Smale [**10**, Theorem F], [**11**].[1] That is to say, we have

$$V^8 = D^8 \cup_f D^4 \times D^4 ,$$

where $f: \partial D^4 \times D^4 \to \partial D^8$ is a differentiable imbedding and $D^8 \cup_f D^4 \times D^4$ denotes the differentiable 8-manifold-with-boundary obtained from the disjoint union of D^8 and $D^4 \times D^4$ by identifying each point of $\partial D^4 \times D^4$ with its image under f, making use of the device of straightening the angle. ∂V^8 is diffeomorphic to S^7 with the natural differentiable structure. Clearly V^8 has the homotopy type of S^4.

Let $j_1: D^4 \to D^8$ be a continuous map such that

$$j_1(\text{Int } D^4) \subset \text{Int } D^8 , \qquad j_1(x) = f(x, 0) \quad (x \in \partial D^4) ,$$

and let $j_2: D^4 \to D^4 \times D^4$ be the map defined by

$$j_2(x) = (x, 0) \in D^4 \times 0 \quad (x \in D^4) .$$

Define a continuous map $j: S^4 \to V^8$ by j_1 (resp. j_2) on the upper (lower) hemisphere of S^4. Then $j(S^4)$ represents a generator of $\pi_4(V^8) \approx H_4(V^8; Z) \approx Z$. We can assume without loss of generality that j is a differentiable imbedding (Milnor [**8**; Theorem 5.9]).

Now we shall show that V^8 is diffeomorphic to $\bar{B}^8_{m,1}$. Let N be a closed tubular neighborhood of $j(S^4)$. N is a differentiable 8-manifold with the boundary ∂N. Let $(\partial N, j(S^4), S^3, \pi)$ be a S^3 bundle over S^4 associated with the normal bundle $(N, j(S^4), D^4, \bar{\pi})$. N has the homotopy type of S^4. Consider the Mayer-Vietoris homology sequence of a triad $(V^8;\ N,\ V^8 - \text{Int } N)$:

$$\begin{aligned} \cdots \to H_{q+1}(V^8; Z) \to H_q(\partial N; Z) \to H_q(N; Z) + H_q(V^8 - \text{Int } N; Z) \to H_q(V^8; Z) \\ \to H_{q-1}(\partial N; Z) \to H_{q-1}(N; Z) + H_{q-1}(V^8 - \text{Int } N; Z) \to H_{q-1}(V^8; Z) \to \cdots . \end{aligned}$$

The exactness of this sequence yields

$$\psi_*: H_q(\partial N; Z) \approx H_q(V^8 - \text{Int } N; Z) \qquad q = 0, 1, 2, 4, 5, 6, 7, 8 ,$$

where $\psi: \partial N \to V^8 - \text{Int } N$ is the inclusion map. Moreover we have

$$H_3(\partial N; Z) = 0 .$$

In fact $\psi_*(H_3(\partial N; Z)) = 0$ holds, because a 3-cycle $\psi(\pi^{-1}(0, 0))$ $((0, 0) \in D^4 \times 0 \subset j(S^4))$ is homotopic to $0 \times \partial D^4 \subset \partial V^8 = S^7$. Therefore the normal bundle $(N, j(S^4), D^4, \bar{\pi})$ is $\xi_{m,1}$ and ∂N is homeomorphic to S^7 (Section 1). (More exactly $(N, j(S^4),$

1) I am indebted to Prof. S. Smale for his kindness of sending me a copy of the manuscript of his paper before publication.

$D^4, \bar{\pi}$) is $\bar{\xi}_{m,1}$ or $\bar{\xi}_{m',-1}$. But $\bar{\xi}_{m',-1}$ is equivalent to $\bar{\xi}_{m'-1,1}$.) As is easily verified, $V^8-\operatorname{Int} N$ is simply connected. It follows now that ψ is a homotopy equivalence and that ∂N is a deformation retract of $V^8-\operatorname{Int} N$. $V^8-\operatorname{Int} N$ has the same homotopy type as S^7.

On the other hand, let $\psi'\colon \partial V^8 \to V^8-\operatorname{Int} N$ be the inclusion map. Both ∂V^8 and $V^8-\operatorname{Int} N$ have the homotopy type of S^7 and $\psi'(\partial V^8)$ is homologous to $\psi(\partial N)$ which represents a generator of $H_7(V^8-\operatorname{Int} N; Z)$. Therefore ψ' is a homotopy equivalence and ∂V^8 is a deformation retract of $V^8-\operatorname{Int} N$. Hence $V^8-\operatorname{Int} N$ defines the J-equivalence relation between ∂V^8 and ∂N (Milnor [**8**], Thom [**14**]).

Since a recent result of Smale [**10**], [**11**] implies that ∂V^8 and ∂N are diffeomorphic and that $V^8-\operatorname{Int} N$ is diffeomorphic to $\partial V^8 \times I = \partial N \times I$, it follows that V^8 is diffeomorphic to $N = \bar{B}^8_{m,1}$. Thus we have

$$M^8 = \bar{B}^8_{m,1} \cup_i D^8 ,$$

where $i\colon \partial D^8 \to \partial \bar{B}^8_{m,1} = B^7_{m,1}$ is an onto diffeomorphism.

Pontrjagin classes of M^8 satisfy the following two relations (A), (B). ($[M^n]$ denotes the fundamental homology class of M^n.) Firstly, the index theorem (Hirzebruch [**3**]) implies

$$\text{(A)} \qquad 45(\alpha_4 \cup \alpha_4)[M^8] = (7p_2(M^8) - p_1^2(M^8))[M^8] .$$

Secondly, the integrality of $\hat{A}$-genus $\hat{A}(M^8) = \frac{1}{2^7 \cdot 45}(-4p_2(M^8) + 7p_1^2(M^8))[M^8]$ (Atiyah and Hirzebruch [**1**], Borel and Hirzebruch [**2**]) implies

$$\text{(B)} \qquad (4p_2(M^8) - 7p_1^2(M^8))[M^8] \equiv 0 \mod 2^7 \cdot 45 .$$

Since the first Pontrjagin class of $M^8 = \bar{B}^8_{m,1} \cup_i D^8$ is given by (Section 1)

$$p_1(M^8) = p_1(\bar{B}^8_{m,1}) = \pm 2(2m+1)\alpha_4 ,$$

(A), (B) yield

$$7p_2(M^8)[M^8] = (2^2(2m+1)^2 + 45)(\alpha_4 \cup \alpha_4)[M^8] , \qquad (*)$$

$$p_2(M^8)[M^8] \equiv 7(2m+1)^2(\alpha_4 \cup \alpha_4)[M^8] \mod 2^5 \cdot 45 .$$

Therefore we have

$$7^2(2m+1)^2 \equiv 2^2(2m+1)^2 + 45 \mod 2^5 \cdot 45 ,$$

hence

$$m(m+1) \equiv 0 \mod 8. \qquad (**)$$

Moreover ($*$) implies

$$m(m+1) \equiv 0 \mod 7. \qquad (***)$$

Thus we have the following theorem.

THEOREM 1. *Let M^8 be a 3-connected compact unbounded differentiable 8-manifold such that $H_4(M^8; Z) \approx Z$. Then M^8 is diffeomorphic to $\bar{B}^8_{m,1} \cup_i D^8$ with m satisfying $m(m+1) \equiv 0$ mod 56, where $i\colon \partial D^8 \to \partial \bar{B}^8_{m,1} = B^7_{m,1}$ is an onto dif-*

feomorphism.

The following theorem is an immediate consequence of the above theorem and the fact that $\{\pi_{m,1}\}=\{\nu_4\}+m\{\gamma_4\}$ (Section 1).

THEOREM 2. *Let M^8 be a 3-connected compact unbounded differentiable 8-manifold such that $H_4(M^8;Z)\approx Z$. Then M^8 has the homotopy type of $S^4\cup_g e^8$, where $g\colon S^7\to S^4$ is a map such that $\{g\}=\{\nu_4\}+m\{\gamma_4\}\in\pi_7(S^4)$ with m satisfying* $m(m+1)\equiv 0 \bmod 4$.

For a 7-connected compact unbounded differentiable 16-manifold M^{16}, by a similar argument, making use of two relations (A′), (B′):

$$\text{(A}'\text{)}\qquad 3^4\cdot 5^2\cdot 7(\alpha_8\cup\alpha_8)[M^{16}]=(381p_4(M^{16})-19p_2{}^2(M^{16}))[M^{16}]\,,$$

$$\text{(B}'\text{)}\qquad (2^7\cdot 3p_4(M^{16})-2^5\cdot 13p_2{}^2(M^{16}))[M^{16}]\equiv 0 \qquad \bmod 2^{16}\cdot 3^4\cdot 5^2\cdot 7\,,$$

we obtain the following theorems.

THEOREM 1′. *Let M^{16} be a 7-connected compact unbounded differentiable 16-manifold such that $H_8(M^{16};Z)\approx Z$. Then M^{16} is diffeomorphic to $\bar{B}^{16}_{m,1}\cup_i D^{16}$ with m satisfying* $m(m+1)\equiv 0 \bmod 16256$, *where $i\colon\partial D^{16}\to\partial\bar{B}^{16}_{m,1}=B^{15}_{m,1}$ is an onto diffeomorphism.*

THEOREM 2′. *Let M^{16} be a 7-connected compact unbounded differentiable 16-manifold such that $H_8(M^{16};Z)\approx Z$. Then M^{16} has the homotopy type of $S^8\cup_g e^{16}$, where $g\colon S^{15}\to S^8$ is a map such that $\{g\}=\{\nu_8\}+m\{\gamma_8\}\in\pi_{15}(S^8)$ with m satisfying* $m(m+1)\equiv 0 \bmod 8$.

It is known that an $(n-1)$-connected compact unbounded differentiable 2n-manifold with the nth Betti number 1 exists only for $n=2,4,8$ (Milnor [**7**]). The quaternion (resp. Cayley) projective plane is homeomorphic to $\bar{B}^8_{0,1}\cup_i D^8$ (resp. $\bar{B}^{16}_{0,1}\cup_i D^{16}$).

Now let $\bar{B}^8_{m,1}\cup D^8$ denote the space obtained from the disjoint union of $\bar{B}^8_{m,1}$ and D^8 by identifying $\partial\bar{B}^8_{m,1}=B^7_{m,1}$ with ∂D^8 topologically. Then $\bar{B}^8_{m,1}\cup D^8$ is a compact unbounded triangulable topological 8-manifoid (Milnor [**6**], Thom [**14**]). $\bar{B}^8_{m,1}\cup D^8$ has the homotopy type of $S^4\cup_h e^8$, where $h\colon S^7\to S^4$ is a map such that $\{h\}=\{\nu_4\}+m\{\gamma_4\}\in\pi_7(S^4)$ (Section 1). Thus the following theorem is an immediate consequence of Theorem 2.

THEOREM 3. *If* $m(m+1)\not\equiv 0 \bmod 4$, *$\bar{B}^8_{m,1}\cup D^8$ does not admit any differentiable structure.*

REMARK. Choose a C^∞ triangulation of $\bar{B}^8_{m,1}$ and extend it to a triangulation of $\bar{B}^8_{m,1}\cup D^8$ naturally. Then $\bar{B}^8_{m,1}\cup D^8$ and $\bar{B}^8_{m',1}\cup D^8$ are combinatorially distinct if $m'\neq m,\ -m-1$ (Thom [**14**]).

Furthermore by Theorem 2′ we have

THEOREM 3′. *Let $\bar{B}^{16}_{m,1}\cup D^{16}$ denote the compact unbounded triangulable topological 16-manifold obtained from the disjoint union of $\bar{B}^{16}_{m,1}$ and D^{16} by identifying $\partial\bar{B}^{16}_{m,1}=B^{15}_{m,1}$ with ∂D^{16} topologically. Then if* $m(m+1)\not\equiv 0 \bmod 8$, *$\bar{B}^{16}_{m,1}\cup D^{16}$ does* *not admit any differentiable structure.*

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References

[1] M. F. Atiyah and F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds, Bull. Amer. Math. Soc., **65** (1959), 276-281.

[2] A. Borel and F. Hirzebruch, Characteristic classes and homogeneous spaces III, Amer. J. Math., **82** (1960), 491-504.

[3] F. Hirzebruch, Neue topologische Methoden in der algebraischen Geometrie, Springer, 1956.

[4] M. Kervaire, A manifold which does not admit any differentiable structure, Comm. Math. Helv., **34** (1960), 257-270.

[5] J. Milnor, On manifolds homeomorphic to the 7-sphere, Ann. of Math., **64** (1956) 399-405.

[6] J. Milnor, On the relationship between differentiable manifolds and combinatorial manifolds, (mimeographed), Princeton University, 1956.

[7] J. Milnor, Some consequences of a theorem of Bott, Ann. of Math., **68** (1958), 444-449.

[8] J. Milnor, Differentiable manifolds which are homotopy spheres, (mimeographed), Princeton University, 1959.

[9] N. Shimada, Differentiable structures on 15-sphere and Pontrjagin classses of certain manifolds, Nagoya Math. J., **12** (1957), 59-69.

[10] S. Smale, The generalized Poincaré conjecture in higher dimensions, Bull. Amer. Math. Soc., **66** (1960), 373-375.

[11] S. Smale, Generalized Poincaré's conjecture in dimensions greater than four, Ann. of Math., **74** (1961), 391-406.

[12] I. Tamura, On Pontrjagin classes and homotopy types of manifolds, J. Math. Soc. Japan, **9** (1957), 250-262.

[13] I. Tamura, Homeomorphy classification of total spaces of sphere bundles over spheres, J. Math. Soc. Japan, **10** (1958), 29-43.

[14] R. Thom, Les classes caractéristiques de Pontrjagin des variétés triangulées, Topologia Algebraica, Mexico, 1958, 54-67.