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Approximation by reduced fractions

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1. Let $\{\delta(n)\}$ be a sequence of non-negative numbers. J. W. S. Cassels [1] proved that the set of real numbers x in $0 \leqq x < 1$ for which

$$\left| x - \frac{m}{n} \right| < \delta(n) \tag{1}$$

for infinitely many integers m, n has measure 0 or 1. R. J. Duffin and A. C. Schaeffer [2] had shown that for some sequences $\{\delta(n)\}$, this set has measure 1 while the set of x for which (1) holds for infinitely many relatively prime integers m, n has measure 0. Using an extension of Cassels' method, we will prove

THEOREM 1. *For each sequence of non-negative numbers* $\{\delta(n)\}$, *the set* $\mathcal{E}$ *of* x *in* $0 \leqq x < 1$ *for which*

$$\left| x - \frac{m}{n} \right| < \delta(n), \qquad (m, n) = 1 \tag{2}$$

for infinitely many m, n *has measure* 0 *or* 1.

We may suppose in the proof that $\delta(n) \to 0$. Otherwise each x satisfies (2) for infinitely many n. In fact, suppose that $n_1 < n_2 < \cdots$ is a sequence for which $\delta(n_\nu) \geqq \delta > 0$. For (2) to be satisfied with $n = n_\nu$ it is sufficient for there to exist an m prime to n_ν in the interval $|n_\nu x - m| < n_\nu \delta$. The existence of such an m, for all x and all large ν, follows from the following lemma.

LEMMA 1. *The length* L_n *of the longest interval of consecutive integers not prime to* n *satisfies* $L_n = o(n)$.

PROOF. Let $(m, n) > 1$ for $m_1 < m \leqq m_2$. Then

$$0 = \sum_{m_1 < m \leqq m_2} \sum_{d | (m,n)} \mu(d) = \sum_{d | n} \mu(d) \sum_{d | m, m_1 < m \leqq m_2} 1$$

$$= \sum_{d | n} \mu(d) \left(\left[\frac{m_2}{d} \right] - \left[\frac{m_1}{d} \right] \right) = (m_2 - m_1) \sum_{d | n} \frac{\mu(d)}{d} + O(d(n))$$

$$= (m_2 - m_1) \frac{\phi(n)}{n} + O(d(n)).$$

Here $d(n)$ is the number of divisors of n. It is known that $d(n) = O(n^\varepsilon)$, and $n\phi(n)^{-1} = O(n^\varepsilon)$. Choosing m_1 and m_2 so that $m_2 - m_1 = L_n$, we have $L_n = o(n)$.

2 In this section we give two lemmas which are used in the proof. The first is due to Cassels [1]. The measure of a measurable set $\mathcal{A}$ will be denoted by $|\mathcal{A}|$.

LEMMA 2. *Let* $\{I_k\}$ *be a sequence of intervals and let* $\{U_k\}$ *be a sequence of measurable sets such that, for some positive* $\varepsilon<1$,

(3) $$U_k\subset I_k\,,\qquad |U_k|\geqq\varepsilon|I_k|\,,\qquad |I_k|\to 0\,.$$

Then the set of points which belong to infinitely many of the I_k *has the same measure as the set of points which belong to infinitely many of the* U_k.

PROOF. Let

$$\mathcal{I}=\bigcap_{K=1}^{\infty}\bigcup_{k\geqq K}I_k\,,\qquad \mathcal{U}_k=\bigcup_{k\geqq K}U_k\,,\qquad \mathcal{D}_k=\mathcal{I}-\mathcal{U}_k\,.$$

The lemma states that $\bigcup\mathcal{D}_k$ has measure 0. In fact, each $\mathcal{D}_k$ has measure 0. If not, let x_0 be a density point of $\mathcal{D}_k$ in $\mathcal{D}_k$. Then since $x_0\in I_k$ for infinitely many k, and $|I_k|\to 0$,

(4) $$|\mathcal{D}_k\cap I_k|\sim|I_k|\quad\text{as}\quad k\to\infty\,,\qquad x_0\in I_k\,.$$

On the other hand, let $k\geqq K$. Then $\mathcal{D}_k\cap U_k=\phi$, so U_k and $\mathcal{D}_k\cap I_k$ are disjoint subsets of I_k. Therefore,

$$|I_k|\geqq|U_k|+|\mathcal{D}_k\cap I_k|\geqq\varepsilon|I_k|+|\mathcal{D}_k\cap I_k|\,,$$

or

(5) $$|\mathcal{D}_k\cap I_k|\leqq(1-\varepsilon)|I_k|\,,\qquad k\geqq K\,,$$

contrary to (4).

A transformation of $0\leqq x<1$ into itself is *metrically transitive* if each measurable subset which goes into itself under the transformation has measure 0 or 1.

LEMMA 3. *For each pair of integers* q, s *with* $q\geqq 2$, *the transformation*

$$x\to qx+\frac{s}{q}\qquad(\text{mod }1)$$

is metrically transitive.

PROOF. Let $\mathcal{A}$ be a measurable set which goes into itself under this transformation. Then $\mathcal{A}$ also goes into itself under the ν-th iterate $x\to q^\nu x+\frac{s}{q}$ (mod 1). Letting ϕ be the characteristic function of $\mathcal{A}$, we have $\phi(x)\leqq\phi\left(q^\nu x+\frac{s}{q}\right)$.

Suppose $|\mathcal{A}|>0$. Let x_0 be a density point of $\mathcal{A}$, and let I_ν be the interval of length $q^{-\nu}$ centered at x_0. Then

$$|\mathcal{A}\cap I_\nu|=\int_{I_\nu}\phi(x)dx\leqq\int_{I_\nu}\phi\left(q^\nu x+\frac{s}{q}\right)dx=\frac{1}{q^\nu}\int_0^1\phi(x)dx=|I_\nu|\cdot|\mathcal{A}|\,.$$

Since x_0 is a density point of $\mathcal{A}$, and $I_\nu\to 0$, the left side is asymptotically $|I_\nu|$.

Therefore $|\mathcal{A}|=1$.

3. PROOF OF THEOREM 1. For each prime number p, and each integer $\nu \geqq 1$, we consider the approximation

$$\left|x-\frac{m}{n}\right|<p^{\nu-1}\delta(n) \qquad (m, n)=1 \tag{6}$$

and define two increasing sequences of sets $\mathcal{A}(p^\nu)$ and $\mathcal{B}(p^\nu)$ as follows:

$x \in \mathcal{A}(p^\nu)$ if x satisfies (6) for infinitely many n with $p \nmid n$;

$x \in \mathcal{B}(p^\nu)$ if x satisfies (6) for infinitely many n with $p \| n$.

The sets $\mathcal{A}(p)$, $\mathcal{B}(p)$ are subsets of $\mathcal{E}$.

By Lemma 2, since $\delta(n)\to 0$ we have $|\mathcal{A}(p^\nu)|=|\mathcal{A}(p)|$. Therefore the union $\mathcal{A}^*(p)$ of the $\mathcal{A}(p^\nu)$ also has measure $|\mathcal{A}(p)|$.

If x satisfies (6) with $p \nmid n$, then

$$\left|px-\frac{pm}{n}\right|<p^\nu\delta(n), \qquad (pm, n)=1\,.$$

It follows that the transformation $x\to px$ (mod 1) takes $\mathcal{A}(p^\nu)$ into $\mathcal{A}(p^{\nu+1})$ and thus takes $\mathcal{A}^*(p)$ into itself. By Lemma 3, $\mathcal{A}^*(p)$ has measure 0 or 1. Therefore $\mathcal{A}(p)$ has measure 0 or 1.

A similar argument shows that $\mathcal{B}(p)$ has measure 0 or 1. One uses the transformation $x\to px+\frac{1}{p}$ (mod 1): If x satisfies (6) with $p\| n$, then

$$\left|px+\frac{1}{p}-\frac{pm+\frac{n}{p}}{n}\right|<p^\nu\delta(n), \qquad \left(pm+\frac{n}{p}, n\right)=1\,.$$

Should either $\mathcal{A}(p)$ or $\mathcal{B}(p)$ have positive measure for some prime p, then $|\mathcal{E}|=1$ and the proof is complete. Therefore we may suppose that for all p,

$$|\mathcal{A}(p)|=0, \qquad |\mathcal{B}(p)|=0\,. \tag{7}$$

Now let $\mathcal{C}(p)$ be the set of x for which (2) holds for infinitely many n with $p^2 \mid n$.

Obviously $\mathcal{E}=\mathcal{A}(p)\cup\mathcal{B}(p)\cup\mathcal{C}(p)$. It follows from (7) that for all p, $|\mathcal{E}|=|\mathcal{C}(p)|$.

If m, n and x satisfy (2) with $p^2 \mid n$, then

$$\left|x\pm\frac{1}{p}-\frac{m\pm\frac{n}{p}}{n}\right|<\delta(n), \qquad \left(m\pm\frac{n}{p}, n\right)=1\,.$$

Therefore the set $\mathcal{C}(p)$ has period $\frac{1}{p}$. Since $\mathcal{E}$ differs from $\mathcal{C}(p)$ by a set of measure 0, if follows that for each interval I_p of length $\frac{1}{p}$,

$$|\mathcal{E}\cap I_p|=|I_p|\cdot|\mathcal{E}|\,.$$

Now suppose $|\mathcal{E}|>0$. Let x_0 be a density point of $\mathcal{E}$. Let $\{I_p\}$ be the sequence of intervals of length $\frac{1}{p}$, centered at x_0. By the density point theorem,

$$|\mathcal{E}\cap I_p|\sim|I_p| \quad \text{as} \quad p\to\infty .$$

Therefore $|\mathcal{E}|=1$. This completes the proof.

The result of this paper is part of the author's dissertation, Princeton (1959). The author wishes to express here his thanks to Professor D.C. Spencer for his kind encouragement.

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