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On affinely connected manifolds admitting groups of affine motions with complex reducible linear isotropy groups.

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With respect to affinely connected manifolds admitting groups of affine motions of various types and with respect to the groups themselves, especially on their dimensions, there are many papers, for instance, by I. P. Egorov, H. C. Wang and K. Yano [5], Y. Mutō [6] and the others.

In this paper, we study affinely connected manifolds admitting groups of affine motions of some types with complex reducible linear isotropy groups, that is, with linear isotropy groups which are real representations of complex linear homogeneous groups.

The main purpose is to prove Theorems 4.1, 4.2 and 4.3 in § 4, as applications of Theorem 3.1 and Corollary 3.1 in § 3.

§ 1. Preliminary remarks.

The notations $GL(n, R)$, $GL(m, C)$, $SL(n, R)$, $SL(m, C)$ are as usual and furthermore we denote the real representations of $GL(m, C)$ and $SL(m, C)$ by $RGL(m, C)$ and $RSL(m, C)$ respectively. The other notations are as follows.

- E_N : unit martrix of degree N.
- H^1 : real one dimensional homothetic group: $x \rightarrow rx$ (x, r: real; $r > 0$).
- H_N : real one dimensional group composed of all $(N \times N)$-matirces aE_N (a: positive real).
- T^1 : one dimensional torus group: $z \rightarrow \sigma z$ (σ, z: complex; $|\sigma| = 1$).
- T_m : one dimensional group composed of all complex $(m \times m)$-matrices σE_m (σ: complex; $|\sigma| = 1$).
- $R(T_m)$: real representation of T_m.
- A_{2m} : $2m$-dimensional affinely connected manifold of class C^∞.
- G : Lie group of affine motions of A_{2m}.
- $G(P)$: isotropy group of G leaving invariant a generic point P of A_{2m}.
- $G_0(P)$: linear isotropy group of G at a generic point P, which is the faithful linear representation of $G(P)$. We mean the connected component of the identity.

Under the above notations we give several remarks.

I. $RGL(m, C)$ and $RSL(m, C)$. The matrices M_{2m} of $RGL(m, C)$ are of the form (with respect to suitable bases):

$$M_{2m}=\begin{pmatrix} A_m & -B_m \\ B_m & A_m \end{pmatrix}, \tag{1.1}$$

where A_m and B_m are real matrices of degree m. (1.1) is equivalent to the fact that M_{2m} leaves invariant a matrix $\begin{pmatrix} 0 & E_m \\ -E_m & 0 \end{pmatrix}$. If we consider a matrix

$$P=\frac{1}{\sqrt{2}}\begin{pmatrix} E_m & E_m \\ -\sqrt{-1}\,E_m & \sqrt{-1}\,E_m \end{pmatrix},$$

then we see that

$$P^{-1}M_{2m}P=\begin{pmatrix} A_m+\sqrt{-1}\,B_m & 0 \\ 0 & A_m-\sqrt{-1}\,B_m \end{pmatrix}, \tag{1.2}$$

which gives a transformation of $RSL(m, C)$ with respect to complex bases. If $M_{2m}\in RSL(m, C)$, then $\det|A_m+\sqrt{-1}\,B_m|=\det|A_m-\sqrt{-1}\,B_m|=1$. The real representation $R(T_m)$ of T_m is given by the matrices of the form

$$\begin{pmatrix} aE_m & -bE_m \\ bE_m & aE_m \end{pmatrix} \qquad (a, b:\ \text{real};\ a^2+b^2=1).$$

II. Decomposition of $GL(m, C)$ and $RGL(m, C)$. It is easily seen that

$$GL(m, C)=H^1\otimes T^1\otimes SL(m, C),$$

where $\otimes$ denotes the Kronecker products of the groups. Then, a matrix of $RGL(m, C)$ is given by the form: $h\cdot t\cdot s$, where

$$h=aE_{2m}\in H_{2m} \qquad (a:\ \text{positive real}),$$

$$t=\begin{pmatrix} aE_m & -bE_m \\ bE_m & aE_m \end{pmatrix}\in R(T_m) \qquad (a, b:\ \text{real};\ a^2+b^2=1)$$

and $s\in RSL(m, C)$. h, t and s are mutually commutative.

III. Groups of affine motions and linear isotropy groups. Let

$$x'^A=f^A(x^1, \cdots, x^N;\ a^1, \cdots, a^r) \qquad (A, B, C, \cdots=1, \cdots, N) \tag{1.3}$$

be a transformation of a local group of affine motions in an N-dimensional affinely connected manifold A_N, where (x^A) is the coordinate neighborhood of A_N and $a^1, \cdots, a^r$ are the parameters of G, then we have

$$\Gamma_B{}^A{}_C(x')=\Gamma'_B{}^A{}_C(x'), \tag{1.4}$$

where $\Gamma'_B{}^A{}_C(x')$ are the quantities obtained from $\Gamma_B{}^A{}_C(x)$ by considering (1.3) as a coordinate transformation. Conversely, a transformation $(x)\to(x')$ satisfying (1.4) is an affine motion of A_N.

We have from (1.4),

$$T^A{}_{BC}(x')=T'^A{}_{BC}(x')\,, \tag{1.5}$$

$$R^A{}_{BCD}(x')=R'^A{}_{BCD}(x')\,, \tag{1.6}$$

where $T^A{}_{BC}$ and $R^A{}_{BCD}$ are torsion and curvature tensors respectively, i.e.,

$$\begin{cases} T^A{}_{BC}=\dfrac{1}{2}(\Gamma_B{}^A{}_C-\Gamma_B{}^A{}_C)\,, \\ R^A{}_{BCD}=\partial\Gamma_B{}^A{}_C/\partial x^D-\partial\Gamma_B{}^A{}_D/\partial x^C+\Gamma_B{}^E{}_C\Gamma_E{}^A{}_D-\Gamma_B{}^E{}_D\Gamma_E{}^A{}_C\,. \end{cases}$$

Now, let

$$x'^A=g^A(x^1,\cdots,x^N;\ b^1,\cdots,b^{r'}) \tag{1.7}$$

be a transformation of the isotropy group $G(P_0)$ leaving invariant a point P_0 $(x_0{}^1,\cdots,x_0{}^N)$, where $b^1,\cdots,b^{r'}$ $(r'\leqq r)$ are the parameters of $G(P_0)$ and satisfy

$$x_0{}^A=g^A(x_0{}^1,\cdots,x_0{}^N;\ b^1,\cdots,b^{r'})\,. \tag{1.8}$$

If we consider a transfomation (1.7), we have

$$T^A{}_{BC}(x')=\frac{\partial x'^A}{\partial x^P}\frac{\partial x^Q}{\partial x'^B}\frac{\partial x^R}{\partial x'^C}T^P{}_{QR}(x)\,,$$

and at the point P_0 $(x_0{}^1,\cdots,x_0{}^N)$, these become

$$T^A{}_{BC}(x_0)=\Big(\frac{\partial x'^A}{\partial x^P}\Big)_0\Big(\frac{\partial x^Q}{\partial x'^B}\Big)_0\Big(\frac{\partial x^R}{\partial x'^C}\Big)_0T^P{}_{QR}(x_0)\,, \tag{1.9}$$

where $(\ )_0$ denotes the value at P_0, and similarly we have

$$R^A{}_{BCD}(x_0)=\Big(\frac{\partial x'^A}{\partial x^P}\Big)_0\Big(\frac{\partial x^Q}{\partial x'^B}\Big)_0\Big(\frac{\partial x^R}{\partial x'^C}\Big)_0\Big(\frac{\partial x^S}{\partial x'^D}\Big)_0R^P{}_{QRS}(x_0)\,. \tag{1.10}$$

The matrices $(\partial x'^A/\partial x^P)_0$ appearing in (1.9) and (1.10) give the matrices of the linear isotropy group $G_0(P_0)$.

IV. H_N denotes, as mentioned in the above, the (real) one dimensional group whose matrices are given by aE_N (a: positive real). If $G_0(P)$ contains this H_N, then whether it is a real representation of a complex linear group or not, we see that at any generic point of A_N, $T^A{}_{BC}=0$, $R^A{}_{BCD}=0$ $(A,B,C,D,\cdots=1,\cdots,N)$, which is already known (Ishihara and Obata [**7**, Theorem 2 and 3]). The outline of the proof is as follows.

At any generic point P_0 $(x_0{}^1,\cdots,x_0{}^N)$, (1.9) hold good, where the matrices $(\partial x'^A/\partial x^P)_0$ give transformations of the linear isotropy group $G_0(P_0)$. When $G_0(P_0)$ contains H_N, we can consider a transformation $(\partial x'^A/\partial x^P)_0=a\delta^A_P$ (a: positive real $\neq 1$). If we apply this transformation to $T^A{}_{BC}$, we have from (1.9)

$$T^A{}_{BC}(x_0) = aT^A{}_{BC}(x_0) \qquad (a \neq 1)$$

and therefore $T^A{}_{BC}(x_0) = 0$. Consequently, at any generic point of A_N, we have $T^A{}_{BC} = 0$ and similarly $R^A{}_{BCD} = 0$. If A_N is connected, these hold true all over the A_N.

Throughout this paper, if otherwise stated, the ranges of indices are as follows:

$$i, j, k, i_1, j_1, \cdots, i_p, j_p, \cdots, a, b, c, \cdots = 1, \cdots\cdots\cdots, 2m\,;$$

$$\alpha, \beta, \gamma, \cdots, \lambda, \mu, \nu, \lambda_1, \mu_1, \cdots, \lambda_p, \mu_p, \cdots = 1, \cdots\cdots, m\,;$$

$$\bar\alpha, \bar\beta, \bar\gamma, \cdots, \bar\lambda, \bar\mu, \bar\nu, \bar\lambda_1, \bar\mu_1, \cdots, \bar\lambda_p, \bar\mu_p, \cdots = \alpha+m, \beta+m, \gamma+m, \cdots\cdots\cdots.$$

And we adopt the summation convention.

§ 2. Remarks on the dimension of subgroups of *RSL*(*m*, *C*) and *RGL*(*m*, *C*).

Let $\mathfrak{sl}(m, R)$ and $\mathfrak{sl}(m, C)$ be the Lie algebra of $SL(m, R)$ and $SL(m, C)$ respectively, and we consider $\mathfrak{sl}(m, C)$ in its real representation. We have the following lemma.

LEMMA 2.1. *Let* $\mathfrak{g}$ *be a real Lie subalgebra of* $\mathfrak{sl}(m, C)$ $(m > 1)$ *and let* r *be the (real) dimension of* $\mathfrak{g}$. *If* $r > 2m^2-m-1$, *then*

$$\mathfrak{g} = \mathfrak{sl}(m, C)\,.$$

PROOF. If we put $\mathfrak{sl}(m, R) = \mathfrak{s}$, then we can put $\mathfrak{sl}(m, C) = \mathfrak{s} + \sqrt{-1}\,\mathfrak{s}$ (direct sum) up to an isomorphism. Let

$$\pi : \mathfrak{sl}(m, C) \rightarrow \sqrt{-1}\,\mathfrak{s}$$

be a projection from $\mathfrak{sl}(m, C)$ to $\sqrt{-1}\,\mathfrak{s}$ such that

$$\pi(X) = Z$$

where

$$X = Y + Z \qquad (Y \in \mathfrak{s},\ Z \in \sqrt{-1}\,\mathfrak{s})\,.$$

If we consider π on $\mathfrak{g}$, then the kernel of π in $\mathfrak{g}$ is $\mathfrak{g} \cap \mathfrak{s}$. Since $\pi(\mathfrak{g})$ is in $\sqrt{-1}\,\mathfrak{s}$, we have

$$\dim \sqrt{-1}\,\mathfrak{s} \geqq \dim \pi(\mathfrak{g}) = \dim \mathfrak{g} - \dim (\mathfrak{g} \cap \mathfrak{s})\,,$$

from which

$$\dim (\mathfrak{g} \cap \mathfrak{s}) \geqq \dim \mathfrak{g} - \dim \sqrt{-1}\,\mathfrak{s} = r - (m^2-1)$$

$$> (2m^2-m-1) - (m^2-1) = m^2 - m\,.$$

Hence $\mathfrak{g} \cap \mathfrak{s}$ is a Lie subalgebra of $\mathfrak{s}$ whose dimension is $> m^2-m$ and by virtue of the well known results[1] on $\mathfrak{s}$ $(=\mathfrak{sl}(m, R))$, we have $\mathfrak{g}\cap\mathfrak{s}=\mathfrak{s}$, that is, $\mathfrak{g}\supset\mathfrak{s}$.

Next, put $\mathfrak{g}_1=\sqrt{-1}\,\mathfrak{g}\cap\mathfrak{s}$, then $\mathfrak{g}_1\neq\{0\}$, since $\dim\mathfrak{g}>2m^2-m-1>m^2-1=\dim\mathfrak{s}$ (Note that $m>1$). Furthermore, $\mathfrak{g}_1$ is an ideal of $\mathfrak{s}$, for

$$[\mathfrak{g}_1, \mathfrak{s}]\subset[\sqrt{-1}\,\mathfrak{g}, \mathfrak{s}]\cap[\mathfrak{s}, \mathfrak{s}]\subset\sqrt{-1}\,\mathfrak{g}\cap\mathfrak{s}=\mathfrak{g}_1,$$

taking account of

$$[\sqrt{-1}\,\mathfrak{g}, \mathfrak{s}]=\sqrt{-1}\,[\mathfrak{g}, \mathfrak{s}]\subset\sqrt{-1}\,\mathfrak{g}.$$

Since $\mathfrak{s}$ is a simple Lie algebra, we have $\mathfrak{g}_1=\mathfrak{s}$, from which and from the fact that $\mathfrak{g}\supset\mathfrak{s}$, we get

$$\mathfrak{g}\supset\mathfrak{s}+\sqrt{-1}\,\mathfrak{s}=\mathfrak{sl}(m, C),$$

that is

$$\mathfrak{g}=\mathfrak{sl}(m, C). \qquad \text{Q. E. D.}$$

This Lemma tells us that if the dimension of a Lie subgroup g of $RSL(m, C)$ is $>2m^2-m-1$, then $g=RSL(m, C)$.

LEMMA 2.2. *Let g be a subgroup of the real representation $RGL(m, C)$ of the complex general linear group $GL(m, C)$. If (real) $\dim g>2m^2-m+1$, then necessarily $\dim g\geqq 2m^2-2$ and g is one of the followings (for $m>3$):*

(I)	$g=RGL(m, C)$	$(\dim g=2m^2)$,
(II)	$g=R(H^1\otimes SL(m, C))$[2]	$(\dim g=2m^2-1)$,
(III)	$g=R(T^1\otimes SL(m, C))$	$(\dim g=2m^2-1)$,
(IV)	$g=RSL(m, C)$	$(\dim g=2m^2-2)$.

For $m=3$, the case (IV) *and for $m=2$, the cases* (II), (III), (IV) *drop down respectively.*

PROOF. Let g_1 be the subgroup of g contained in $RSL(m, C)$, that is, let $g_1=g\cap RSL(m, C)$. Then,

$$\dim g_1>(2m^2-m+1)-2=2m^2-m-1,$$

and hence for $m>3$ we have $g_1=RSL(m, C)$ by virtue of Lemma 2.1, from which the conclusion of the Lemma follows immediately, omitting the cases (IV) for $m=3$. The case $m=2$ is trivial. Q. E. D.

§ 3. An algebraic theorem.

LEMMA 3.1. *Let $T^{\lambda_1\lambda_2\cdots\lambda_p}_{\mu_1\mu_2\cdots\mu_q}$ be $m^p\times m^q$ quantities, where $p\not\equiv q \pmod{m}$. If*

1) The proof is at first given by S. Lie: Theorie der Transformationsgruppen, I, p. 564, Theorem 100. A refined proof is recently given by T. Satō in his paper which will shortly appear.

2) In general, we denote the real representation of a group g with complex variables by $R(g)$.

$T^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}$ *satisfy*

$$\sum_{a=1}^{p}\delta^{\lambda_a}_{\beta}\, T^{\lambda_1\cdots\overset{a}{\widehat{\alpha}}\cdots\lambda_p}_{\mu_1\cdots\cdots\mu_q}-\sum_{b=1}^{q}\delta^{\alpha}_{\mu_b}\, T^{\lambda_1\cdots\cdots\cdots\lambda_p}_{\mu_1\cdots\underset{b}{\underbrace{\beta}}\cdots\mu_q}=\frac{p-q}{m}\,\delta^{\alpha}_{\beta}\, T^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\,{}^{3)}, \tag{3.1}$$

then $T^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\equiv 0.$

PROOF. Put $\alpha=\beta$ in (3.1), not applying the summation convention. If any of $\lambda_1,\cdots,\lambda_p$ and $\mu_1,\cdots,\mu_q$ are not equal to α $(=\beta)$, then the left hand side of (3.1) vanishes, to obtain

$$T^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}=0\,,$$

since $p-q\neq 0$. If some of $\lambda_1,\cdots,\lambda_p$ and some of $\mu_1,\cdots,\mu_q$ are equal to α $(=\beta)$, for instance, if $\lambda_1=\alpha$; $\mu_2,\mu_3=\alpha$ and the other λ's and μ's are not equal to α, then we have

$$T^{\alpha\lambda_2\cdots\cdots\lambda_p}_{\mu_1\alpha\alpha\cdots\mu_q}-T^{\alpha\lambda_2\cdots\cdots\lambda_p}_{\mu_1\alpha\alpha\cdots\mu_q}-T^{\alpha\lambda_2\cdots\cdots\lambda_p}_{\mu_1\alpha\alpha\cdots\mu_q}=\frac{p-q}{m}\,T^{\alpha\lambda_2\cdots\cdots\lambda_p}_{\mu_1\alpha\alpha\cdots\mu_q}\,,$$

from which we get $T^{\alpha\lambda_2\cdots\cdots\lambda_p}_{\mu_1\alpha\alpha\cdots\mu_q}=0$. The other cases can be proved similarly, since $(p-q)/m$ is not an integer.

Let $SL(m,R)\times SL(m,R)$ be the diagonal product of $SL(m,R)$, the representative matrix being of the form

$$\begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}, \qquad \det|A|=1\,,$$

where A is a real $(m\times m)$-matrix. This is of course a Lie subgroup of $GL(2m,R)$ and conjugate to a subgroup of $RSL(m,C)$ in $GL(2m,R)$, since a matrix of $RSL(m,C)$ is of the form (1.1) (the determinant $=1$) with respect to suitable bases. Then we have the following

THEOREM 3.1. *Let* $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ *be a tensor with respect to* $GL(2m,R)$ *invariant under a subgroup* g *of* $GL(2m,R)$ *containing* $SL(m,R)\times SL(m,R)$. *If* $p\not\equiv q$ $(\mathrm{mod}\, m)$, *then* $T^{i_1\cdots i_p}_{j_1\cdots j_q}\equiv 0.$

PROOF. With respect to suitable bases, the infinitesimal transformations of $SL(m,R)\times SL(m,R)$ are given by $\delta^i_j+\varepsilon^i_j$ satisfying

$$\varepsilon^{\alpha}_{\beta}=\varepsilon^{\bar\alpha}_{\bar\beta}\,,\qquad \varepsilon^{\alpha}_{\alpha}\,(=\varepsilon^{\bar\alpha}_{\bar\alpha})=0\,,\qquad \varepsilon^{\alpha}_{\bar\beta}=\varepsilon^{\bar\alpha}_{\beta}=0\,, \tag{3.2}$$

ε^i_j being arbitrary infinitesimal except the above restrictions. Since $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ is invariant under g, it is of course invariant under the infinitesimal transformations of $SL(m,R)\times SL(m,R)$ satisfying (3.2), which is expressed by

$$\sum_{a=1}^{p}\varepsilon^{i_a}_{k}\, T^{i_1\cdots\overset{a}{\widehat{k}}\cdots i_p}_{j_1\cdots\cdots j_q}-\sum_{b=1}^{q}\varepsilon^{k}_{j_b}\, T^{i_1\cdots\cdots i_p}_{j_1\cdots\underset{b}{\underbrace{k}}\cdots j_q}=0\,. \tag{3.3}$$

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3) $\overset{a}{\widehat{\alpha}}$ means that the α is in the a-th position from λ_1.

Putting in this equation $i_1=\lambda_1, \cdots, i_p=\lambda_p; j_1=\mu_1, \cdots, j_q=\mu_q$ and taking account of (3.2), we have

$$\sum_{a=1}^{p} \varepsilon_{\alpha}^{\lambda_a} T_{\mu_1\cdots\cdots\mu_q}^{\lambda_1\cdots\overset{a}{\widehat{\alpha}}\cdots\lambda_p} - \sum_{b=1}^{q} \varepsilon_{\mu_b}^{\beta} T_{\mu_1\cdots\underset{b}{\underbrace{\beta}}\cdots\mu_q}^{\lambda_1\cdots\cdots\cdots\lambda_p} = 0 .$$

This equation being consistent for any $\varepsilon_{\alpha}^{\lambda_a}$'s satisfying $\varepsilon_{\alpha}^{\alpha}=0$, there exist quantities $X_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p}$ such that

$$\sum_{a=1}^{p} \varepsilon_{\alpha}^{\lambda_a} T_{\mu_1\cdots\cdots\mu_q}^{\lambda_1\cdots\overset{a}{\widehat{\alpha}}\cdots\lambda_p} - \sum_{b=1}^{q} \varepsilon_{\mu_b}^{\beta} T_{\mu_1\cdots\underset{b}{\underbrace{\beta}}\cdots\mu_q}^{\lambda_1\cdots\cdots\cdots\lambda_p} = \varepsilon_{\alpha}^{\alpha} X_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p}$$

or

$$\varepsilon_{\alpha}^{\beta}\Big(\sum_{a=1}^{p} \delta_{\beta}^{\lambda_a} T_{\mu_1\cdots\cdots\mu_q}^{\lambda_1\cdots\overset{a}{\widehat{\alpha}}\cdots\lambda_p} - \sum_{b=1}^{q} \delta_{\mu_b}^{\alpha} T_{\mu_1\cdots\underset{b}{\underbrace{\beta}}\cdots\mu_q}^{\lambda_1\cdots\cdots\cdots\lambda_p}\Big) = \varepsilon_{\alpha}^{\beta}\, \delta_{\beta}^{\alpha}\, X_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p}$$

are identically satisfied for quite arbitrary $\varepsilon_{\alpha}^{\beta}$. Therefore we have

$$\sum_{a=1}^{p} \delta_{\beta}^{\lambda_a} T_{\mu_1\cdots\cdots\mu_q}^{\lambda_1\cdots\overset{a}{\widehat{\alpha}}\cdots\lambda_p} - \sum_{b=1}^{q} \delta_{\mu_b}^{\alpha} T_{\mu_1\cdots\underset{b}{\underbrace{\beta}}\cdots\mu_q}^{\lambda_1\cdots\cdots\cdots\lambda_p} = \delta_{\beta}^{\alpha}\, X_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p} . \tag{3.4}$$

Contracting with respect to α and β, we get

$$X_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p} = \frac{p-q}{m} T_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p} ,$$

and substituting this in (3.4), we obtain

$$\sum_{a=1}^{p} \delta_{\beta}^{\lambda_a} T_{\mu_1\cdots\cdots\mu_q}^{\lambda_1\cdots\overset{a}{\widehat{\alpha}}\cdots\lambda_p} - \sum_{b=1}^{q} \delta_{\mu_b}^{\alpha} T_{\mu_1\cdots\underset{b}{\underbrace{\beta}}\cdots\mu_q}^{\lambda_1\cdots\cdots\cdots\lambda_p} = \frac{p-q}{m}\, \delta_{\beta}^{\alpha}\, T_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p} .$$

Since $p \not\equiv q \pmod{m}$, the Lemma 3.1 is applicable, to obtain

$$T_{\mu_1\cdots\mu_q}^{\lambda_1\cdots\lambda_p} = 0 .$$

Next, putting in (3.3) $i_1=\lambda_1, i_2=\lambda_2, \cdots, i_p=\lambda_p; j_1=\mu_1, \cdots, j_q=\mu_q$, and taking account of (3.2), we have

$$\varepsilon_{\alpha}^{\lambda_1} T_{\mu_1\cdots\mu_q}^{\bar{\alpha}\lambda_2\cdots\lambda_p} + \sum_{a=2}^{p} \varepsilon_{\alpha}^{\lambda_a} T_{\mu_1\cdots\cdots\mu_q}^{\bar{\lambda}_1\cdots\overset{a}{\widehat{\alpha}}\cdots\lambda_p} - \sum_{b=1}^{q} \varepsilon_{\mu_b}^{\beta} T_{\mu_1\cdots\underset{b}{\underbrace{\beta}}\cdots\mu_q}^{\bar{\lambda}_1\lambda_2\cdots\cdots\lambda_p} = 0^{4)} .$$

If we put

$$T_{\mu_1\cdots\cdots\mu_q}^{\bar{\lambda}_1\lambda_2\cdots\lambda_p} = \overset{*}{T}{}_{\mu_1\cdots\cdots\mu_q}^{\lambda_1\lambda_2\cdots\lambda_p}$$

for simplicity, then the above equation is written as

$$\sum_{a=1}^{p} \varepsilon_{\alpha}^{\lambda_a} \overset{*}{T}{}_{\mu_1\cdots\cdots\mu_q}^{\lambda_1\cdots\overset{a}{\widehat{\alpha}}\cdots\lambda_p} - \sum_{b=1}^{q} \varepsilon_{\mu_b}^{\beta} \overset{*}{T}{}_{\mu_1\cdots\underset{b}{\underbrace{\beta}}\cdots\mu_q}^{\lambda_1\cdots\cdots\cdots\lambda_p} = 0 .$$

4) We adopt a new summation convention such that

$$u_{\lambda}v^{\bar{\lambda}} = u_1 v^{\bar{1}} + \cdots + u_m v^{\bar{m}} .$$

Therefore by means of the same process as in the above we get

$$\overset{*}{T}{}^{\lambda_1\lambda_2\cdots\lambda_p}_{\mu_1\cdots\cdots\mu_q}\equiv 0 \qquad \text{that is} \qquad T^{\bar{\lambda}_1\lambda_2\cdots\lambda_p}_{\mu_1\cdots\cdots\mu_q}\equiv 0\,.$$

Analogously we can see that the other components of $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ all vanish.

Q. E. D.

Since $RSL(m, C)$ contains a subgroup conjugate to $SL(m, R)\times SL(m, R)$ in $GL(2m, R)$, we have

COROLLARY 3.1. *Let* $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ *be a tensor with respect to* $GL(2m, R)$ *invariant under a subgroup* g *of* $GL(2m, R)$ *containing* $RSL(m, C)$. *If* $p\not\equiv q \pmod m$, *then* $T^{i_1\cdots i_p}_{j_1\cdots j_q}\equiv 0$.

REMARK. The Theorem 3.1 and Corollary 3.1 are valid for a tensor $T^{i_1\cdots i_p}_{j_1\cdots j_q}$ in general with respect to $GL(2m, R)$. It means that those are applicable for affinely connected manifolds even if we do not know, for instance, whether they are complex analytic or not.

EXAMPLE. Let $T^i{}_{jk}$ and $R^i{}_{jkh}$ be tensors with respect to $GL(2m, R)$ invariant under $RSL(m, C)$. Whether there are tensorial relations among the components of $T^i{}_{jk}$ or $R^i{}_{jkh}$, or not, we have $T^i{}_{jk}=0$ if $m>1$ and $R^i{}_{jkh}=0$ if $m>2$ by virtue of Corollary 3.1.

§ 4. Applications.

From Theorem 3.1 and Corollary 3.1, we have immediately

THEOREM 4.1. *Let* A_{2m} *be an affinely connected manifold admitting a group of affine motions. If the linear isotropy group* $G_0(P)$ *of* G *contains* $SL(m, R)\times SL(m, R)$, *or if it contains* $RSL(m, C)$, *then we have* $T^i{}_{jk}=0$, $R^i{}_{jkh}=0$ *for* $m>2$.

REMARK. If $m=2$, we can easily see that $T^i{}_{jk}=0$, $\nabla_l R^i{}_{jkh}=0$[5] since $G_0(P)$ contains a transformation given by $-\delta^i_j$. For $m=1$, the assumptions of the Theorem are meaningless.

THEOREM 4.2. *Let* A_{2m} *be an affinely connected manifold admitting an almost complex structure and let* G *be a group of affine motions of* A_{2m} *leaving invariant the almost complex structure. If* $\dim G>2m^2+m+1$, *then only one of the following cases can occur (for* $m>3$*)*:

(I)	$\dim G=2m^2+2m$,	$G_0(P)=RGL(m, C)$;
(II)	$\dim G=2m^2+2m-1$,	$G_0(P)=R(H^1\otimes SL(m, C))$;
(III)	$\dim G=2m^2+2m-1$,	$G_0(P)=R(T^1\otimes SL(m, C))$;
(IV)	$\dim G=2m^2+2m-2$,	$G_0(P)=RSL(m, C)$.

In each case, $T^i{}_{jk}=0$ *and* $R^i{}_{jkh}=0$ *at each generic point of* A_{2m}, *where* $T^i{}_{jk}$ *and* $R^i{}_{jkh}$ *are the torsion and the curvature tensors of* A_{2m}.

5) ∇_l denotes the covariant differentiation with respect to the affine connection of A_{2m}.

PROOF. Put $\dim G = r$ and $\dim G_0(P) = r_0$ and let

$$X_\theta = \xi_\theta{}^i(x) \frac{\partial}{\partial x^i} \qquad (\theta = 1, \cdots, r)$$

be the bases of the Lie algebra of G. If the rank of $\| \xi_\theta{}^i \|$ is q, then $q \leqq 2m$ and we have

$$r_0 = r - q > (2m^2 + m + 1) - 2m = 2m^2 - m + 1 .$$

Since $G_0(P)$ is a linear homogeneous group leaving invariant the almost complex structure at the tangent space of P, it is a subgroup of $RGL(m, C)$. Hence, by virtue of Lemma 2.2, $G_0(P)$ is one of the followings:

$$G_0(P) = RGL(m, C) ,$$
$$G_0(P) = R(H^1 \otimes SL(m, C)) ,$$
$$G_0(P) = R(T^1 \otimes SL(m, C)) ,$$
$$G_0(P) = RSL(m, C) .$$

In each case $G_0(P)$ contains $RSL(m, C)$ and hence we have $T^A{}_{BC} = 0$ and $R^A{}_{BCD} = 0$ by virtue of Theorem 4.1. Q. E. D.

REMARK. In the former two cases $G_0(P)$ contains H_{2m} and we get also the same conclusion by virtue of the remark IV of § 1.

To consider the cases $m \leqq 3$, we state the following Lemma.

LEMMA 4.1. *Let A_{2m} be a $2m$-dimensional affinely connected manifold admitting a group of affine motions G and assume that the linear isotropy group $G_0(P)$ of G contains $R(T_m)$. Then A_{2m} is an affine symmetric space, that is,*

$$T^i{}_{jk} = 0 , \qquad \nabla_l R^i{}_{jkh} = 0 ,$$

and G is transitive.

PROOF. Since $G_0(P)$ contains $R(T_m)$, there is in $G_0(P)$ a transformation given by $-\delta^i_j$ since the matrices of the transformation of $R(T_m)$ are of the form $\begin{pmatrix} aE_m & -bE_m \\ bE_m & aE_m \end{pmatrix}$ $(a^2 + b^2 = 1)$. Then we can easily see that $T^A{}_{BC} = 0$ (cf. Fukami [**3**, Lemma 3]). And further, since $R^i{}_{jkh}$ is invariant under G, $\nabla_l R^i{}_{jkh}$ is also invariant under G, hence we have $\nabla_l R^i{}_{jkh} = 0$. The transitivity of G easily follows.

For $m \leqq 3$, $2m^2 + m + 1 \geqq 2m^2 + 2m - 2$ and we get

THEOREM 4.2. *Let A_{2m} $(m \geqq 3)$ be an affinely connected manifold admitting an almost complex structure and let G be a group of affine motions leaving invariant the almost complex structure and assume that* $\dim G \geqq 2m^2 + 2m - 2$. *Then,*

(I) *For $m = 3$, we have $T^i{}_{jk} = 0$ and $R^i{}_{jkh} = 0$.*

(II) *For $m = 2$, A_{2m} $(= A_4)$ is an affine symmetric space, the almost complex structure being necessarily a complex analytic structure parallel with respect to* *the affine connection; or* $T^i_{jk}=0$, $R^i_{jkh}=0$.

(III) *For* $m=1$, *G is simply transitive; or* $T^i_{jk}=0$, $R^i_{jkh}=0$; *or* A_2 *is a Riemannian manifold with constant curvature.*

PROOF. (I) $m=3$. If $\dim G=2m^2+2m$, then we can easily see that G is transitive and $G_0(P)=GL(2m, R)$, and hence $T^i_{jk}=0$, $R^i_{jkh}=0$ (cf. IV of § 1).

If $\dim G=2m^2+2m-1$, then

$$\dim G_0(P)=r-q \geqq (2m^2+2m-1)-2m=2m^2-1\,,$$

where q has the same meaning as in the proof of Theorem 4.2, and hence we have

$$G_0(P)=RGL(m, C), \qquad R(H^1 \otimes SL(m, C)) \qquad \text{or} \qquad R(T^1 \otimes SL(m, C))\,.$$

In each cace, since $G_0(P)$ contains $RSL(m, C)$, we have $T^i_{jk}=0$ and $R^i_{jkh}=0$ by virtue of Corollary 3.1.

If $\dim G=2m^2+2m-2$, then we have $\dim G_0(P) \geqq (2m^2+2m-2)-2m=2m^2-2$ $(=16)$. Hence $\dim G_0(P)=2m^2, 2m^2-1$ or $2m^2-2$. If $\dim G_0(P)=2m^2$, which is the maximal dimension, $G_0(P)$ contains $R(T_m)$ and it is transitive $(q=2m)$ by Lemma 4.1. Hence $\dim G=2m^2+2m$, which is impossible. If $\dim G_0(P)=2m^2-1$, it is necessarily one of the followings:

$$R(H^1 \otimes SL(m, C)), \qquad R(T^1 \otimes SL(m, C)), \qquad H_{2m} \times R(T_m) \times g\,,$$

where g is a subgroup of $RSL(m, C)$ of dimension $2m^2-3$. In the former two cases, $G_0(P)$ contains $RSL(m, C)$ and hence G is transitive. For, if otherwise $G_0(P)$ leaves invariant a sublinear space tangent to the trajectory of G passing through P, which is impossible. Therefore $\dim G=2m^2+2m-1$, but it is a contradiction. In the last case $G_0(P)$ contains $R(T_m)$ and hence $G_0(P)$ is also transitive by virtue of Lemma 4.1, which is also a contradiction. Consequently, the only one possible case is that $\dim G_0(P)=2m^2-2$ and G is transitive. In this case $G_0(P)$ is one of the following types:

$$(4.1) \qquad H_{2m} \times g_1, \qquad R(T_m) \times g_2, \qquad H_{2m} \times R(T_m) \times g_3, \qquad RSL(m, C)\,,$$

where g_1, g_2 and g_3 are subgroups of $RSL(m, C)$ of dimension $2m^2-3$, $2m^2-3$ and $2m^2-4$ respectively. But in the former two cases we must have $g_1, g_2 = RSL(m, C)$ by Lemma 2.1 (for $m=3$), which is a contradiction. In the last two cases, we have $T^i_{jk}=0$ and $R^i_{jkh}=0$ by virtue of IV of § 1 and Corollary 3.1 respectively.

(II) $m=2$. If $\dim G=2m^2+2m$, then $T^i_{jk}=0$, $R^i_{jkh}=0$ as in the case $m=3$.

If $\dim G=2m^2+2m-1$, then

$$\dim G_0(P) \geqq (2m^2+2m-1)-2m=2m^2-1 \ (=7)\,;$$

we have $\dim G_0(P)=2m^2$ $(=8)$ or $2m^2-1$ $(=7)$. If $\dim G_0(P)=2m^2$, it is of the maximal dimension and contains $R(T_m)$, from which we see that G is transitive by Lemma 4.1. Hence $\dim G = 2m^2+2m$ $(=12)$, but this contradicts to the assumption that $\dim G = 2m^2+2m-1$ $(=11)$. Consequently, it gives rise the only one case: $\dim G_0(P) = 2m^2-1$ and G is transitive $(q = 2m)$. Then, $G_0(P)$ is one of the following three types:

$$R(H^1 \otimes SL(m, C)), \qquad R(T^1 \otimes SL(m, C)), \qquad H_{2m} \times R(T_m) \times g,$$

where g in the last type is a subgroup of $RSL(m, C)$ of dimension $2m^2-3$ $(=5)$. In each case, we have $T^i{}_{jk} = 0$ and $R^i{}_{jkh} = 0$ by virtue of IV of § 1 or Corollary 3.1.

If $\dim G = 2m^2+2m-2$, we see that $G_0(P)$ is one of the types of (4.1), by the same considerations as in the case $m = 3$. In the second and the fourth case, we see that A_{2m} $(= A_4)$ is affine symmetric by Lemma 4.1 and by the remark to Theorem 4.1 respectively. The almost complex structure $\phi_j{}^i$ gives a complex analytic structure since the Nijenhuis tensor $N_{jk}{}^i$ vanishes[6)] by virtue of the same reason that $T^i{}_{jk}$ vanishes. Further, since $\phi_j{}^i$ is invariant under G, $\nabla_k \phi_j{}^i$ is also invariant under G, hence we have $\nabla_k \phi_j{}^i = 0$. In the remaining case of (4.1) we have $T^i{}_{jk} = 0$, $R^i{}_{jkh} = 0$ by virtue of IV of § 1.

(III) $m = 1$. We have $\dim G_0(P) = 0, 1$, or 2. In the first case, G is simply transitive and in the last case we see that $T^i{}_{jk} = 0$, $R^i{}_{jkh} = 0$. If $\dim G_0(P) = 1$, then $G_0(P) = H_2$ or $R(T_2) = SO(2)$. In the first case, we also have $T^i{}_{jk} = 0$, $R^i{}_{jkh} = 0$. In the second case we have $T^i{}_{jk} = 0$, $\nabla_l R^i{}_{jkh} = 0$, $\nabla_k R_{ij} = 0$, $\nabla_k \phi_j{}^i = 0$, where R_{ij} is the Ricci tensor. If $R_{ij} \equiv 0$, then we get $R^i{}_{jkh} = 0$. If $R_{ij} \not\equiv 0$, we can easily see from $\nabla_k R_{ij} = 0$ and $\nabla_k \phi_j{}^i = 0$ that the restricted homogeneous holonomy group is $SO(2)$. Hence A_2 is a Riemannian manifold, the affine connection under consideration giving the Riemannian connection. Further since it is Riemannian symmetric, it is of constant curvature. Q. E. D.

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6) If A_{2m} is real analytic, $N_{jk}{}^i = 0$ implies immediately the complex analyticity of A_{2m}. If A_{2m} is of class C^∞, we owe to [**11**].

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