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On the Goldbach problem in an algebraic number field I.

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§ 1. Introduction.

The famous but yet unsolved problem of Goldbach is to decide whether the following conjecture is true: every even positive rational integer except 2 and 4 will be represented as the sum of two odd prime numbers.

Concerning this problem, Vinogradov [7] proved in 1937 that every large odd integer is represented as the sum of three prime numbers, and obtained also an asymptotic formula for the number of representations. Estermann [1] proved then, in 1938, using the result of [7], that almost all even rational integers are represented as the sum of two prime numbers.

The purpose of this paper is to generalize these results to the case of algebraic number fields. Our final results will be stated as Theorem 10.1 and Theorem 11.1 in § 10 and § 11 respectively, but we shall give here an outline of our results.

Let K be an algebraic number field of degree n. This and the following notations will be used throughout this paper.

$K^{(1)}, K^{(2)}, \cdots, K^{(r_1)}$ are the real conjugates of K; $K^{(r_1+1)}, \cdots, K^{(r_1+r_2)}, K^{(r_1+r_2+1)} = \bar{K}^{(r_1+1)}, \cdots, K^{(n)} = \bar{K}^{(r_1+r_2)}$ are the complex conjugates of K.

We denote by $\mathfrak{o}$ the ideal consisting of all integers of K, by $\mathfrak{d}$ the *differente* of K and by $D = N(\mathfrak{d})$ (norm of $\mathfrak{d}$) the absolute value of the *discriminant* of K.

Let γ be a number of K and put $\mathfrak{d}\gamma = \mathfrak{b}/\mathfrak{a}$ with integral ideals $\mathfrak{a}$ and $\mathfrak{b}$ such that $(\mathfrak{a}, \mathfrak{b}) = 1$. We call $\mathfrak{a}$ the *denominator* of γ and denote this relation by $\gamma \to \mathfrak{a}$.

If μ is a number of K, we have an n-dimensional complex vector $(\mu^{(1)}, \mu^{(2)}, \cdots, \mu^{(n)})$ with real $\mu^{(q)}$ $(q = 1, 2, \cdots, r_1)$ and complex $\mu^{(p+r_2)} = \bar{\mu}^{(p)}$ $(p = r_1+1, \cdots, r_1+r_2)$, where $\mu^{(i)}$ is the conjugate of μ in $K^{(i)}$ $(i = 1, 2, \cdots, n)$. We shall denote this vector also by μ. We shall consider more generally any n-dimensional complex vector $\xi = (\xi_1, \xi_2, \cdots, \xi_n)$ with real $\xi_1, \cdots, \xi_{r_1}$ and complex $\xi_{p+r_2} = \bar{\xi}_p$ $(p = r_1+1, \cdots, r_1+r_2)$. For such ξ, we write

$$S(\xi) = \sum_{j=1}^{n} \xi_j, \qquad N(\xi) = \prod_{i=1}^{n} \xi_i$$

and put

$$X_q(\xi) = \xi_q \qquad (q=1, 2, \cdots, r_1)$$

$$X_p(\xi) = \Re(\xi_p), \qquad X_{p+r_2}(\xi) = \Im(\xi_p) \qquad (p = r_1+1, \cdots, r_1+r_2).$$

We denote by $x(\xi)$ the n-dimensional real vector:

$$x(\xi) = (X_1(\xi),\ X_2(\xi), \cdots, X_n(\xi)).$$

We call an integer ω of K a *prime number*, if the principal ideal (ω) is a prime ideal.

We call a number γ of K *totally positive number*, if $\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(r_1)}$ are positive. When $r_1=0$, totally positive number means non-vanishing number.

Now let $\delta_1, \delta_2, \cdots, \delta_n$ be a basis of $\mathfrak{d}^{-1}$ and put

$$z_j = \sum_{i=1}^{n} x_i \delta_i^{(j)} \qquad (j=1, 2, \cdots, n)$$

for real numbers $x_1, x_2, \cdots, x_n$. We define a set E of $z=(z_1, z_2, \cdots, z_n)$ as follows:

$$E = \{z\,;\ z_j = \sum_{i=1}^{n} x_i \delta_i^{(j)};\quad -1/2 < x_i \leqq 1/2 \qquad (i=1, 2, \cdots, n)\}$$

and denote by $\mathfrak{E}$ a set in n-dimensional euclidean space as follows:

$$\mathfrak{E} = \{x(z)\,;\ z \in E\}.$$

Let λ be a totally positive integer of K and $\Omega(\lambda)$ be the set of prime numbers ω such that

$$0 < \omega^{(q)} \leqq \lambda^{(q)} \qquad (q=1, 2, \cdots, r_1)$$

$$|\omega^{(p)}| \leqq |\lambda^{(p)}| \qquad (p=r_1+1, \cdots, r_1+r_2).$$

We define a trigonometrical sum as follows:

$$S(z\,;\lambda) = \sum_{\omega \in \Omega(\lambda)} e^{2\pi i\, S(\omega z)}. \tag{1.1}$$

Then the integral

$$\begin{aligned} I_s(\lambda) &= \int_{-1/2}^{1/2}\!\!\cdots\!\int S(z\,;\lambda)^s e^{-2\pi i\, S(\lambda z)} dx_1 dx_2 \cdots dx_n \\ &= 2^{r_2}\sqrt{D} \int_{\mathfrak{E}}\!\cdots\!\int S(z\,;\lambda)^s e^{-2\pi i\, S(\lambda z)} dX_1(z) \cdots dX_n(z) \end{aligned} \tag{1.2}$$

with rational integer $s \geqq 3$ is equal to the number of the s-tuples $(\omega_1, \omega_2, \cdots, \omega_s)$ of prime numbers such that

$$\lambda = \omega_1 + \omega_2 + \cdots + \omega_s$$

$$\omega_j \in \Omega(\lambda) \qquad (j=1, 2, \cdots, s).$$

Assuming that $N(\lambda)$ is sufficiently large, we shall obtain in § 10 an asymptotic formula for $I_s(\lambda)$, which is a generalization of Vinogradov's theorem.

An integer of K will be called *even*, if it is divisible by every prime ideal of K, whose norm is exactly 2, and *odd* if it is prime to any such prime ideal. Then we shall prove in § 11 that almost all totally positive even integers of K are represented as the sum of two totally positive odd prime numbers of K.

We shall now sketch the contents of §§ 2-10. But before doing this, we shall first give an outline of the proof of Vinogradov, which will help understanding of the whole arguments.

We denote by $S_N(x)$ a trigonometrical sum of the following form:

$$S_N(x) = \sum_{p \leqq N} e^{2\pi i px}, \qquad (0 \leqq x \leqq 1) \tag{1.3}$$

where p runs through all prime numbers not exceeding a large integer N. Then the integral

$$I(N) = \int_0^1 S_N(x)^3 e^{-2\pi i Nx} dx \tag{1.4}$$

is equal to the number of representations of N as the sum of three prime numbers.

In order to estimate $I(N)$, we consider the Farey dissection of the interval $[0, 1]$. In our case, however, it is convenient to take $[-\tau, 1-\tau]$ with $\tau = (\log N)^{3h}/N$ $(h \geqq 3)$ instead of $[0, 1]$ and divide $[-\tau, 1-\tau]$ into two parts I_1 and I_2 as follows: I_1 is the sum of subintervals $[-\tau + a/q, \tau + a/q]$, where a and q are integers such that $0 \leqq a < q \leqq (\log N)^{3h}$ and $(a, q) = 1$.

$$I_2 = [-\tau, 1-\tau] - I_1 .$$

If x belongs to I_1, then, writing $x = \frac{a}{q} + y$, $|y| \leqq \tau$, we have

$$S_N(x) = S_N\left(\frac{a}{q} + y\right) = \sum_{\substack{l=0 \\ (l,q)=1}}^{q-1} e^{2\pi i \frac{la}{q}} \sum_{\substack{p \leqq N \\ p \equiv l (q)}} e^{2\pi i py} + O(\sum_{p|q} 1) . \tag{1.5}$$

To estimate the inner sum, we apply the prime number theorem for an arithmetic progression. This is stated as follows:

If we denote by $\pi(x; k, l)$ the number of the prime numbers p such that $p \leqq x$ and $p \equiv l \pmod k$ with $(k, l) = 1$, then we have

$$\pi(x; k, l) = \frac{1}{\varphi(k)} \int_2^x \frac{dt}{\log t} + O(xe^{-c\sqrt{\log x}}) \qquad (c > 0) . \tag{1.6}$$

Moreover, the constants in the error term are independent of k, provided that $k \leqq (\log x)^A$ for a positive constant A.

This important form of prime number theorem was proved by Siegel [5] and Walfisz [8].

By (1.6), the inner sum of the right-hand side of (1.5) is approximated by

$$\frac{1}{\varphi(q)}\int_2^N \frac{e^{2\pi i yt}}{\log t}\,dt=\frac{1}{\varphi(q)}J(y) \tag{1.7}$$

and finally we have

$$S_N\Big(\frac{a}{q}+y\Big)=\frac{\mu(q)}{\varphi(q)}J(y)+O\Big(\frac{N}{(\log N)^{15h+1}}\Big)$$

and so

$$\begin{aligned}&\int_{-\tau+a/q}^{\tau+a/q} S_N(x)^3 e^{-2\pi i Nx}dx\\ &=\frac{\mu(q)}{\varphi(q)^3}\,e^{-2\pi i \frac{a}{q}N}\int_{-\tau}^{\tau} J(y)^3 e^{-2\pi i Ny}dy+\text{error term.}\end{aligned} \tag{1.8}$$

If x belongs to I_2, we cannot make use of function-theoretical methods, and it was for the treatment of this case that Vinogradov originated a new method.

We put $D=\prod_{p\leqq\sqrt{N}} p$, then we have

$$\begin{aligned}S_N(x)&=\sum_{\substack{m=2\\(m,D)=1}}^{N} e^{2\pi i mx}+O(\sqrt{N})\\ &=\sum_{d\leqq N,\,d\mid D}\mu(d)\sum_{m=2,\,d\mid m}^{N} e^{2\pi i mx}+O(\sqrt{N}).\end{aligned}$$

After some techniques and the refined estimations of some trigonometrical sums, Vinogradov obtained

$$S_N(x)=O\Big(\frac{N}{(\log N)^{h-2}}\Big). \tag{1.9}$$

Hence we have

$$\begin{aligned}\int_{I_2} S_N(x)^3 e^{-2\pi i Nx}dx&=O\Big(\frac{N}{(\log N)^{h-2}}\int_0^1 |S_N(x)|^2dx\Big)\\ &=O\Big(\frac{N^2}{(\log N)^{h-1}}\Big)\end{aligned}$$

and so

$$I(N)=R(N)\sum_{1\leqq q\leqq(\log N)^h}\frac{\mu(q)}{\varphi(q)^3}\sum_{\substack{a=1\\(a,q)=1}}^{q} e^{-2\pi i \frac{a}{q}N}+\text{error term.} \tag{1.10}$$

In order to determine $R(N)$, Vinogradov considered a sum

$$T(N)=\sum_{N_1} I(N_1), \tag{1.11}$$

where N_1 runs through the integers such that

$$N-\frac{N}{(\log N)^{1/2}}<N_1\leqq N.$$

Then we have

$$T(N)=R(N)\Big(\frac{N}{(\log N)^{1/2}}+O(1)\Big)+O\Big(\frac{N^3}{(\log N)^4}\Big). \tag{1.12}$$

On the other hand, we know that $T(N)$ is the number of triples (p_1, p_2, p_3) of the prime numbers such that

$$N-\frac{N}{(\log N)^{1/2}}<p_1+p_2+p_3\leqq N.$$

Now we denote by $p(k)$ the k-th prime number. The correspondence between $p(k)$ and k is one-to-one. If $p(k)\leqq N$, then

$$p(k)=k\log N+\delta CN\frac{\log\log N}{\log N}$$

with a suitable positive constant C and $|\delta|\leqq 1$.

Therefore, $T(N)$ does not exceed the number T_1 of the triples (k_1, k_2, k_3) of positive integers such that

$$N-\frac{N}{(\log N)^{1/2}}-3C\frac{N\log\log N}{\log N}<(k_1+k_2+k_3)\log N\leqq N+3C\frac{N\log\log N}{\log N}$$

and will not be less than the number T_2 of the triples (k_1, k_2, k_3) such that

$$N-\frac{N}{(\log N)^{1/2}}+3C\frac{N\log\log N}{\log N}<(k_1+k_2+k_3)\log N\leqq N-3C\frac{N\log\log N}{\log N}.$$

Thus the estimation of $T(N)$ is reduced to that of T_1 and T_2, which are much easier. In fact, after some calculations, we have

$$T(N)=\frac{N^3}{2(\log N)^{3+1/2}}\Big(1+O\Big(\frac{\log\log N}{(\log N)^{1/2}}\Big)\Big). \tag{1.13}$$

Comparing this results (1.13) with (1.12), we obtain

$$R(N)=\frac{N^2}{2(\log N)^3}\Big(1+O\Big(\frac{\log\log N}{(\log N)^{1/2}}\Big)\Big)$$

and consequently we have the desired asymptotic formula for $I(N)$:

$$I(N)=\frac{N^2}{2(\log N)^3}S(N)+O\Big(\frac{N^2\log\log N}{(\log N)^{3+1/2}}\Big). \tag{1.14}$$

$S(N)$ is the singular series. It is written in the form of an infinite product:

$$S(N)=\prod_{p\mid N}\Big(1-\frac{1}{(p-1)^2}\Big)\prod_{p\nmid N}\Big(1+\frac{1}{(p-1)^3}\Big).$$

If N is even, then $S(N)=0$, and if N is odd, then $S(N)\geqq c>0$.

Therefore $I(N)$ is positive for sufficiently large odd number N and $I(N)$ has an asymptotic formula (1.14). This is the theorem of Vinogradov.

Now we return to the sketch of our §§ 2-10. We begin with some explanations of notations.

Let X and Y be two quantities and $Y>0$. If an inequality $|X| \leqq AY$ is true for a suitable positive constant A depending on K alone, then we write

$$X=O(Y) \quad \text{or} \quad X \ll Y .$$

A small Romen letter c means positive constants, the values of which may vary but depend on K alone. We also use $c_1, c_2, \cdots$ in the same meaning.

We denote by $\|x\|$ for a real number x the least difference between x and rational integers.

In § 2, we shall define, using a pair of two numbers H and T, a division of E, which is a generalization of $[0,1]$ in rational case to the case of K, into E^0 and E_γ ($\gamma \in \Gamma$), where Γ is a certain set of the numbers of K. This division of E originated by Siegel [**6**] is sufficient for our purpose. In fact, Lemma 2.2 is very useful for our study of trigonometrical sums.

We shall call this division *the Farey division of E with respect to* (H, T).

In § 3, we shall prove some results concerning trigonometrical sums. Lemma 3.1 and Lemma 3.2 are mere preliminaries, but Lemma 3.3 is more important in the sense that it will be more contributive to the proof of Theorem 3.1.

Let M be a set of positive rational integers m_i $(i=1,2,\cdots,s)$ not exceeding T. Then it is obvious that

$$\sum_{i=1}^{s} \frac{1}{m_i} \ll \log(T+1) .$$

Lemma 3.3 is a simple extension of this inequality.

Theorem 3.1 is to estimate the sum Z of following type:

$$Z=\sum_{\mu} \min_{1 \leqq j \leqq n} (U, \|S(\eta_j \mu z)\|^{-1}) , \tag{1.15}$$

where z belongs to E^0 which is defined by the Farey division with respect to (H, T), $\eta_1, \eta_2, \cdots, \eta_n$ a basis of $\mathfrak{a}_0\mathfrak{c}$, a product of ideals, such that $|\eta_j^{(l)}| \leqq cN(\mathfrak{c})^{1/n}$ $(j, l=1,2,\cdots,n)$, $U \geqq 1$ and μ runs through a certain set of the elements of $\mathfrak{a}_0^{-1}$.

To prove this Theorem, we shall make full use of Lemma 2.2. The proof is partly due to Siegel, but we shall need more detailed technique.

After two Theorems and three Lemmas, we shall estimate, in Theorem 3.4, a trigonometrical sum of the following form:

$$S=\sum_{\mathfrak{b} \in M_2} \sum_{\substack{\nu \in \mathfrak{M} \\ (\nu)/\mathfrak{c}\mathfrak{b} \in M_1}} e^{2\pi i S(\nu z)} , \tag{1.16}$$

where $z \in E^0$, M_1 and M_2 are some sets of ideals and $\mathfrak{M}$ is a set of integers ν such that

$$N_0 < \nu^{(q)} \leqq N_q \qquad (q=1,2,\cdots,r_1)$$
$$N_0 < |\nu^{(p)}| \leqq N_p \qquad (p=r_1+1,\cdots,r_1+r_2).$$

Making use of Lemmas and some techniques, we shall be able to reduce the estimation of S to that of the sums like Z in Theorem 3.1. This Theorem 3.1 plays a fundamental role in the sequel and will also be applied to the study of the problems of Waring and Goldbach-Waring etc.

From § 4 on, we shall consider $S(z;\lambda)$. It is obvious that $I_s(\lambda)=I_s(\eta\lambda)$ for any totally positive unit η, therefore we may assume that

$$c_1 N(\lambda)^{1/n} \leqq |\lambda^{(j)}| \leqq c_2 N(\lambda)^{1/n} \qquad (j=1,2,\cdots,n).$$

We put

$$N=\max(\lambda^{(1)},\cdots,\lambda^{(r_1)},|\lambda^{(r_1+1)}|,\cdots,|\lambda^{(n)}|)$$

and

$$H=\frac{N}{(\log N)^{\sigma_1}}, \qquad T=(\log N)^{\sigma_2}$$

with suitable positive constants σ_1 and σ_2. Then we consider the Farey division with respect to this pair (H, T).

For the later use, we shall have to define an integral

$$I_s(\mu;\lambda)=\int_{-1/2}^{1/2}\cdots\int S(z;\lambda)^s e^{-2\pi i S(\mu z)}dx_1 dx_2\cdots dx_n$$

with a totally positive integer μ.

In § 4, we shall define a division of E into B^0 and B_γ $(\gamma \in \Gamma)$ such that $B^0 \subset E^0$, $E_\gamma \subset B_\gamma$ $(\gamma \in \Gamma)$ and $B_{\gamma_1} \cap B_{\gamma_2} = \phi$ $(\gamma_1 \neq \gamma_2)$, which will be more convenient than the Farey division defined in § 2.

In § 5, we shall estimate $S(z;\lambda)$ for $z \in E^0$, which corresponds to $S_N(x)$ for $x \in I_2$. Our result is that

$$S(z;\lambda) \ll \frac{N^n}{(\log N)^\sigma} \qquad (\sigma \geqq 3), \tag{1.17}$$

provided that we choose suitably σ_1 and σ_2 for σ. The proof is partly analogous to that of Vinogradov [7]. As the estimations of the fundamental parts will have been obtained in § 3, the contents of § 5 will not be so long.

In § 6, we shall estimate $S(z;\lambda)$ for $z \in B_\gamma$ with $\gamma \to \mathfrak{a}$. Let $\Omega_1(\lambda)$ be the set of the prime numbers ω such that

$$\sqrt{N} < \omega^{(q)} \leqq \lambda^{(q)} \qquad (q=1,2,\cdots,r_1)$$
$$\sqrt{N} < |\omega^{(p)}| \leqq |\lambda^{(p)}| \qquad (p=r_1+1,\cdots,r_1+r_2),$$

then we have

$$S(z\,;\lambda)=\sum_{\rho} e^{2\pi i S(\rho\gamma)} \sum_{\substack{\omega\equiv\rho(\mathfrak{a})\\ \omega\in\Omega_1(\lambda)}} e^{2\pi i S(\omega y)}+O(N^{n-1/2}),$$

where y is a point of B_0 and ρ in the first sum runs through the complete system of residues mod $\mathfrak{a}$ which are totally positive and prime to $\mathfrak{a}$.

The inner sum will be approximated by

$$\frac{w}{2^{r_1}hR\varphi(\mathfrak{a})}\ J(y\,;\lambda),$$

where it is easily seen that

$$J(y\,;\lambda)\ll\frac{N^n}{\log N}\,. \tag{1.18}$$

The exact form of $J(y\,;\lambda)$ will be given in § 6, but for the moment it will not interest us. In the proof of Vinogradov, it is necessary to estimate $J(y)$ in (1.7) more exactly, but, as it will be shown later, the estimation (1.18) will be sufficient for us. This is a little profit of our method in §§ 6-9.

As in the case of rational number field, we shall have to make use of the prime number theorem in K which will be quoted from [**3**] as Lemma 6.1.

We shall not stop here to describe the details on this theorem, but we shall have

$$S(z\,;\lambda)=\frac{w\mu(\mathfrak{a})}{2^{r_1}hR\varphi(\mathfrak{a})}\ J(y\,;\lambda)+O\Big(\frac{N^n}{(\log N)^{a-b+1}}\Big),$$

where $b=(n-1)\sigma_2+\sigma_1$ and a is a sufficiently large constant.

Collecting the results up to § 6, we shall have in § 7

$$I_s(\mu\,;\lambda)=\frac{2^{r_1}\sqrt{D}}{W^s}\ R(\mu,\lambda)\sum_{N\mathfrak{a}\leq\gamma^n}\frac{\mu(\mathfrak{a})^s}{\varphi(\mathfrak{v})^s}\,G(\mathfrak{a}\,;\mu)+O\Big(\frac{N^{n(s-1)}}{(\log N)^{s+1}}\Big), \tag{1.19}$$

where $W=2^{r_1}hR/w$ and

$$G(\mathfrak{a}\,;\mu)=\sum_{\substack{\gamma\to\mathfrak{a}\\ \gamma\ \mathrm{mod}\ \mathfrak{a}\mathfrak{b}^{-1}}} e^{-2\pi i S(\mu\gamma)}$$

$$R(\mu,\lambda)=\int_{\mathfrak{B}_0}\!\!\cdots\!\int J(z\,;\lambda)^s e^{-2\pi i S(\mu z)}dX_1(z)\cdots dX_n(z)\,.$$

In order to determine $R(\lambda,\lambda)$, we shall sum up the both sides of (1.19) over all integers μ such that

$$\lambda^{(q)}-\frac{N}{(\log N)^{\kappa}}<\mu^{(q)}\leqq\lambda^{(q)}\qquad(q=1,2,\cdots,r_1)$$

$$|\lambda^{(p)}-\mu^{(p)}|\leqq\frac{N}{(\log N)^{\kappa}}\qquad(p=r_1+1,\cdots,r_1+r_2),$$

where $\kappa=b(n+1)+1$.

Then we shall have, after some calculations,

$$T(\lambda)=\frac{2^{2r_2}\pi^{r_2}N^n R(\lambda,\lambda)}{W^s(\log N)^{n\kappa}}\left(1+O\left(\frac{(\log N)^{\kappa+1}}{N}\right)\right)$$
(1.20)
$$+O\left(\frac{N^{ns}}{(\log N)^{n\kappa+s+1}}\right).$$

On the other hand, $T(\lambda)$ is the number of the s-tuples $(\omega_1,\omega_2,\cdots,\omega_s)$ of prime numbers which satisfy the following conditions

$$\lambda^{(q)}-\frac{N}{(\log N)^{\kappa}}<\omega_1^{(q)}+\omega_2^{(q)}+\cdots+\omega_s^{(q)}\leqq\lambda^{(q)}\qquad(q=1,2,\cdots,r_1),$$
(C_ω)
$$|\lambda^{(p)}-(\omega_1^{(p)}+\omega_2^{(p)}+\cdots+\omega_s^{(p)})|\leqq\frac{N}{(\log N)^{\kappa}}\quad(p=r_1+1,\cdots,r_1+r_2),$$
$$\omega_j\in\Omega(\lambda)\qquad(j=1,2,\cdots,s).$$

Similarly to the case of rational number field, we shall have to reduce the conditions (C_ω) to those connected with integers.

If $n\geqq 2$, however, there will be no one-to-one correspondence between the integers and the prime numbers which is suitable for our purpose.

To avoid this difficulty, we shall, in § 8, construct two sets $\mathfrak{L}_1$ and $\mathfrak{L}_2$ of integers and the mappings $\tilde{\phi}:\Omega(\lambda)\to\mathfrak{L}_1$ and $\tilde{\psi}:\mathfrak{L}_2\to\Omega(\lambda)$, which satisfy the following conditions

$$\tilde{\phi}(\omega)\neq\tilde{\phi}(\omega_1)\qquad\text{if}\quad\omega\neq\omega_1,$$
$$\tilde{\psi}(\nu)\neq\tilde{\psi}(\nu_1)\qquad\text{if}\quad\nu\neq\nu_1,$$
$$\omega-\frac{Y}{N_0}\tilde{\phi}(\omega)\ll\frac{N}{(\log N)^{\kappa+1}}\qquad\text{for}\quad\omega\in\Omega(\lambda),$$
$$\tilde{\psi}(\nu)-\frac{Z}{N_0}\nu\ll\frac{N}{(\log N)^{\kappa+1}}\qquad\text{for}\quad\nu\in\mathfrak{L}_2,$$

where

$$Y=C_0N_0(\log N_0)^{1/n},\qquad Z=Y\left(1+(\kappa_0+1)\frac{\log\log N}{n\log N}\right)$$

with $C_0=(2^{r_2}\pi^{r_2}nW/\sqrt{D})^{1/n}$, $\kappa_0=\kappa+1+\dfrac{1}{n}$ and $N_0=N/(\log N)^{\kappa_0}$.

In order to obtain such sets and mappings, we shall again make use of the prime number theorem in K.

By the help of this technique, we shall be able to estimate $T(\lambda)$ from above and below: $T_1\geqq T(\lambda)\geqq T_2$, each of T_1 and T_2 is the number of the s-tuples of integers which satisfy some conditions. To obtain asymptotic formulas for T_1 and T_2 is reduced to a special case of Waring's problem in K, which is more easily treated than that for $T(\lambda)$.

In § 9, we shall treat this problem generally and in later part of this paragraph we shall arrange the results to a form which is easily applicable to the estimations of T_1 and T_2. Similar problems concerning integers were partly solved by Siegel [**6**].

In the beginning of § 10, we shall see that T_1 and T_2 have the same asymptotic formula, and we shall finally obtain an asymptotic formula for $T(\lambda)$, that is,

$$T(\lambda)=\frac{(2^{1-s}\pi^{1-s}\sigma(s))^{r_1}}{n^s W^s((s-1)!)^{r_1}}\cdot\frac{N^n N(\lambda)^{s-1}}{(\log N)^{n\kappa+s}}\left(1+O\left(\frac{\log\log N}{\log N}\right)\right), \tag{1.21}$$

where $\sigma(s)$ is a positive constant depending on s alone.

Comparing this result (1.21) with (1.20), we shall have an asymptotic formula for $R(\lambda,\lambda)$ and then that for $I_s(\lambda\,;\lambda)=I_s(\lambda)$, which is a generalization of Vinogradov's theorem to an algebraic number field.

Our results will be collected and stated in Theorem 10.1 at the end of § 10.

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§ 2. Farey division.

Let $\delta_1,\delta_2,\cdots,\delta_n$ be a basis of $\mathfrak{d}^{-1}$. We put

$$z_j=x_1\delta_1^{(j)}+x_2\delta_2^{(j)}+\cdots+x_n\delta_n^{(j)}\qquad(j=1,2,\cdots,n) \tag{2.1}$$

for real numbers $x_1,x_2,\cdots,x_n$ and define a set E of $z=(z_1,z_2,\cdots,z_n)$ as follows:

$$E=\left\{z\,;\,z_j=\sum_{i=1}^{n}x_i\delta_i^{(j)}\,;\quad -\frac{1}{2}<x_i\leqq\frac{1}{2}\quad(i=1,2,\cdots,n)\right\}. \tag{2.2}$$

Let H and T be real numbers such that

$$H>2DT,\qquad T>1 \tag{2.3}$$

and let Γ be the set of numbers γ of K such that $(\gamma^{(1)}, \gamma^{(2)}, \cdots, \gamma^{(n)}) \in E$ and $\gamma \to \mathfrak{a}$ with $N(\mathfrak{a}) \leqq T^n$.

For every $\gamma \in \Gamma$ with $\gamma \to \mathfrak{a}$ we define a subset E_γ of E as follows.

(2.4) $$E_\gamma = \left\{z \,;\, z \in E,\ N(\max(H|z-\gamma_1|, T^{-1})) \leqq \frac{1}{N(\mathfrak{a})} \text{ for any } \gamma_1 \equiv \gamma \pmod{\mathfrak{d}^{-1}}\right\}$$

and put

$$E^0 = E - \bigcup_{\gamma \in \Gamma} E_\gamma .$$

This division of E into E^0 and E_γ $(\gamma \in \Gamma)$ depends on the pair (H, T). We shall call this division *the Farey division of E with respect to (H, T)*.

The following Lemmas 2.1 and 2.2 were proved by Siegel [**6**], which will play a fundamental role in the following paragraphs.

LEMMA 2.1. *If γ_1 and γ_2 belong to Γ and $\gamma_1 \neq \gamma_2$, then we have*

$$E_{\gamma_1} \cap E_{\gamma_2} = \phi .$$

LEMMA 2.2. *If $z = (z_1, \cdots, z_n)$ is a point of E^0, then there exist an integer $\alpha \in \mathfrak{o}$ and a number $\beta \in \mathfrak{d}^{-1}$ satisfying the following four conditions:*

(2.5) $$|\alpha^{(j)} z_j - \beta^{(j)}| < H^{-1},\ 0 < |\alpha^{(j)}| < H \qquad (j = 1, 2, \cdots, n),$$

(2.6) $$\max(|\alpha^{(1)}|, |\alpha^{(2)}|, \cdots, |\alpha^{(n)}|) > T,$$

(2.7) $$\max(H|\alpha^{(j)} z_j - \beta^{(j)}|, |\alpha^{(j)}|) \geqq D^{-1/2} \qquad (j = 1, 2, \cdots, n),$$

(2.8) $$N((\alpha, \beta\mathfrak{d})) \leqq D^{1/2}.$$

§ 3. Trigonometrical sums.

LEMMA 3.1. *Let $\mathfrak{f}$ be a fractional or integral ideal. Then we can take a basis $\lambda_1, \lambda_2, \cdots, \lambda_n$ of $\mathfrak{f}$ such that*

(3.1) $$|\lambda_j^{(k)}| \leqq c N(\mathfrak{f})^{1/n} \qquad (j, k = 1, 2, \cdots, n).$$

Moreover, we can choose a basis $\eta_1, \eta_2, \cdots, \eta_n$ of $(\mathfrak{f}\mathfrak{d})^{-1}$ such that

$$S(\lambda_i \eta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (i, j = 1, 2, \cdots, n)$$

and

$$|\eta_j^{(k)}| \leqq c N(\mathfrak{f})^{-1/n} \qquad (j, k = 1, 2, \cdots, n).$$

PROOF. Let $\mathfrak{C}$ be the ideal class containing $\mathfrak{f}$. Then $\mathfrak{f}$ is a product of a fixed ideal $\mathfrak{a}_0$ in $\mathfrak{C}$ and a number α of K; $\mathfrak{f} = \alpha\mathfrak{a}_0$. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be a basis of $\mathfrak{a}_0$, then $\lambda_j = \alpha\alpha_j$ $(j = 1, 2, \cdots, n)$ is a basis of $\mathfrak{f}$.

On the other hand, by the theory of units, we may assume that

$$c_1 |N(\alpha)|^{1/n} \leqq |\alpha^{(j)}| \leqq c_2 |N(\alpha)|^{1/n} \qquad (j = 1, 2, \cdots, n)$$

with positive constants c_1 and c_2 depending on K alone. Therefore we have

$$|\lambda_j^{(k)}| = |\alpha^{(k)}\alpha_j^{(k)}| \leqq c\,|N(\alpha)|^{1/n} \leqq cN(\mathfrak{f})^{1/n} \qquad (j, k = 1, 2, \cdots, n).$$

Now let $\beta_1, \beta_2, \cdots, \beta_n$ be a basis of $(\mathfrak{a}_0\mathfrak{d})^{-1}$ such that

$$S(\alpha_i\beta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (i, j = 1, 2, \cdots, n).$$

Then $\eta_j = \beta_j/\alpha$ $(j = 1, 2, \cdots, n)$ is a basis of $(\mathfrak{f}\mathfrak{d})^{-1}$ and we have

$$S(\lambda_i\eta_j) = S(\alpha_i\beta_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad (i, j = 1, 2, \cdots, n)$$

and

$$|\eta_j^{(k)}| = \left|\frac{\beta_j^{(k)}}{\alpha^{(k)}}\right| \leqq c\,|N(\alpha)|^{-1/n} \leqq cN(\mathfrak{f})^{-1/n} \qquad (j, k = 1, 2, \cdots, n).$$

Thus we complete the proof.

In the following lines, we shall often make use of this Lemma 3.1, without special references. Besides, we shall use a notation p' in the meaning $p' = p + r_2$ $(p \geqq r_1 + 1)$ when p and p' appear in the same expression.

LEMMA 3.2. *Let $\mathfrak{f}$ be a fractional or integral ideal. We take positive numbers $A_1, A_2, \cdots, A_n$ such that $A_{p'} = A_p$ $(p = r_1 + 1, \cdots, r_1 + r_2)$ and α_p, β_p $(p = r_1 + 1, \cdots, r_1 + r_2)$ such that*

$$\beta_p < \alpha_p \leqq 2\pi + \beta_p \qquad (p = r_1 + 1, \cdots, r_1 + r_2).$$

We denote by $n(\mathfrak{f}; A, \alpha, \beta)$ the number of the elements ν of $\mathfrak{f}$ satisfying the conditions

$$0 < \nu^{(q)} \leqq A_q \qquad (q = 1, 2, \cdots, r_1),$$

$$\left.\begin{array}{l} |\nu^{(p)}| \leqq A_p \\ \beta_p \leqq \arg \nu^{(p)} \leqq \alpha_p \end{array}\right. \qquad (p = r_1 + 1, \cdots, r_1 + r_2).$$

Then we have

$$n(\mathfrak{f}; A, \alpha, \beta) = \frac{1}{\sqrt{D}\,N(\mathfrak{f})} \prod_{j=1}^{n} A_j \prod_{p=r_1+1}^{r_1+r_2} (\alpha_p - \beta_p) + O\left(\frac{A_0^{n-1}}{N(\mathfrak{f})^{1-1/n}}\right), \tag{3.2}$$

where

$$A_0 = \max(N(\mathfrak{f})^{1/n}, (A_1A_2 \cdots A_n)^{1/n}).$$

PROOF. By the theory of units, we can choose a unit ε_0 such that

$$c_1(A_1A_2 \cdots A_n)^{1/n} \leqq A_j|\varepsilon_0^{(j)}| \leqq c_2(A_1A_2 \cdots A_n)^{1/n} \qquad (j = 1, 2, \cdots, n).$$

It is obvious that

$$n(\mathfrak{f}; A, \alpha, \beta) = n(\mathfrak{f}; A|\varepsilon_0|, \alpha + \arg\varepsilon_0, \beta + \arg\varepsilon_0).$$

Therefore, taking $A_j|\varepsilon_0^{(j)}|$ instead of A_j, we may assume that

$$c_1(A_1A_2\cdots A_n)^{1/n}\leqq A_j\leqq c_2(A_1A_2\cdots A_n)^{1/n}\qquad (j=1,2,\cdots,n).$$

Let $\lambda_1,\lambda_2,\cdots,\lambda_n$ be a basis of $\mathfrak{f}$ such that

$$|\lambda_j^{(k)}|\leqq cN(\mathfrak{f})^{1/n}\qquad (j,k=1,2,\cdots,n).$$

The vectors $x(\lambda_1), x(\lambda_2),\cdots, x(\lambda_n)$ in n-dimensional euclidean space E^n are linearly independent and they span a parallelepiped whose volume is $N(\mathfrak{f})\sqrt{D}/2^{r_2}$ and diameter is less than $c_0N(\mathfrak{f})^{1/n}$ with a positive constant c_0.

We shall define a domain V_0 in E^n by the following conditions

$$V_0:\quad \begin{array}{ll} 0\leqq x_q\leqq A_q & (q=1,2,\cdots,r_1) \\ x_p^2+x_{p'}^2\leqq A_p^2 & \\ \beta_p\leqq \arg(x_p+ix_{p'})\leqq\alpha_p & (p=r_1+1,\cdots,r_1+r_2), \end{array}$$

$(x_1,x_2,\cdots,x_n)$ being the points of E^n. Then $n(\mathfrak{f};A,\alpha,\beta)$ is equal to the number of the lattice points in V_0 with respect to vectors $x(\lambda_1), x(\lambda_2),\cdots,x(\lambda_n)$.

We denote by $\rho(\mathfrak{x}_1,\mathfrak{x}_2)$ the distance between two points $\mathfrak{x}_1$ and $\mathfrak{x}_2$ in E^n and define two domains V_1 and V_2 in E^n as follows:

$$V_1=\{\mathfrak{x}_1;\ \rho(\mathfrak{x}_1,\mathfrak{x}_2)\leqq c_0N(\mathfrak{f})^{1/n}\quad\text{for any } \mathfrak{x}_2\in V_0\},$$
$$V_2=\{\mathfrak{x}_1;\ \mathfrak{x}_1\in V_0,\ \rho(\mathfrak{x}_1,\mathfrak{x}_2)\geqq c_0N(\mathfrak{f})^{1/n}\quad\text{for all } \mathfrak{x}_2\notin V_0\}.$$

Denoting the volumes of V_0, V_1 and V_2 by $\sigma(V_0)$, $\sigma(V_1)$ and $\sigma(V_2)$ respectively, we have, by a simple calculation,

$$\sigma(V_1)-\sigma(V_2)\ll A_0^{n-1}N(\mathfrak{f})^{1/n} \tag{3.3}$$

and

$$\sigma(V_0)=\frac{1}{2^{r_2}}\prod_{j=1}^{n}A_j\prod_{p=r_1+1}^{r_1+r_2}(\alpha_p-\beta_p). \tag{3.4}$$

On the other hand, we see that

$$\frac{2^{r_2}}{N(\mathfrak{f})\sqrt{D}}\sigma(V_2)\leqq n(\mathfrak{f};A,\alpha,\beta)\leqq\frac{2^{r_2}}{N(\mathfrak{f})\sqrt{D}}\sigma(V_1). \tag{3.5}$$

Therefore we obtain the proof.

LEMMA 3.3. *Let $T_0, T_1,\cdots,T_m$ $(m\geqq 1)$ be rational integers and M be the set of the m-tuples $(t_1,t_2,\cdots,t_m)$ of rational integers such that*

$$T_j\leqq t_j\leqq T_j+T_0\qquad (j=1,2,\cdots,m),$$
$$(t_1,t_2,\cdots,t_m)\neq(0,0,\cdots,0).$$

We take a subset M_0 of M and define a sum S as follows:

$$S=\sum_{(t_1,\cdots,t_m)\in M_0}\min\Big(\frac{1}{|t_1|},\frac{1}{|t_2|},\cdots,\frac{1}{|t_m|}\Big). \tag{3.6}$$

Then we have

(3.7) $$S \ll A^{1-1/m} \log(1+T_0),$$

where A is the number of the elements of M_0.

PROOF. Without the loss of generality, we may assume that $T_j \geqq 0$ $(j=1, \cdots, m)$. Let M_k $(1 \leqq k \leqq m)$ be the subset of M_0 consisting of $(t_1, t_2, \cdots, t_m)$ such that $t_k \geqq t_l$ $(l=1,2,\cdots,m)$. We put

$$A_k = \sum_{(t_1,\cdots,t_m)\in M_k} 1$$

and

$$S_k = \sum_{(t_1,\cdots,t_m)\in M_k} \min\left(\frac{1}{t_1}, \frac{1}{t_2}, \cdots, \frac{1}{t_m}\right).$$

First we shall consider S_1 and prove

(3.8) $$S_1 \ll A_1^{1-1/m} \log(1+T_0).$$

We shall consider m-dimensional euclidean space E^m and denote by $(u_1, u_2, \cdots, u_m)$ the points of E^m. Taking a positive number t, we denote by $D(t)$ a domain in E^m which is defined by the conditions

$$D(t): \quad \begin{array}{ll} t \geqq u_1 > 0 & \\ u_1 \geqq u_l \geqq 0 & (l=2,3,\cdots,m), \end{array}$$

by $M(t)$ the set of the points in $D(t)$ with integral coordinates and by $n(t)$ the number of the elements of $M(t)$. Moreover, let $M_0(t)$ be a subset of M_1 consisting of $(t_1, t_2, \cdots, t_m)$ such that $t \geqq t_1$ and $n_0(t)$ be the number of the elements of $M_0(t)$.

It is obvious that for any t

$$n(t) \geqq n_0(t).$$

Therefore, if we choose a rational integer t_0 such that

$$n(t_0) \leqq A_1 = n_0(T_0) < n(t_0+1),$$

then we can construct a mapping φ from $M_0(T_0)$ to $M(t_0+1)$ which satisfies a condition that, for every $(t_1, t_2, \cdots, t_m) \in M_0(T_0)$, the first coordinate of $(t_1, t_2, \cdots, t_m)$ is not less than that of φ-image $\varphi((t_1, t_2, \cdots, t_m)) = (s_1, s_2, \cdots, s_m)$ of $(t_1, t_2, \cdots, t_m)$, that is, $s_1 \leqq t_1$. Hence we have

$$S_1 = \sum_{(t_1,\cdots,t_m)\in M_1} \frac{1}{t_1} = \sum_{(t_1,\cdots,t_m)\in M_0(T_0)} \frac{1}{t_1}$$

$$\leqq \sum_{(s_1,\cdots,s_m)\in M(t_0+1)} \frac{1}{s_1} \leqq (t_0+2)^{m-1}(1+\log(t_0+1)).$$

Since

$$A_1 \geqq n(t_0) = 2^{m-1} + 3^{m-1} + \cdots + (t_0+1)^{m-1} \geqq c(t_0+1)^m,$$

we have

$$S_1 \ll A_1^{1-1/m}(\log(1+t_0)+1) \ll A_1^{1-1/m}\log(1+T_0).$$

In the similar way, we obtain

$$S_k \ll A_k^{1-1/m}\log(1+T_0) \qquad (k=1,2,\cdots,m).$$

Therefore we complete the proof, since $S \leqq S_1+S_2+\cdots+S_m$.

THEOREM 3.1. *Let* $\mathfrak{a}_0$ *be an ideal fixed together with* K, $\mathfrak{c}$ *be an ideal and* $\eta_1, \eta_2, \cdots, \eta_n$ *be a basis of* $\mathfrak{a}_0\mathfrak{c}$ *which satisfy the inequalities*

$$|\eta_j^{(k)}| \leqq cN(\mathfrak{a}_0\mathfrak{c})^{1/n} \qquad (j,k=1,2,\cdots,n).$$

Let M *be a parallelotope in* n*-dimensional euclidean space* E^n *which is defined as follows:*

$$M=\{(x_1,x_2,\cdots,x_n);\ a_j \leqq x_j \leqq b_j \quad (j=1,2,\cdots,n)\}.$$

We take a point $z=(z_1,z_2,\cdots,z_n)$ *of* E^0 *which is defined by the Farey division with respect to* (H,T). *We put*

$$V=\max(1, b_1-a_1, b_2-a_2, \cdots, b_n-a_n)$$

and assume that

$$V<\frac{H}{2^{3+r_2}(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}}. \tag{3.9}$$

Now we define a sum of the following form:

$$Z=\sum_{\substack{\mu\in\mathfrak{a}_0^{-1}\\ x(\mu)\in M}} \min_{1\leqq j\leqq n}(U, \|S(\eta_j\mu z)\|^{-1}), \tag{3.10}$$

where U *is a given number* $\geqq 1$ *and the sum is taken over all* $\mu\in\mathfrak{a}_0^{-1}$ *such that* $x(\mu)\in M$.

Then we have

$$Z \ll V^n UN(\mathfrak{c})\left(\frac{1}{T}+\frac{1}{V}+\frac{H\log H}{VUN(\mathfrak{c})^{1/n}}+\frac{\log H}{U}\right). \tag{3.11}$$

PROOF. We write $S(\eta_j\mu z)=s_j+d_j$ $(j=1,2,\cdots,n)$ with rational integers s_j and $-1/2 \leqq d_j < 1/2$ $(j=1,2,\cdots,n)$ and put

$$\vartheta=\sum_{j=1}^n s_j\lambda_j, \qquad \zeta=\sum_{j=1}^n d_j\lambda_j, \tag{3.12}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ is a basis of $(\mathfrak{a}_0\mathfrak{c}\mathfrak{d})^{-1}$ such that

$$S(\lambda_j\eta_k)=\begin{cases}1 & \text{if } j=k\\ 0 & \text{if } j\neq k\end{cases} \qquad (j,k=1,2,\cdots,n)$$

and

$$|\lambda_j^{(k)}| \leqq cN(\mathfrak{c})^{-1/n} \qquad (j,k=1,2,\cdots,n).$$

ϑ and ζ are the functions of μ and we have

$$\vartheta \in (\mathfrak{a}_0\mathfrak{c}\mathfrak{d})^{-1}, \qquad \mu z = \vartheta + \zeta,$$

$$\| S(\eta_j \mu z) \| = |d_j| \qquad (j=1,2,\cdots,n),$$

$$|X_j(\zeta)| \leqq \sum_{k=1}^{n} |d_k \lambda_k^{(j)}| \leqq cN(\mathfrak{c})^{-1/n} \sum_{k=1}^{n} |d_k| \qquad (j=1,2,\cdots,n).$$

Hence we have

$$Z \ll \sum_{\mu} \min_{1\leqq j\leqq n} (U, |d_j|^{-1}) \ll \sum_{\mu} \min_{1\leqq j\leqq n} (U, N(\mathfrak{c})^{-1/n} |X_j(\zeta)|^{-1})$$
$$\ll N(\mathfrak{c})^{-1/n} \sum_{\mu} \min_{1\leqq j\leqq n} (UN(\mathfrak{c})^{1/n}, |X_j(\zeta)|^{-1}),$$

where the summation $\sum_{\mu}$ has the meaning stated in Theorem 3.1.

Therefore, it suffices for us to estimate a sum

$$Z^* = \sum_{\substack{\mu \in \mathfrak{a}_0^{-1} \\ x(\mu) \in M}} \min_{1\leqq j\leqq n} \left(U, \frac{1}{|X_j(\zeta)|}\right). \tag{3.13}$$

By suitable choice of a positive constant b_1, we obtain the inequality

$$|X_j(\zeta)| \leqq b_1 N(\mathfrak{c})^{-1/n} \qquad (j=1,2,\cdots,n) \tag{3.14}$$

for all μ in the sum Z^*. We shall put

$$b_0 = 2b_1(DN(\mathfrak{a}_0))^{1/n}. \tag{3.15}$$

Taking b_1 suitably, we may assume that

$$b_0 > D^{1/2}. \tag{3.16}$$

We know, by Lemma 2.2, that there exist $\alpha \in \mathfrak{o}$ and $\beta \in \mathfrak{d}^{-1}$ which satisfy (2.5), (2.6), (2.7) and (2.8) for $(z_1, z_2, \cdots, z_n)$ in our Theorem.

To each μ in the sum Z^* we assign a vector

$$y(\mu) = (C_1 X_1(\zeta), C_2 X_2(\zeta), \cdots, C_n X_n(\zeta)) \tag{3.17}$$

in E^n with

$$C_j = 2(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n} |\alpha^{(j)}| \qquad (j=1,2,\cdots,n).$$

All $y(\mu)$ are contained in a parallelotope

$$\{(x_1, x_2, \cdots, x_n);\ |x_j| \leqq b_0 |\alpha^{(j)}| \quad (j=1,2,\cdots,n)\}.$$

Now we shall divide a set $\{1,2,\cdots,n\}$ into three parts J_1, J_2 and J_3 by the conditions as follows:

$$i \in J_1 \quad \text{if and only if} \quad \frac{V}{H}(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n} \geqq 2b_0 |\alpha^{(i)}|, \tag{3.18}$$

$$j \in J_2 \quad \text{if and only if} \quad \frac{1}{2} \geqq 2b_0 |\alpha^{(j)}| > \frac{V}{H}(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}, \tag{3.19}$$

$$k \in J_3 \quad \text{if and only if} \quad 2b_0 |\alpha^{(k)}| > \frac{1}{2}. \tag{3.20}$$

J_1 or J_2 may be empty, but J_3 is not empty on account of (2.6). Moreover, we see from (3.18), (3.19) and (3.16) that

$$|\alpha^{(j)}| \leqq b_0^{-1} < D^{-1/2} \qquad (j \in J_1 + J_2). \tag{3.21}$$

Therefore, putting

$$\delta^{(j)} = \alpha^{(j)} z_j - \beta^{(j)} \qquad (j = 1, 2, \cdots, n),$$

we have by (2.7)

$$|\delta^{(j)}|^{-1} \leqq D^{1/2} H \qquad (j \in J_1 + J_2). \tag{3.22}$$

We shall put

$$\tau_i = \frac{2V}{H} (DN(\mathfrak{a}_0 \mathfrak{c}))^{1/n} \qquad (i \in J_1), \tag{3.23}$$

$$\tau_j = 4b_0 |\alpha^{(j)}| \qquad (j \in J_2). \tag{3.24}$$

Since

$$\prod_{j \in J_1 + J_2} \tau_j \prod_{k \in J_3} (b_0 |\alpha^{(k)}|) \geqq b_0^n |N(\alpha)| \geqq b_0^n \geqq 1,$$

we can choose positive numbers τ_k for $k \in J_3$ such that

$$b_0 |\alpha^{(k)}| \geqq \tau_k \geqq 2^{-2-r_2} \qquad (k \in J_3), \tag{3.25}$$

$$\tau_{p'} = \tau_p \qquad (p \geqq r_1 + 1,\ p \in J_3),$$

$$\tau_1 \tau_2 \cdots \tau_n = 2^{-2-r_2}.$$

Let $g_1, g_2, \cdots, g_n$ be rational integers and $B(g) = B(g_1, g_2, \cdots, g_n)$ be a parallelotope in E^n which is defined as follows:

$$B(g) = \left\{(x_1, x_2, \cdots, x_n);\ \tau_j\left(g_j - \frac{1}{2}\right) < x_j \leqq \tau_j\left(g_j + \frac{1}{2}\right),\ (j = 1, 2, \cdots, n)\right\}. \tag{3.26}$$

We shall consider $B(g)$ containing at least one $y(\mu)$.

If $y(\mu)$ and $y(\mu_1)$ are contained in the same $B(g)$, then, decomposing μz and $\mu_1 z$ as in (3.12), $\mu z = \vartheta + \zeta$ and $\mu_1 z = \vartheta_1 + \zeta_1$, we have

$$|C_j(X_j(\zeta) - X_j(\zeta_1))| < \tau_j \qquad (j = 1, 2, \cdots, n)$$

so that

$$|\alpha^{(j)}(X_j(\zeta) - X_j(\zeta_1))| < \frac{\tau_j}{2(DN(\mathfrak{a}_0 \mathfrak{c}))^{1/n}} \qquad (j = 1, 2, \cdots, n). \tag{3.27}$$

On the other hand, in view of (3.23), (3.24), (3.25) and (3.9), we have

$$|\delta^{(j)}(X_j(\mu) - X_j(\mu_1))| \leqq H^{-1} V \leqq \frac{\tau_j}{2(DN(\mathfrak{a}_0 \mathfrak{c}))^{1/n}} \qquad (j = 1, 2, \cdots, n). \tag{3.28}$$

We now put

$$\kappa = \alpha(\vartheta - \vartheta_1) - \beta(\mu - \mu_1),$$

then

$$\kappa \in (\mathfrak{a}_0 \mathfrak{c} \mathfrak{d})^{-1},$$

$$\kappa = -\alpha(\zeta - \zeta_1) + \delta(\mu - \mu_1)$$

and we have, by (3.27) and (3.28),

$$|\kappa^{(q)}| < \frac{\tau_q}{(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}} \qquad (q=1,2,\cdots,r_1),$$

$$|\kappa^{(p)}| < \frac{\sqrt{2}\,\tau_p}{(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}} \qquad (p=r_1+1,\cdots,r_1+r_2)$$

so that

$$|N(\kappa)| < \frac{2^{r_2}\tau_1\tau_2\cdots\tau_n}{DN(\mathfrak{a}_0\mathfrak{c})} < \frac{1}{DN(\mathfrak{a}_0\mathfrak{c})}. \tag{3.29}$$

Since $\kappa \in (\mathfrak{a}_0\mathfrak{c}\mathfrak{d})^{-1}$, this inequality (3.29) implies $\kappa=0$.

Hence

$$\alpha(\zeta-\zeta_1)=\delta(\mu-\mu_1) \tag{3.30}$$

and

$$\beta(\mu-\mu_1)=\alpha(\vartheta-\vartheta_1)\in\frac{\alpha}{\mathfrak{a}_0\mathfrak{c}\mathfrak{d}}. \tag{3.31}$$

We denote by $\mathfrak{a}$ the denominator of β/α, that is, $\beta/\alpha\to\mathfrak{a}$. Then (2.8) means that

$$|N(\alpha)| \leqq D^{1/2}N(\mathfrak{a}).$$

Since $\mathfrak{d}\beta$ and $\mathfrak{a}_0\mathfrak{c}(\mu-\mu_1)$ are integral ideals, it follows from (3.31) that

$$\mathfrak{a}_0\mathfrak{c}(\mu-\mu_1)\subset\mathfrak{a} \tag{3.32}$$

and

$$\mathfrak{a}_1\mathfrak{a}_0\mathfrak{c}(\mu-\mu_1)\subset(\alpha),$$

where $\mathfrak{a}_1=(\alpha)/\mathfrak{a}$. Therefore we have

$$\rho(\mu-\mu_1)\in(\alpha) \tag{3.33}$$

with a suitable element ρ of $\mathfrak{c}$ such that

$$|\rho^{(j)}| \leqq cN(\mathfrak{c})^{1/n} \qquad (j=1,2,\cdots,n). \tag{3.34}$$

Now we denote by $W(g)=W(g_1,g_2,\cdots,g_n)$ the number of μ such that

$$\mu\in\mathfrak{a}_0^{-1},\quad x(\mu)\in M,\quad y(\mu)\in B(g)=B(g_1,g_2,\cdots,g_n). \tag{3.35}$$

If we choose a number μ_1 satisfying (3.35), then we see from (3.33) that $W(g)$ does not exceed the number of integers ν such that

$$|\nu^{(i)}| \leqq \max_{\mu}\left|\frac{\rho^{(i)}(\mu^{(i)}-\mu_1^{(i)})}{\alpha^{(i)}}\right| \qquad (i=1,2,\cdots,n), \tag{3.36}$$

where μ runs through all numbers of K satisfying the condition (3.35).

We shall estimate the right-hand side of (3.36). If $j\in J_2+J_3$, then we have by (3.34)

$$\max_{\mu}\left|\frac{\rho^{(j)}(\mu^{(j)}-\mu_1^{(j)})}{\alpha^{(j)}}\right| \ll \frac{VN(\mathfrak{c})^{1/n}}{|\alpha^{(j)}|} \tag{3.37}$$

and if $i \in J_1$, then we have, by (3.30), (3.22), (3.34) and (3.14),

$$\max_{\mu} \left| \frac{\rho^{(i)}(\mu^{(i)}-\mu_1^{(i)})}{\alpha^{(i)}} \right| = \max_{\mu} \left| \frac{\rho^{(i)}(\zeta^{(i)}-\zeta_1^{(i)})}{\delta^{(i)}} \right| \ll H. \tag{3.38}$$

From (3.37), (3.38) and Lemma 3.2 follows

$$W(g) \ll 1 + H^{q_1} N(\mathfrak{c})^{1-q_1/n} V^{n-q_1} \prod_{j \in J_2+J_3} |\alpha^{(j)}|^{-1}, \tag{3.39}$$

where q_1 is the number of the elements of J_1. We put

$$W_0 = H^{q_1} N(\mathfrak{c})^{1-q_1/n} V^{n-q_1} \prod_{j \in J_2+J_3} |\alpha^{(j)}|^{-1}. \tag{3.40}$$

Now we shall return to Z^*. We write

$$Z^* = \sum_{g_1,\cdots,g_n} \sum_{y(\mu) \in B(g)} \min_{1 \leqq j \leqq n} \left(U, \frac{1}{|X_j(\zeta)|}\right),$$

where $g_1, g_2, \cdots, g_n$ run through all n rational integers for which each $B(g)$ contains at least one $y(\mu)$ and the inner sum is taken over all μ such that

$$\mu \in \mathfrak{a}_0^{-1}, \quad x(\mu) \in M, \quad y(\mu) \in B(g).$$

Let G_j be the least rational integer satisfying the inequality

$$b_0 |\alpha^{(j)}| < \tau_j\left(G_j + \frac{1}{2}\right) \qquad (1 \leqq j \leqq n).$$

Since the j-th coordinate $C_j X_j(\zeta)$ of the vector $y(\mu)$ satisfies the inequality

$$|C_j X_j(\zeta)| < b_0 |\alpha^{(j)}| \qquad (j = 1, 2, \cdots, n),$$

the range of $g_1, g_2, \cdots, g_n$ in Z^* is roughly given by the conditions

$$|g_j| \leqq G_j \qquad (j = 1, 2, \cdots, n).$$

Easily we have

$$\begin{aligned} G_j &= 0 \qquad (j \in J_1 + J_2), \\ 1 \leqq G_k &\leqq \frac{2b_0 |\alpha^{(k)}|}{\tau_k} \qquad (k \in J_3). \end{aligned} \tag{3.41}$$

Therefore, we can write

$$Z^* = \sum_{\{g_k\}} \sum_{y(\mu) \in B(g)} \min_{1 \leqq j \leqq n} \left(U, \frac{1}{|X_j(\zeta)|}\right),$$

where $\sum_{\{g_k\}}$ means that this sum is taken over all g_k with $k \in J_3$.

We shall divide Z^* into two parts:

$$Z^* = \sum_{y(\mu) \in B(0,\cdots,0)} + \sum_{\{g_k\} \neq \{0\}} \sum_{y(\mu) \in B(g)} = Z_1 + Z_2.$$

First we shall estimate Z_1.

If $J_1 + J_2 = \phi$, then it follows from (2.6) and (3.20) that $|N(\alpha)| \geqq cT$. Therefore, by (3.39) and (3.40), we have

$$Z_1 \ll (1+W_0)U \ll U + V^n U N(\mathfrak{c}) \frac{1}{|N(\alpha)|} \ll U + \frac{V^n U N(\mathfrak{c})}{T}. \tag{3.42}$$

Now we assume that $J_1+J_2 \neq \phi$. Then

$$Z_1 \ll \sum_{y(\mu)\in B(0)} \min_{1\leqq j\leqq n}\left(U, \frac{1}{|\zeta^{(j)}|}\right), \tag{3.43}$$

since an inequality

$$\min\left(\frac{1}{|\Re(\xi)|}, \frac{1}{|\Im(\xi)|}\right) \leqq \frac{\sqrt{2}}{|\xi|}$$

is true for any complex number $\xi \neq 0$.

If $W(0)<2$, then Z_1 is trivially estimated; $Z_1 \ll U$.

We assume that $W(0)\geqq 2$. If we fix a number μ_1 such that $y(\mu_1)\in B(0)$, then other μ with $y(\mu)\in B(0)$ satisfies following relations:

$$\alpha(\zeta-\zeta_1)=\delta(\mu-\mu_1),$$

$$\mu-\mu_1 \in \mathfrak{a}^* = \frac{\mathfrak{a}}{\mathfrak{a}_0\mathfrak{c}}.$$

Therefore we have from (3.43)

$$\begin{aligned} Z_1 &\ll \sum_{\substack{y(\mu)\in B(0)\\ x(\mu)\in M}} \min_{j\in J_1+J_2}\left(U, \frac{1}{|\zeta^{(j)}|}\right) \\ &\ll \sum_{\substack{\mu-\mu_1\in\mathfrak{a}^*\\ x(\mu)\in M}} \min_{j\in J_1+J_2}\left(U, \left|\frac{\delta^{(j)}}{\alpha^{(j)}}(\mu^{(j)}-\mu_1^{(j)})+\zeta_1^{(j)}\right|^{-1}\right). \end{aligned} \tag{3.44}$$

We define an n-dimensional cube

$$M_1=\{(x_1, x_2, \cdots, x_n);\ |x_j|\leqq V, \quad (j=1,2,\cdots,n)\}$$

and put

$$\xi_0^{(j)} = \begin{cases} \dfrac{\alpha^{(j)}}{\delta^{(j)}}\zeta_1^{(j)} & \text{if} \quad j\in J_1+J_2 \\ 0 & \text{if} \quad j\in J_3. \end{cases}$$

Then we have from (3.44), (3.21) and (3.22)

$$\begin{aligned} Z_1 &\ll \sum_{\substack{\mu\in\mathfrak{a}^*\\ x(\mu)\in M_1}} \min_{j\in J_1+J_2}\left(U, \frac{H}{|\mu^{(j)}+\xi_0^{(j)}|}\right) \\ &\ll \sum_{\substack{\mu\in\mathfrak{a}^*\\ x(\mu)\in M_1}} \min_{j\in J_1+J_2}\left(U, \frac{H}{|X_j(\mu+\xi_0)|}\right). \end{aligned} \tag{3.45}$$

The last inequality follows from the inequality

$$\frac{1}{|\xi|} \leqq \min\left(\frac{1}{|\Re(\xi)|}, \frac{1}{|\Im(\xi)|}\right)$$

for any complex number $\xi \neq 0$.

Let $t_1, t_2, \cdots, t_n$ be rational integers and define an n-dimensional cube

$$B^*(t) = B^*(t_1, t_2, \cdots, t_n)$$

$$= \left\{(x_1, x_2, \cdots, x_n);\ \frac{N(\mathfrak{a}^*)^{1/n}}{3}\left(t_j - \frac{1}{2}\right) < x_j \leqq \frac{N(\mathfrak{a}^*)^{1/n}}{3}\left(t_j + \frac{1}{2}\right) \quad (j=1, 2, \cdots, n)\right\}.$$

For every $B^*(t)$ the number of $\mu \in \mathfrak{a}^*$ such that $x(\mu+\xi_0) \in B^*(t)$ is at most one. Moreover we have

$$\min_{j\in J_1+J_2}\left(\frac{1}{|X_j(\mu+\xi_0)|}\right) \ll \min_{j\in J_1+J_2}\left(\frac{1}{|t_j|N(\mathfrak{a}^*)^{1/n}}\right) \tag{3.46}$$

for $\mu+\xi_0$ such that $x(\mu+\xi_0) \in B^*(t)$.

Therefore we have from (3.45) and (3.46)

$$Z_1 \ll \sum_{t_1,\cdots,t_n} \min_{j\in J_1+J_2}\left(U, \frac{H}{|t_j|N(\mathfrak{a}^*)^{1/n}}\right), \tag{3.47}$$

where $t_1, t_2, \cdots, t_n$ run through all n rational integers for which there exists $\mu \in \mathfrak{a}^*$ such that

$$x(\mu+\xi_0) \in B^*(t), \qquad x(\mu) \in M_1.$$

The range of $\{t_1, t_2, \cdots, t_n\}$ is given as follows:

$$|t_k| \ll T_0 = 1 + \frac{V}{N(\mathfrak{a}^*)^{1/n}} \ll VN(\mathfrak{c})^{1/n} \qquad (k \in J_3), \tag{3.48}$$

$$T_j \leqq t_j \leqq T_j' \qquad (j \in J_1+J_2) \tag{3.49}$$

with $T_j' - T_j \ll T_0$ $(j \in J_1+J_2)$.

We shall divide the sum in the right-hand side of (3.47) into two parts:

$$\sum_{t_1,\cdots,t_n} = \sum\nolimits_1 + \sum\nolimits_2, \tag{3.50}$$

where $\sum_1$ is the sum taken over all $t_1, t_2, \cdots, t_n$ with $t_j = 0$ for all $j \in J_1+J_2$ and $\sum_2$ consists of other terms.

Since J_3 is not empty, it follows from (3.48) that

$$\sum\nolimits_1 \ll UT_0^{n-1} \ll V^{n-1}UN(\mathfrak{c}). \tag{3.51}$$

As for $\sum_2$, noting the existence of an index $l \in J_1+J_2$ for which $t_l \neq 0$, we have by (3.47)

$$\sum\nolimits_2 \ll \frac{H}{N(\mathfrak{a}^*)^{1/n}} \sum{}' \min_{j\in J_1+J_2}\left(\frac{1}{|t_j|}\right),$$

where $\sum'$ means a sum taken over all possible n rational integers t_k $(k=1, 2, \cdots, n)$ whose range is given by (3.49).

Therefore we have, by Lemma 3.3,

$$\sum\nolimits_2 \ll \frac{H}{N(\mathfrak{a}^*)^{1/n}} T_0^{n-1} \log(1+T_0) \ll V^{n-1} N(\mathfrak{c}) H \log H. \tag{3.52}$$

From (3.51) and (3.52) follows

$$Z_1 \ll V^n U N(\mathfrak{c}) \Big(\frac{1}{V} + \frac{H \log H}{VU}\Big) \tag{3.53}$$

and consequently, by (3.42) and (3.53),

$$Z_1 \ll V^n U N(\mathfrak{c}) \Big(\frac{1}{T} + \frac{1}{V} + \frac{H \log H}{VU}\Big). \tag{3.54}$$

This is the desired estimation for Z_1.

Now we shall estimate

$$Z_2 = \sum_{\{g_k\} \neq \{0\}} \sum_{y(\mu) \in B(g)} \min_{1 \leqq j \leqq n} \Big(U, \frac{1}{|X_j(\zeta)|}\Big).$$

By the definition of $y(\mu)$, we have, for μ such that $y(\mu) \in B(g)$,

$$\min_{1 \leqq j \leqq n} \Big(\frac{1}{|X_j(\zeta)|}\Big) \ll \min_{1 \leqq j \leqq n} \Big(\frac{C_j}{\tau_j |g_j|}\Big) \ll N(\mathfrak{c})^{1/n} \min_{k \in J_3} \Big(\frac{|\alpha^{(k)}|}{\tau_k |g_k|}\Big),$$

which gives

$$Z_2 \ll N(\mathfrak{c})^{1/n} \sum_{\{g_k\} \neq \{0\}} W(g) \min_{k \in J_3} \Big(\frac{|\alpha^{(k)}|}{\tau_k |g_k|}\Big). \tag{3.55}$$

First we assume that $W_0 > 1$. Then (3.55) gives

$$Z_2 \ll N(\mathfrak{c})^{1/n} W_0 \sum_{\{g_k\} \neq \{0\}} \min_{k \in J_3} \Big(\frac{|\alpha^{(k)}|}{\tau_k |g_k|}\Big). \tag{3.56}$$

Since the range of g_k is given by (3.41), we have

$$\sum_{\{g_k\} \neq \{0\}} \min_{k \in J_3} \Big(\frac{|\alpha^{(k)}|}{\tau_k |g_k|}\Big) \ll \sum_{k \in J_3} \overset{(k)}{\sum}{}' \sum_{g_k=1}^{G_k} \frac{|\alpha^{(k)}|}{\tau_k g_k},$$

where $\overset{(k)}{\sum}{}'$ means a sum taken over g_l $(l \in J_3, l \neq k)$. Therefore

$$\begin{aligned} \sum_{\{g_k\} \neq \{0\}} \min_{k \in J_3} \Big(\frac{|\alpha^{(k)}|}{\tau_k |g_k|}\Big) &\ll \sum_{k \in J_3} \overset{(k)}{\sum}{}' 1 \cdot \sum_{g=1}^{G_k} \frac{|\alpha^{(k)}|}{\tau_k g} \\ &\ll \prod_{k \in J_3} \frac{|\alpha^{(k)}|}{\tau_k} \sum_{k \in J_3} \log(1+G_k) \ll \prod_{k \in J_3} \frac{|\alpha^{(k)}|}{\tau_k} \log H. \end{aligned}$$

Putting this result in (3.56) and using (3.40), we obtain

$$\begin{aligned} Z_2 &\ll N(\mathfrak{c})^{1/n} W_0 \prod_{k\in J_3} \frac{|\alpha^{(k)}|}{\tau_k} \log H \\ &= N(\mathfrak{c})^{1/n} H^{q_1} N(\mathfrak{c})^{1-q_1/n} V^{n-q_1} \log H \cdot \prod_{k\in J_3} \tau_k^{-1} \prod_{j\in J_3} |\alpha^{(j)}|^{-1} \\ &\ll N(\mathfrak{c})^{1/n} H^{q_1} N(\mathfrak{c})^{1-q_1/n} V^{n-q_1} \log H \prod_{j\in J_1+J_2} \tau_j \prod_{j\in J_2} |\alpha^{(j)}|^{-1}, \end{aligned}$$

which gives

$$Z_2 \ll V^n N(\mathfrak{c})^{1+1/n} \log H, \tag{3.57}$$

since

$$\tau_i \ll \frac{V}{H} N(\mathfrak{c})^{1/n} \qquad (i\in J_1),$$

$$\tau_j \ll |\alpha^{(j)}| \qquad (j\in J_2).$$

Finally we assume that

$$W_0 \leqq 1.$$

Then there exists a positive constant C_0 such that

$$W(g) \leqq C_0$$

for all $W(g)$. Let G_0 be the set of $\{g_k\ (k\in J_3)\}$ such that

$$W(g) \neq 0, \qquad \{g_k\} \neq \{0\}.$$

Then, noting that

$$|\alpha^{(j)}| < H \qquad (j=1, 2, \cdots, n),$$

and

$$\tau_k \geqq 2^{-2-r_2} \qquad (k\in J_3),$$

we have from (3.55)

$$Z_2 \ll N(\mathfrak{c})^{1/n} H \sum_{\{g_k\}\in G_0} \min_{k\in J_3}\left(\frac{1}{|g_k|}\right).$$

The sum in this right-hand side is of the similar type as in Lemma 3.3. The value of $|g_k|$ in G_0 does not exceed $G_k \ll H$. Therefore we have, by Lemma 3.3,

$$Z_2 \ll N(\mathfrak{c})^{1/n} H \cdot A^{1-1/n} \log H,$$

where A is the number of the elements of G_0.

A is, however, easily estimated;

$$A \leqq \sum_{\{g_k\}\neq\{0\}} W(g) \leqq \sum_{\substack{\mu\in\mathfrak{a}_0^{-1} \\ x(\mu)\in M}} 1 \ll V^n.$$

Thus we obtain

$$Z_2 \ll N(\mathfrak{c})^{1/n} H V^{n-1} \log H. \tag{3.58}$$

Combining (3.57) and (3.58), we have

$$Z_2 \ll V^n U N(\mathfrak{c})\left(\frac{H \log H}{VU} + \frac{N(\mathfrak{c})^{1/n} \log H}{U}\right) \tag{3.59}$$

so that

$$Z^* = Z_1 + Z_2 \ll V^n U N(\mathfrak{c})\left(\frac{1}{T} + \frac{1}{V} + \frac{H \log H}{VU} + \frac{N(\mathfrak{c})^{1/n} \log H}{U}\right). \tag{3.60}$$

Replacing U by $UN(\mathfrak{c})^{1/n}$ in this right-hand side and then multiplying $N(\mathfrak{c})^{-1/n}$ to the whole, we obtain the desired estimation for Z:

$$Z \ll V^n U N(\mathfrak{c})\left(\frac{1}{T} + \frac{1}{V} + \frac{H \log H}{VUN(\mathfrak{c})^{1/n}} + \frac{\log H}{U}\right).$$

Thus we complete the proof of our Theorem.

THEOREM 3.2. *We take ideals $\mathfrak{a}_0$ and $\mathfrak{c}$, a basis $\eta_1, \eta_2, \cdots, \eta_n$ of $\mathfrak{a}_0\mathfrak{c}$, a domain M and a point $z \in E^0$ as in Theorem* 3.1 *and we assume that the inequality* (3.9) *holds. Morever, we assume that M is contained in an n-dimensional cube*

$$\{(x_1, x_2, \cdots, x_n);\ |x_j| \leqq V/2 \quad (j = 1, 2, \cdots, n)\}$$

and that

$$V^n < \frac{T}{(4b_0)^{n-1}\sqrt{D}\, N(\mathfrak{a}_0\mathfrak{c})}. \tag{3.61}$$

Now we take a positive number V_0 and define a sum as follows:

$$Z' = \sum_{\substack{\mu \in \mathfrak{a}_0^{-1},\ x(\mu) \in M \\ |\mu| \geqq V_0}} \min_{1 \leqq j \leqq n} (U,\ \|S(\eta_j \mu z)\|^{-1}), \tag{3.62}$$

where the sum is taken over all elements μ of $\mathfrak{a}_0^{-1}$ such that $x(\mu) \in M$ and

$$|\mu^{(j)}| \geqq V_0 \qquad (j = 1, 2, \cdots, n).$$

Then we have

$$Z' \ll V^n U N(\mathfrak{c})\left(\frac{H}{V_0 U N(\mathfrak{c})^{1/n}} + \frac{H \log H}{VUN(\mathfrak{c})^{1/n}} + \frac{\log H}{U}\right). \tag{3.63}$$

PROOF. Similarly to Z^*, Z_1 and Z_2 in the proof of Theorem 3.1, we define sums $Z^{*\prime}, Z_1'$ and Z_2'. We shall also use the same notations.

As for Z_2', we obtain the same estimation as (3.59), that is,

$$Z_2' \ll V^n U N(\mathfrak{c})\left(\frac{H \log H}{VU} + \frac{N(\mathfrak{c})^{1/n} \log H}{U}\right). \tag{3.64}$$

Assume that $y(\mu) \in B(0)$. Then we have

$$|C_j X_j(\zeta)| \leqq \frac{\tau_j}{2} \qquad (j = 1, 2, \cdots, n),$$

that is,

$$|\alpha^{(j)} X_j(\zeta)| \leqq \frac{\tau_j}{4(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}} \qquad (j = 1, 2, \cdots, n).$$

On the other hand, by our assumption about the domain M, we have

$$|\delta^{(j)}X_j(\mu)| \leqq \frac{V}{2H} \leqq \frac{\tau_j}{4(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}} \qquad (j=1,2,\cdots,n).$$

Therefore, by the same way as we derived (3.30) and (3.32) from (3.27) and (3.28), we see that

$$\alpha\zeta=\delta\mu, \qquad \mu\in\mathfrak{a}^*=\frac{\mathfrak{a}}{\mathfrak{a}_0\mathfrak{c}}.$$

If $J_1+J_2\neq\phi$, then we have

$$\begin{aligned} Z_1' &\ll \sum_{\substack{\mu\in\mathfrak{a}^*,\ x(\mu)\in M\\ |\mu|\geqq V_0}} \min_{j\in J_1+J_2}\left(U,\ \frac{H}{|\mu^{(j)}|}\right) \\ &\ll \left(1+\frac{V^n}{N(\mathfrak{a}^*)}\right)\frac{H}{V_0} \ll V^nN(\mathfrak{c})\frac{H}{V_0}. \end{aligned} \tag{3.65}$$

Finally we assume that $J_1+J_2=\phi$. Then it follows from (2.6) and (3.20) that

$$N(\mathfrak{a}^*)=N\left(\frac{\mathfrak{a}}{\mathfrak{a}_0\mathfrak{c}}\right)\geqq\frac{|N(\alpha)|}{\sqrt{D}\,N(\mathfrak{a}_0\mathfrak{c})}\geqq\frac{T}{(4b_0)^{n-1}\sqrt{D}\,N(\mathfrak{a}_0\mathfrak{c})},$$

since (2.8) means that $|N(\alpha)|\leqq D^{1/2}N(\mathfrak{a})$. Therefore, if $x(\mu)$ $(\mu\neq0)$ were a point in M and corresponding point $y(\mu)$ belonged to $B(0)$, then the following two inequalities

$$|N(\mu)|\leqq V^n,$$

$$|N(\mu)|\geqq N(\mathfrak{a}^*)\geqq\frac{T}{(4b_0)^{n-1}\sqrt{D}\,N(\mathfrak{a}_0\mathfrak{c})}$$

would be obtained. But these contradict to (3.61).

Hence Z_1' is empty when $J_1+J_2=\phi$. Therefore our Theorem follows from (3.64) and (3.65).

THEOREM 3.3. *We take ideals* $\mathfrak{a}_0$ *and* $\mathfrak{c}$, *a basis* $\eta_1,\eta_2,\cdots,\eta_n$ *of* $\mathfrak{a}_0\mathfrak{c}$ *and a point* z *of* E^0 *as in Theorem* 3.1 *and we assume that*

$$\frac{H}{2^{3+r_2}(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}}>1. \tag{3.66}$$

Let Q *be an* n*-dimensional cube in* E^n:

$$Q=\{(x_1,x_2,\cdots,x_n);\ |x_j|\leqq W \quad (j=1,2,\cdots,n)\}$$

with $W\geqq1$.

Now we define a sum of the following form:

$$L=\sum_{\substack{\mu\in\mathfrak{a}_0^{-1}\\ x(\mu)\in Q}}\min_{1\leqq j\leqq n}(U,\ \|S(\eta_j\mu z)\|^{-1}), \tag{3.67}$$

where U *is a given number* $\geqq1$ *and the sum is taken over all* $\mu\in\mathfrak{a}_0^{-1}$ *such that* $x(\mu)\in Q$.

Then we have

$$L \ll W^n U N(\mathfrak{c}) \left(\frac{1}{T} + \frac{1}{W} + \frac{N(\mathfrak{c})^{1/n}}{H} + \frac{H \log H}{W U N(\mathfrak{c})^{1/n}} + \frac{\log H}{U} \right). \tag{3.68}$$

PROOF. If

$$2W < \frac{H}{2^{3+r_2}(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}},$$

then we have by Theorem 3.1

$$L \ll W^n U N(\mathfrak{c}) \left(\frac{1}{T} + \frac{1}{W} + \frac{H \log H}{W U N(\mathfrak{c})^{1/n}} + \frac{\log H}{U} \right). \tag{3.69}$$

Suppose that

$$2W \geqq \frac{H}{2^{3+r_2}(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}}.$$

We put

$$V = \max\left(1, \frac{H}{2^{4+r_2}(DN(\mathfrak{a}_0\mathfrak{c}))^{1/n}}\right)$$

and cover Q with at most $O(W^n/V^n)$ parallelotopes the sides of which do not exceed V. Then we can divide the sum L into at most $O(W^n/V^n)$ parts, each of which is of the same type as the sum Z in Theorem 3.1. Therefore we have

$$L \ll \frac{W^n}{V^n} U V^n N(\mathfrak{c}) \left(\frac{1}{T} + \frac{1}{V} + \frac{H \log H}{V U N(\mathfrak{c})^{1/n}} + \frac{\log H}{U} \right)$$
$$\ll W^n U N(\mathfrak{c}) \left(\frac{1}{T} + \frac{N(\mathfrak{c})^{1/n}}{H} + \frac{\log H}{U} \right).$$

Our Theorem follows from this result and (3.69).

LEMMA 3.4. *Let $a_1, a_2, \cdots, a_n$ be positive numbers such that*

$$a_1 a_2 \cdots a_n \geqq 1,$$
$$a_{p'} = a_p \qquad (p = r_1 + 1, \cdots, r_1 + r_2)$$

and N_0 be the number of the units ε which satisfy the conditions

$$|\varepsilon^{(j)}| \leqq a_j \qquad (j = 1, 2, \cdots, n).$$

Then

$$N_0 \ll (1 + \log(a_1 a_2 \cdots a_n))^r,$$

where $r = r_1 + r_2 - 1$.

PROOF. It suffices to prove our Lemma under the assumption

$$a_1 = a_2 = \cdots = a_n = a_0 \geqq 1.$$

The unit ε in question satisfies inequalities

$$a_0 \geqq |\varepsilon^{(j)}| = \prod_{\substack{k=1 \\ k \neq j}}^{n} |\varepsilon^{(k)}|^{-1} \geqq a_0^{1-n} \qquad (j = 1, 2, \cdots, n).$$

Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r$ be the fundamental units of K, then N_0 is equal to the product of the number of the roots of unity in K and the number of the r-tuples $(t_1, t_2, \cdots, t_r)$ of rational integers which satisfy the following conditions

$$a_0 \geqq |\prod_{k=1}^{r} \varepsilon_k^{(j)t_k}| \geqq a_0^{1-n} \qquad (j=1, 2, \cdots, n)$$

or

$$\log a_0 \geqq \sum_{k=1}^{r} t_k \log|\varepsilon_k^{(j)}| \geqq (1-n)\log a_0 \qquad (j=1, 2, \cdots, n). \tag{3.70}$$

Since the rank of a matrix

$$A = \begin{pmatrix} \log|\varepsilon_1^{(1)}|, & \log|\varepsilon_1^{(2)}|, & \cdots, & \log|\varepsilon_1^{(n)}| \\ \log|\varepsilon_2^{(1)}|, & \log|\varepsilon_2^{(2)}|, & \cdots, & \log|\varepsilon_2^{(n)}| \\ & \cdots\cdots\cdots\cdots & & \\ \log|\varepsilon_r^{(1)}|, & \log|\varepsilon_r^{(2)}|, & \cdots, & \log|\varepsilon_r^{(n)}| \end{pmatrix}$$

is equal to r, there exists a matrix

$$B = \begin{pmatrix} b_{11}, & b_{12}, & \cdots, & b_{1r} \\ b_{21}, & b_{22}, & \cdots, & b_{2r} \\ & \cdots\cdots\cdots & & \\ b_{n1}, & b_{n2}, & \cdots, & b_{nr} \end{pmatrix}$$

such that the product AB is the unit matrix of degree r.

Therefore we have by (3.70)

$$|t_l| = |\sum_{j=1}^{n} b_{jl} \sum_{k=1}^{r} t_k \log|\varepsilon_k^{(j)}|\ | \leqq n\,|\sum_{j=1}^{n} b_{jl}| \log a_0 \qquad (l=1, 2, \cdots, r)$$

so that

$$N_0 \ll (1+\log a_0)^r \ll (1+\log(a_1 a_2 \cdots a_n))^r .$$

Thus we complete the proof.

LEMMA 3.5. *Let $\xi_1, \xi_2, \cdots, \xi_{r_1}$ be r_1 real numbers, $\xi_{r_1+1}, \xi_{r_1+2}, \cdots, \xi_n$ be $2r_2$ complex numbers such that $\xi_{p'} = \bar{\xi}_p$ $(p=r_1+1, \cdots, r_1+r_2)$ and $A_1, A_2, \cdots, A_n$ be positive numbers such that $A_{p'} = A_p$ $(p=r_1+1, \cdots, r_1+r_2)$.*

We define a trigonometrical sum I as follows:

$$I = \sum_{\alpha \in \mathfrak{f}}{}' e^{2\pi i S(\xi\alpha)},$$

where $\mathfrak{f}$ is an integral or fractional ideal and the summation means that α runs through all $\alpha \in \mathfrak{f}$ such that

$$\begin{aligned} 0 < \alpha^{(q)} &\leqq A_q && (q=1, 2, \cdots, r_1), \\ |\alpha^{(p)}| &\leqq A_p && (p=r_1+1, \cdots, r_1+r_2). \end{aligned} \tag{3.71}$$

Then we have

$$I \ll \frac{A_0^{n-1}}{N(\mathfrak{f})^{1-1/n}} \min_{1 \leqq j \leqq n}\left(\frac{A_0}{N(\mathfrak{f})^{1/n}}, \quad \| S(\lambda_j \xi) \|^{-1}\right), \tag{3.72}$$

where

$$A_0 = \max(A_1, A_2, \cdots, A_n, N(\mathfrak{f})^{1/n})$$

and $\lambda_1, \lambda_2, \cdots, \lambda_n$ *is a basis of* $\mathfrak{f}$ *which satisfy the inequality*

$$|\lambda_j^{(k)}| \leqq c N(\mathfrak{f})^{1/n} \qquad (j, k = 1, 2, \cdots, n). \tag{3.73}$$

Proof. First we have by Lemma 3.2

$$I \ll \frac{A_1 A_2 \cdots A_n}{N(\mathfrak{f})} + 1 \ll \frac{A_0^n}{N(\mathfrak{f})}. \tag{3.74}$$

Let λ be one of the $\lambda_1, \lambda_2, \cdots, \lambda_n$, a basis of $\mathfrak{f}$, satisfying (3.73), then

$$I \cdot e^{2\pi i S(\lambda\xi)} = I + \sum_{\alpha}{}^{*} e^{2\pi i S(\xi\alpha)},$$

where $\sum^*$ is a sum taken over all $\alpha \in \mathfrak{f}$ which satisfy the conditions

$$\begin{aligned} &0 < \alpha^{(q)} - \lambda^{(q)} \leqq A_q && (q = 1, 2, \cdots, r_1), \\ &|\alpha^{(p)} - \lambda^{(p)}| \leqq A_p && (p = r_1 + 1, \cdots, r_1 + r_2) \end{aligned} \tag{3.75}$$

but do not satisfy at least one of the inequalities in (3.71).

In view of Lemma 3.2, we see that the number of $\alpha \in \mathfrak{f}$ which satisfies the conditions (3.75) and for a certain index q_0 $(1 \leqq q_0 \leqq r_1)$

$$\alpha^{(q_0)} < 0 \qquad \text{or} \qquad \alpha^{(q_0)} > A_{q_0}$$

is

$$O\left(\frac{A_1 A_2 \cdots A_n}{A_{q_0}} \cdot \frac{|\lambda^{(q_0)}|}{N(\mathfrak{f})} + 1\right) = O\left(\frac{A_0^{n-1}}{N(\mathfrak{f})^{1-1/n}}\right).$$

Similarly, applying Lemma 3.2, we see that the number of $\alpha \in \mathfrak{f}$ which satisfies (3.71) and

$$A_{p_0} < |\alpha^{(p_0)}| \leqq A_{p_0} + |\lambda^{(p_0)}|$$

for a certain index p_0 $(r_1 + 1 \leqq p_0 \leqq r_1 + r_2)$ is

$$O\left(\frac{A_1 A_2 \cdots A_n}{A_{p_0}^2} \cdot \frac{A_{p_0} |\lambda^{(p_0)}|}{N(\mathfrak{f})} + 1\right) = O\left(\frac{A_0^{n-1}}{N(\mathfrak{f})^{1-1/n}}\right).$$

Therefore we have

$$I \cdot (e^{2\pi i S(\lambda\xi)} - 1) \ll \frac{A_0^{n-1}}{N(\mathfrak{f})^{1-1/n}}. \tag{3.76}$$

Since

$$|e^{2\pi i S(\lambda\xi)} - 1| \geqq \| S(\lambda\xi) \|,$$

(3.72) follows from (3.74) and (3.76).

From this Lemma 3.5 follows:

LEMMA 3.6. *We take* $\xi_1, \xi_2, \cdots, \xi_n$ *and* $A_1, A_2, \cdots, A_n$ *as in Lemma* 3.5. *Moreover we take n real numbers* $B_1, B_2, \cdots, B_n$ *such that*

$$0 \leqq B_j < A_j \qquad (j=1,2,\cdots,n),$$
$$B_{p'} = B_p \qquad (p=r_1+1,\cdots,r_1+r_2).$$

We define a trigonometrical sum

$$J = \sum_{\alpha \in \mathfrak{f}}'' e^{2\pi i S(\xi\alpha)},$$

where the summation means that α *runs through all* $\alpha \in \mathfrak{f}$ *such that*

$$B_q < \alpha^{(q)} \leqq A_q \qquad (q=1,2,\cdots,r_1),$$
$$B_p < |\alpha^{(p)}| \leqq A_p \qquad (p=r_1+1,\cdots,r_1+r_2).$$

Then we have

$$J \ll \frac{A_0^{\,n-1}}{N(\mathfrak{f})^{1-1/n}} \min_{1 \leqq j \leqq n} \left(\frac{A_0}{N(\mathfrak{f})^{1/n}},\ \| S(\lambda_j \xi) \|^{-1} \right),$$

where A_0 *and* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *have the same meaning as in Lemma* 3.5.

THEOREM 3.4. *Let* $N_1, N_2, \cdots, N_n$ *and* N_0 *be positive numbers such that*

$$N_0 < N_j \qquad (j=1,2,\cdots,n),$$
$$N_{p'} = N_p \qquad (p=r_1+1,\cdots,r_1+r_2).$$

Let $\mathfrak{M}$ *be the set of integers* μ *of* K *such that*

$$N_0 < \mu^{(q)} \leqq N_q \qquad (q=1,2,\cdots,r_1),$$
$$N_0 < |\mu^{(p)}| \leqq N_p \qquad (p=r_1+1,\cdots,r_1+r_2).$$

Let M_1 *and* M_2 *be the sets of some ideals satisfying the following inequalities:*

$$\begin{aligned} &1 \leqq U_1 \leqq N(\mathfrak{a}) \leqq U_2 \leqq 2U_1 \qquad (\mathfrak{a} \in M_1), \\ &1 \leqq V_1 \leqq N(\mathfrak{b}) \leqq V_2 \leqq 2V_1 \qquad (\mathfrak{b} \in M_2). \end{aligned} \tag{3.77}$$

Moreover, we take an ideal $\mathfrak{c}$ *and a point* $z=(z_1, z_2, \cdots, z_n)$ *of* E^0 *which is defined by the Farey division with respect to* (H, T) *and we assume that the inequality* (3.66) *is true.*

Now we define a trigometrical sum of the following form:

$$S = \sum_{\mathfrak{b} \in M_2} \sum_{\substack{\nu \in \mathfrak{M} \\ \frac{(\nu)}{\mathfrak{cb}} \in M_1}} e^{2\pi i S(\nu z)},$$

where the inner sum is taken over all integers ν *such that*

$$\nu \in \mathfrak{M}, \qquad \frac{(\nu)}{\mathfrak{cb}} \in M_1. \tag{3.78}$$

Then we have

$$S \ll \frac{N^{5n/4}}{N_0^{n/4} N(\mathfrak{c})^{3/4}} (\log N+1)^{3r/4} \Big(\frac{1}{T} + \frac{1}{V_1^{1/n}} + \frac{N(\mathfrak{c})^{1/n}}{H} + \frac{H \log H}{N} + \frac{(V_1 N(\mathfrak{c}))^{1/n} \log H}{N} \Big)^{1/4},$$

where $r = r_1 + r_2 - 1$ *and* $N = \max(N_1, N_2, \cdots, N_n)$.

PROOF. Let $\mathfrak{C}_1, \mathfrak{C}_2, \cdots, \mathfrak{C}_h$ be the ideal classes of K and put

$$M_{i,j} = M_i \cap \mathfrak{C}_j \qquad (i=1,2\,;\ j=1,2,\cdots,h)\,.$$

We can write

$$S = \sum_{j,k=1}^{h} \sum_{\mathfrak{b} \in M_2,_j} \sum_{\substack{\nu \in \mathfrak{M} \\ \frac{(\nu)}{\mathfrak{bc}} \in M_1,_k}} e^{2\pi i S(\nu z)},$$

where the innermost sum is taken over all integers ν such that

$$\text{(3.79)} \qquad \nu \in \mathfrak{M}, \quad \frac{(\nu)}{\mathfrak{bc}} \in M_{1,k} \quad (\mathfrak{b} \in M_{2,j})\,.$$

If there exists an integer ν satisfying (3.79), then the second relation of (3.79) means that

$$\mathfrak{C}_k \mathfrak{C}_j \mathfrak{C}(\mathfrak{c}) = \mathfrak{C}_0\,,$$

where $\mathfrak{C}_0$ is the principal ideal class and $\mathfrak{C}(\mathfrak{c})$ is the class containing $\mathfrak{c}$. Therefore it suffices to consider the sum S with additional conditions:

(i) All ideals in M_1 and M_2 belong to classes $\mathfrak{C}_1$ and $\mathfrak{C}_2$ respectively,

(ii) $\mathfrak{C}_1 \mathfrak{C}_2 \mathfrak{C}(\mathfrak{c}) = \mathfrak{C}_0$.

If we fix the ideals $\mathfrak{a}_0$ in $\mathfrak{C}_1$ and $\mathfrak{b}_0$ in $\mathfrak{C}_2$, then we can put

$$\text{(3.80)} \qquad \begin{aligned} &\mathfrak{a} = \alpha \mathfrak{a}\,; \quad \alpha \in \mathfrak{a}_0^{-1} \qquad (\mathfrak{a} \in M_1)\,, \\ &\mathfrak{b} = \beta \mathfrak{b}_0\,; \quad \beta \in \mathfrak{b}_0^{-1} \qquad (\mathfrak{b} \in M_2)\,. \end{aligned}$$

We denote by A_1^0 the set of the principal ideals (α) and by A_2 the set of β, both of which are defined by (3.80).

Putting

$$\mathfrak{c}\mathfrak{a}_0\mathfrak{b}_0 = (\gamma)\,,$$

we can write

$$\text{(3.81)} \qquad S = \sum_{\beta \in A_2} \sum_{\substack{\nu \in \mathfrak{M} \\ \left(\frac{\nu}{\beta\gamma}\right) \in A_1^0}} e^{2\pi i S(\nu z)}\,.$$

Therefore, denoting by A_1 the set of $\alpha \in \mathfrak{a}_0^{-1}$ derived from (3.80), the sum S is written as follows:

$$\text{(3.82)} \qquad S = \sum_{\beta \in A_2} \sum_{\varepsilon} \sum_{\alpha} e^{2\pi i S(\varepsilon\alpha\beta\gamma z)}\,,$$

where the sum $\sum_{\varepsilon}$ is taken over all units ε for which there exists at least one α such that

$$\alpha \in A_1, \qquad \varepsilon\alpha\beta\gamma \in \mathfrak{M}, \tag{3.83}$$

and after taking $\beta \in A_2$ and a unit ε, the innermost sum $\sum_{\alpha}$ is taken over all α satisfying (3.83).

By multiplying a suitable unit, if necessary, we can assume that γ and $\alpha \in A_1$ satisfy the following inequalities:

$$\begin{aligned} &c_0 N(\mathfrak{a})^{1/n} < |\alpha^{(j)}| < c_1 N(\mathfrak{a})^{1/n} \qquad (\mathfrak{a} = \alpha\mathfrak{a}_0\,;\ j = 1, 2, \cdots, n)\,, \\ &c_0 N(\mathfrak{c})^{1/n} < |\gamma^{(j)}| < c_1 N(\mathfrak{c})^{1/n} \qquad (j = 1, 2, \cdots, n)\,, \end{aligned} \tag{3.84}$$

where c_0 and c_1 are suitable positive constants. We may also assume that $c_0 < 1$.

We now put

$$X_1 = \frac{N_0}{c_1{}^2(U_2 N(\mathfrak{c}))^{1/n}}\,, \qquad X_2 = \frac{N}{c_0{}^2(U_1 N(\mathfrak{c}))^{1/n}}$$

and define a set $\mathfrak{S}$ of $\nu \in \mathfrak{b}_0{}^{-1}$ such that

$$\frac{V_1}{N(\mathfrak{b}_0)} \leqq |N(\nu)| \leqq \frac{V_2}{N(\mathfrak{b}_0)}\,,$$

$$X_1 < |\nu^{(j)}| < X_2 \qquad (j = 1, 2, \cdots, n)\,.$$

We see that the product $\varepsilon\beta$ of a unit ε and a number $\beta \in A_2$ in the sum (3.82) satisfies the inequalities

$$\frac{N_0}{|\alpha^{(j)}\gamma^{(j)}|} < |\varepsilon^{(j)}\beta^{(j)}| \leqq \frac{N_j}{|\alpha^{(j)}\gamma^{(j)}|} \qquad (j = 1, 2, \cdots, n)$$

with a certain number $\alpha \in A_1$. Therefore, noting (3.84), we have

$$X_1 < |\varepsilon^{(j)}\beta^{(j)}| < X_2 \qquad (j = 1, 2, \cdots, n)\,, \tag{3.85}$$

which means that

$$\varepsilon\beta \in \mathfrak{S}\,.$$

Therefore, applying the Schwarz's inequality to the right-hand side of (3.82), we have

$$\begin{aligned} |S|^2 &\leqq \sum_{\beta}\sum_{\varepsilon} 1 \cdot \sum_{\beta}\sum_{\varepsilon} \Big| \sum_{\alpha} e^{2\pi i S(\varepsilon\alpha\beta\gamma z)} \Big|^2 \\ &\leqq \sum_{\nu \in \mathfrak{S}} 1 \cdot \sum_{\nu \in \mathfrak{S}} \Big| \sum_{\alpha}{}' e^{2\pi i S(\alpha\nu\gamma z)} \Big|^2\,, \end{aligned}$$

where the last sum $\sum'$ is taken over all α such that

$$\alpha \in A_1, \quad \nu\alpha\gamma \in \mathfrak{M} \quad (\nu \in \mathfrak{S})\,. \tag{3.86}$$

It follows from Lemma 3.4 that

$$\sum_{\nu \in \mathfrak{S}} 1 \ll V_2(\log N + 1)^r \tag{3.87}$$

so that

$$S^2 \ll V_2(\log N+1)^r \sum_{\nu\in\mathfrak{S}} \sum_{\alpha,\alpha_1}' e^{2\pi i S(\nu\gamma(\alpha-\alpha_1)z)} .$$

We shall change the order of the double sum in this right-hand side;

$$S^2 \ll V_2(\log N+1)^r \sum_{\alpha,\alpha_1} \sum_{\nu\in\mathfrak{S}(\alpha,\alpha_1)} e^{2\pi i S(\nu\gamma(\alpha-\alpha_1)z)} , \tag{3.88}$$

where $\mathfrak{S}(\alpha, \alpha_1)$ is the set of numbers ν such that

$$\nu \in \mathfrak{S}, \quad \nu\alpha\gamma \in \mathfrak{M}, \quad \nu\alpha_1\gamma \in \mathfrak{M} . \tag{3.89}$$

(3.86) shows that α (or α_1) in the sum of (3.88) satisfies the inequality

$$|N(\alpha)| \leqq \frac{N^n}{|N(\nu\gamma)|} \leqq \frac{N^n}{V_1 N(\mathfrak{a}_0\mathfrak{c})} .$$

Therefore, if we define a set A of λ such that

$$\lambda \in \mathfrak{a}_0^{-1}, \quad |\lambda^{(j)}| \leqq \frac{c_1 N}{(V_1 N(\mathfrak{c}))^{1/n}} \qquad (j=1, 2, \cdots, n) ,$$

then we see from (3.84) that α and α_1 in the sum (3.88) run through a certain subset of A.

We can define the subset $\mathfrak{S}(\alpha, \alpha_1)$ of $\mathfrak{S}$ over all pairs (α, α_1) with $\alpha, \alpha_1 \in A$ by the condition (3.89). Hence we have, using again Schwarz's inequality,

$$\begin{aligned} & \Big| \sum_{\alpha\ \alpha_1}' \sum_{\nu\in\mathfrak{S}(\alpha,\alpha_1)} e^{2\pi i S(\nu\gamma(\alpha-\alpha_1)z)} \Big|^2 \\ & \leqq \sum_{\alpha\ \alpha_1}' 1 \cdot \sum_{\alpha,\alpha_1}' \Big| \sum_{\nu\in\mathfrak{S}(\alpha\ \alpha_1)} e^{2\pi i S(\nu\gamma(\alpha-\alpha_1)z)} \Big|^2 \\ & \leqq \sum_{\alpha,\alpha_1\in A} 1 \cdot \sum_{\alpha,\alpha_1\in A} \Big| \sum_{\nu\in\mathfrak{S}(\alpha,\alpha_1)} e^{2\pi i S(\nu\gamma(\alpha-\alpha_1)z)} \Big|^2 . \end{aligned} \tag{3.90}$$

Since we consider the case when the sum S has at least one term, there exists $\nu \in \mathfrak{M}$ such that

$$(\nu) = \mathfrak{c}\mathfrak{b}\mathfrak{a} \qquad (\mathfrak{b} \in M_2, \ \mathfrak{a} \in M_1) .$$

Therefore, the following two inequalities

$$V_1 U_1 N(\mathfrak{c}) \leqq N^n , \tag{3.91}$$

$$N_0^n \leqq V_2 U_2 N(\mathfrak{c}) \tag{3.92}$$

are true. If we put

$$W = \frac{N^n}{V_1 N(\mathfrak{c})} ,$$

then (3.91) shows that $W \geqq U_1 \geqq 1$, which gives

$$\sum_{\alpha,\alpha_1\in A} 1 \ll W^2 .$$

Therefore, putting

$$S_1=\sum_{\alpha,\alpha_1\in A}\Big|\sum_{\nu\in\mathfrak{S}(\alpha,\alpha_1)}e^{2\pi i S(\nu\gamma(\alpha-\alpha_1)z)}\Big|^2,$$

we have, from (3.88) and (3.90),

$$\begin{aligned}S^2&\ll V_2(\log N+1)^r\,(S_1\cdot\sum_{\alpha,\alpha_1\in A}1)^{1/2}\\ &\ll V_2(\log N+1)^r\,\frac{N^n}{V_1N(\mathfrak{c})}\,S_1^{1/2}\\ &\ll\frac{N^n}{N(\mathfrak{c})}(\log N+1)^r\cdot S_1^{1/2}.\end{aligned}\tag{3.93}$$

As for the sum S_1, we write

$$\begin{aligned}S_1&=\sum_{\alpha,\alpha_1\in A}\ \sum_{\nu,\nu_1\in\mathfrak{S}(\alpha,\alpha_1)}e^{2\pi i S((\nu-\nu_1)\gamma(\alpha-\alpha_1)z)}\\ &=\sum_{\alpha_1\in A}\ \sum_{\nu,\nu_1}{}'\ \sum_{\alpha}{}^{*}\,e^{2\pi i S((\nu-\nu_1)\gamma(\alpha-\alpha_1)z)},\end{aligned}$$

where the sum $\sum'_{\nu,\nu_1}$ means that ν and ν_1 run through a certain subset of $\mathfrak{S}$ and the innermost sum $\sum^*_{\alpha}$ is taken over all α such that

$$\alpha\in A,\quad \nu\alpha\gamma\in\mathfrak{M},\quad \nu_1\alpha\gamma\in\mathfrak{M}.\tag{3.94}$$

These conditions for α in (3.94) are also written as follows:

$$\alpha\in\mathfrak{a}_0^{-1},$$

$$\max\Big(\frac{N_0}{|\nu^{(j)}\gamma^{(j)}|},\frac{N_0}{|\nu_1^{(j)}\gamma^{(j)}|}\Big)<|\alpha^{(j)}|\leqq\min\Big(c_1W^{1/n},\frac{N_j}{|\nu^{(j)}\gamma^{(j)}|},\frac{N_j}{|\nu_1^{(j)}\gamma^{(j)}|}\Big)$$
$$(j=1,2,\cdots,n).$$

Therefore we have, applying Lemma 3.6 to the sum $\sum^*_{\alpha}$,

$$\begin{aligned}|S_1|&\leqq\sum_{\alpha_1\in A}\ \sum_{\nu,\nu_1}{}'\ \Big|\sum_{\alpha}{}^{*}e^{2\pi i S(\gamma(\nu-\nu_1)\alpha z)}\Big|\\ &\ll W^{2-1/n}\sum_{\nu,\nu_1\in\mathfrak{S}}\ \min_{1\leqq j\leqq n}(W^{1/n},\ \|S(\rho_j(\nu-\nu_1)\gamma z)\|^{-1}),\end{aligned}$$

where $\rho_1,\rho_2,\cdots,\rho_n$ is a basis of $\mathfrak{a}_0^{-1}$ such that

$$|\rho_j^{(k)}|\leqq c\qquad (j,k=1,2,\cdots,n).$$

If we put $\eta_j=\rho_j\gamma$ $(j=1,2,\cdots,n)$, then $\eta_1,\eta_2,\cdots,\eta_n$ is a basis of $\mathfrak{b}_0\mathfrak{c}$ such that

$$|\eta_j^{(k)}|\leqq cN(\mathfrak{c})^{1/n}\qquad (j=1,2,\cdots,n)$$

and we have by (3.87)

$$\begin{aligned}S_1&\ll W^{2-1/n}\sum_{\nu,\nu_1\in\mathfrak{S}}\ \min_{1\leqq j\leqq n}(W^{1/n},\ \|S(\eta_j(\nu-\nu_1)z)\|^{-1})\\ &\ll W^{2-1/n}V_2(\log N+1)^r\sum_{\substack{\mu\in\mathfrak{b}_0^{-1}\\ |\mu|\leqq 2X_2}}\ \min_{1\leqq j\leqq n}(W^{1/n},\ \|S(\eta_j\mu z)\|^{-1}),\end{aligned}\tag{3.95}$$

where the last sum is taken over all $\mu \in \mathfrak{d}_0^{-1}$ such that

$$|\mu^{(j)}| \leqq 2X_2 \qquad (j=1,2,\cdots,n).$$

We know from (3.91) that

$$X_2 \geqq \frac{V_1^{1/n}}{c_0^2} > V_1^{1/n} \geqq 1. \tag{3.96}$$

Therefore, applying Theorem 3.3 to the last sum of (3.95), we have

$$S_1 \ll W^2 V_2(\log N+1)^r X_2^n N(\mathfrak{c}) \times\left(\frac{1}{T}+\frac{1}{X_2}+\frac{N(\mathfrak{c})^{1/n}}{H}+\frac{H\log H}{X_2 W^{1/n}N(\mathfrak{c})^{1/n}}+\frac{\log H}{W^{1/n}}\right).$$

Moreover, using (3.96) and the following inequality

$$X_2 \ll X_1\frac{N}{N_0} \ll V_2^{1/n}\frac{N}{N_0},$$

which is obtained from (3.92), we have

$$S_1 \ll \frac{N^{3n}}{N_0^n N(\mathfrak{c})}(\log N+1)^r \times\left(\frac{1}{T}+\frac{1}{V_1^{1/n}}+\frac{N(\mathfrak{c})^{1/n}}{H}+\frac{H\log H}{N}+\frac{(V_1 N(\mathfrak{c}))^{1/n}\log H}{N}\right).$$

Our Theorem follows from this result and (3.93).

THEOREM 3.5. *We take positive numbers $N_0, N_1, \cdots, N_n$, a set $\mathfrak{M}$ of integers of K, an ideal $\mathfrak{c}$ and a point $z \in E^0$ as in Theorem 3.4 and we assume the inequality (3.66). We consider two sets M_1^* and M_2^* of ideals which satisfy the following inequalities*

$$1 \leqq U_1^* \leqq N(\mathfrak{a}) \leqq U_2^* \qquad (\mathfrak{a}\in M_1^*),$$
$$1 \leqq V_1^* \leqq N(\mathfrak{b}) \leqq V_2^* \qquad (\mathfrak{b}\in M_2^*).$$

Now we define, similarly as the sum S in Theorem 3.4, a trigonometrical sum S^* as follows:

$$S^* = \sum_{\mathfrak{b}\in M_2^*} \sum_{\substack{\nu\in\mathfrak{M} \\ \frac{(\nu)}{\mathfrak{b}\mathfrak{c}}\in M_1^*}} e^{2\pi i S(\nu z)}.$$

Then we have

$$S^* \ll \frac{N^{5n/4}}{N_0^{n/4}N(\mathfrak{c})^{3/4}}(\log N+1)^{3r/4}(\log U_2^*+1)(\log V_2^*+1) \times\left(\frac{1}{T}+\frac{1}{(V_1^*)^{1/n}}+\frac{N(\mathfrak{c})^{1/n}}{H}+\frac{H\log H}{N}+\frac{(V_2^* N(\mathfrak{c}))^{1/n}\log H}{N}\right)^{1/4}, \tag{3.97}$$

where $r=r_1+r_2-1$ and $N=\max(N_1, N_2, \cdots, N_n)$.

PROOF. We divide two intervals $[U_1^*, U_2^*]$ and $[V_1^*, V_2^*]$ as follows:

$$U = U_1^* < 2U < 2^2U < \cdots < 2^l U < U_2^* \leqq 2^{l+1}U,$$

$$V = V_1^* < 2V < 2^2V < \cdots < 2^m V < V_2^* \leqq 2^{m+1}V.$$

Then S^* becomes the sum of $(l+1)(m+1)$ sums of the types in Theorem 3.4. Therefore we obtain (3.97).

(*To be continued*)

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