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On the genus of the alternating knot II.

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Let k be a knot and let $G(k)$ be the genus of k as defined by Seifert [6]. Let $\Delta(x)=a_0+a_1x+a_2x^2+\cdots+a_{2t}x^{2t}$ be the Alexander polynomial of k. Then Seifert has proved in [6] that we have always

$$t\leqq G(k). \tag{1}$$

In a previous paper [3], we proved that the equality holds in (1) for a knot in a special class of alternating knots. In the present paper we shall show that the equality holds in (1) for all alternating knots (Theorem 4.1). It was also shown in [3] that, for an alternating knot k of the class considered in that paper, the orientable surface spanning k, whose genus is equal to $G(k)$, is obtained by Seifert's construction [6]. It will be shown that this is the case for every alternating knot.

Furthermore we shall show that $\Delta(x)$ is "alternating" for an alternating knot k (Theorem 4.4).

From this theorem, we can immediately deduce the well-known fact that a knot 8_{19} in [2] is not equivalent with an alternating knot. Throughout this paper we shall use the same notations as in [3].

§ 1. Preliminaries.

Let k be a polygonal oriented knot in the 3-sphere S^3 and let S^2 be a 2-sphere in S^3, which does not meet k. Let K be an image of a regular projection of k into S^2.

Let K have n crossing points $c_1, c_2,\cdots, c_n$. Then K divides S^2 into $n+2$ regions $r_0, r_1,\cdots, r_{n+1}$, which are classified into two classes, called "black" or "white", in such a way that every side of K is the common boundary of black and white regions. (Whenever we speak of the classification of regions in "black" and "white", we always mean a classification of this nature.) As is well-known, an integer $I(r_i)$, called the index of r_i, corresponds to each region r_i. We have

LEMMA 1.1. *For two regions r_i and r_j of the same colour, we have*

$$I(r_i)\equiv I(r_j) \qquad (\text{mod } 2)$$

and conversely.

This is proved by the same method as used in the proof of Lemma 3.2 in [3].

Each corner of the two of the four regions[1] meeting at a crossing point c_i is marked with a dot, and we can assume that the signs of the elements distinct from zero in any column of the L-matrices are positive, i. e. either x or 1 (cf. [1], [3]).

§ 2. The loops of the first and of the second kind.

Let us divide K into some oriented loops, called the *standard loops*, in the same way as in [6].

DEFINITION 2.1. If a standard loop L bounds a region r_i, we say L is of the *first kind* and r_i is the region bounded by L. Otherwise L is of the *second kind*.

LEMMA 2.2. *The corners of the regions bounded by a loop L of the first kind are either all dotted or all undotted.*

This is proved in the same way as in Lemma 3.1 in [3].

Conversely it is obvious that

LEMMA 2.3. *If the corners of a region r_i are either all dotted or all undotted, $\dot{r}_i$ is a loop of the first kind.*

Let m be the number of the loops of the second kind of K. The case $m=0$ has been treated in [3]. In the following we assume $m\geqq 1$.

Now let us deform the loops of the second kind into the following loops. Let the loops C_i and C_j of the second kind have a crossing point c. Let ε be a sufficiently small positive number, and a, b and d, e the points of intersection of the circle (in S^2) of radius ε with the center c with C_i and C_j respectively (Fig. 1).

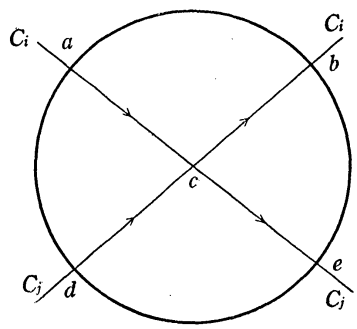

Fig. 1

Then we replace the parts $ac\cup cb$ and $dc\cup ce$ of C_i and C_j by the disjoint arcs ab and de respectively. If we perform this operation at each crossing

1) We may assume that these regions are different from one another. See the note 5) in [3].

point of two loops of the second kind, then we obtain m disjoint loops. These m disjoint loops will be called hereafter loops of the second kind, and if we need to consider the loops of the second kind in the older sense of Def. 2.1, we shall mention it expressly. Then it is obvious that

LEMMA 2.4. *m loops of the second kind divide S^2 into $m+1$ domains*[2] E_0, $E_1, \cdots, E_m$.

LEMMA 2.5. *Let E_j be a domain bounded by some loops, $C_{j_1}, \cdots, C_{j_\nu}$, of the second kind: $\dot{E}_j = C_{j_1} \cup \cdots \cup C_{j_\nu}$. Then the regions $r_\xi, \cdots, r_\eta$ contained in E_j having some sides in common with C_{j_i} have the same index (depending on j_i).*

Furthermore we have

LEMMA 2.6. *The regions contained in the domain E_j can be classified in black and white, and in such a way that the regions having some sides in common with $\dot{E}_j$ have the same colour, say white. All these white regions have then the same index, say p, and the black regions have loops of the first kind as boundaries. Then indices of the black regions are either $p-1$ or $p+1$.*

PROOF. Let $\dot{E}_j = C_{j_1} \cup \cdots \cup C_{j_\nu}$, where C_{j_i}, $i=1, 2, \cdots, \nu$, are loops of the second kind, and $r_\xi, \cdots, r_\eta$ the regions contained in E_j such that each of $\dot{r}_\xi, \cdots, \dot{r}_\eta$ has some sides with C_{j_1} in common. By Lemma 2.5, we have $I(r_\xi) = \cdots = I(r_\eta)$, so that, if we classify $r_0, \cdots, r_{n+1}$ in black and white as said above, $r_\xi, \cdots, r_\eta$ have the same colour, say white, by Lemma 1.1. Let us fix this classification, and let r_λ be one of black regions contained in E_j such that $\dot{r}_\lambda$ has some sides in common with one of $\dot{r}_\xi, \cdots, \dot{r}_\eta$. Then $\dot{r}_\lambda$ is a loop of the first kind, because, if a common side of $\dot{r}_\lambda$ and $\dot{r}_\xi$, say, were a part of a loop of the second kind, r_λ would not be contained in E_j. Now let r_μ be a black region in E_j opposite to r_λ over a crossing point c_k. Then $\dot{r}_\mu$ is also a loop of the first kind (Fig. 2).

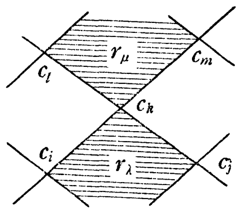

Fig. 2

In fact, if $c_m c_k \cup c_k c_l$ in Fig. 2 were a part of a loop of the second kind, r_μ would not be contained in E_j, and it is impossible that $c_m c_k$ and $c_k c_l$ belong to two different loops of the first kind. Thus it is easily seen that the boundaries of all the black regions in E_j are loops of the first kind. Hence the regions in E_j, whose boundaries have some sides in common with any one of $C_{j_1}, \cdots, C_{j_\nu}$, are white. The remaining part is easy to prove.

2) A domain is connected and is an open subset of S^2.

Hereafter we shall use almost exclusively the classification in black and white of the regions contained in each domain E_j and consider the classification of all regions $r_0, \cdots, r_{n+1}$ only in exceptional cases.

§ 3. Sign of the domain.

Hitherto the numbering of the domains $E_0, E_1, \cdots, E_m$ and loops of the second kind $C_1, C_2, \cdots, C_m$ was made arbitrarily. Now we introduce some rules on the numbering.

There is at least one loop of the second kind such that one of the two parts, into which S^2 is divided by it, does not contain other loops of the second kind. Let us fix one of these loops and denote it by C_1. We denote the domain bounded by C_1 which does not contain any loop of the second kind by E_0, and the domain bounded by C_1 and other loops of the second kind by E_1. Let the domains bounded by loops of the second kind other than E_0, E_1 be numbered arbitrarily. They will be denoted by $E_2, \cdots, E_m$. We define the *outer boundary* C_i of $E_i, i=2, \cdots, m$, as follows. C_i is one of the loops of the second kind bounding E_i such that the following holds: C_i divides S^2 into two parts, one of which contains E_0 and the other E_i. It is clear that the loops of the second kind and the domains bounded by them are thus numbered consistently.

Now let us take a point e_i from each E_i for $i=0, 1, \cdots, m$, and fix it. Let l_{ij} be an arc connecting e_i with e_j not crossing over any crossing point and not touching any loop of the second kind. We shall now define an *intersection number* $I(l_{ij}, C_h, q)$ for a point q at which l_{ij} meets C_h.

DEFINITION 3.1. $I(l_{ij}, C_h, q)=+1$, or -1 according as l_{ij} crosses over C_h at q from the right to the left or from the left to the right with reference to the orientation of C_h. We set $I(l_{ij}, C_h)=\sum_q I(l_{ij}, C_h, q)$. If l_{ij} and C_h are disjoint, we set $I(l_{ij}, C_h)=0$. Then set $e_{ij}=\sum_h I(l_{ij}, C_h)$.

It is easily shown that

LEMMA 3.2. *e_{ij} is uniquely determined by e_i and e_j independently of the choice of l_{ij}.*

Hence we may assume that l_{ij} meets C_h at most at one point for every i, j and h. We can easily show that

$$e_{ij}=-e_{ji} \tag{3.1}$$

$$e_{ij}=e_{ih}+e_{hj} \qquad 0 \leqq i, j, h \leqq m. \tag{3.2}$$

DEFINITION 3.3. We shall call the *sign* of E_j $(j=1, 2, \cdots, m)$ positive or negative according as $I(l_{0j}, C_j, C_j \cap l_{0j})=1$ or -1. The sign of E_0 is defined as the same as that of E_1.

We may assume without loss of generality that $E_0, \cdots, E_d$ are positive and

$E_{d+1},\cdots,E_m$ are negative, where $d\geqq 1$. (We have only to change the orientation of the knot and change the numbering of $E_2,\cdots,E_m$, if necessary.) Let us put $\min_i I(r_i)=p-1$ and $\max_j I(r_j)=p+h+1$. We may suppose $h\geqq 1$.[3)]

Lemma 3.4. *The regions with the maximal and minimal indices are the black regions and the corners of the former are all dotted.*

Proof. Suppose that the region r_{i_0} with the maximal index $p+h+1$ is white. Let r_{i_0} be contained in E_t. Then E_t is positive. For otherwise, the index of a white region in E_s, which is a domain separated from E_t by C_t, will be $p+h+2$, which is a contradiction. Furthermore E_t must contain black regions. For otherwise, there would exist a positive domain E_u whose outer boundary would be $\subset\dot{E}_t$. Hence a white region in E_u would be of the index $p+h+2$, which is a contradiction. Consequently, E_t must contain a black region with the index $p+h+2$. This contradicts the assumption. Hence r_{i_0} is a black region. It will be easily shown that the corners of r_{i_0} are all dotted.

In the same way, we shall see that a region with the minimal index is black, q. e. d.

Remark. More generally, we obtain the following Lemma in the same way as above.

Lemma 3.5. $\max_i I(r_i)-\min_j I(r_j)=\max_{0\leqq i,j\leqq m} e_{ij}+2.$

As this lemma will not be used in following sections, the proof is omitted.

§ 4. Statement of the main theorems.

As mentioned in the introduction, our main theorems are the following:

Theorem 4.1. *The genus of an alternating knot is exactly one half of the degree of its Alexander polynomial.*

This will be proved in § 7. Hence follows in the same way as in § 8 [3]

Theorem 4.2. *The genus of the product knot*[4)] *k_0 of the two alternating knots k_1 and k_2 is exactly one half of the degree of its Alexander polynomial.*

Since the Alexander polynomial of k_0 is the product of those of k_1 and k_2, we have

Corollary 4.3.[5)] *The genus of k_0 is equal to the sum of the genera of k_1 and k_2.*

Furthermore, we have

Theorem 4.4. *If k is an alternating knot, then its Alexander polynomial is*

3) If $h=0$, there is no loop of the second kind. This case was considered in [3].

4) k_0 may not be alternating.

5) This fact is already shown by H. Schubert in [5] for all knots.

of the form

$$\Delta(x)=a_0-a_1x+a_2x^2-\cdots+(-1)^t a_t x^t+\cdots+a_{2t}x^{2t},$$

where $a_i\geqq 0$, *and in particular,* a_0, a_t *and* $a_{2t}\neq 0$, *and* $a_i=a_{2t-i}$ *for* $i=0,1,\cdots,2t$.

§ 5. Preparations for the proofs of theorems.

Let Δ_{pq} be the determinant of the matrix obtained by striking out from the L-matrix of K two columns corresponding to two regions with indices p and q. Since $\Delta_{(q+1)q}=\pm x^{r-q}\Delta_{(r+1)r}$, the determinant of the smallest degree with respect to x among the determinants of the forms $\Delta_{(s+1)s}$ is $\Delta_{(p+h+1)(p+h)}$. Hence the Alexander polynomial $\Delta(x)$ of k is

$$\Delta(x)=\pm x^{-\mu}\Delta_{(p+h+1)(p+h)}, \tag{5.1}$$

where μ is a non-negative integer. Now the determinant of the matrix obtained by striking out from the L-matrix two columns corresponding to two adjacent white regions r_α and r_β contained in E_0 and E_1 respectively, may be denoted by $\Delta_{(p+q+1)(p+q)}$, with a suitable q, $0\leqq q\leqq h-1$, and we have

$$\Delta_{(p+h+1)(p+h)}=\pm x^{q-h}\Delta_{(p+q+1)(p+q)}. \tag{5.2}$$

If λ denotes the number of the black regions with all dotted corners, then we have

$$\Delta_{(p+q+1)(p+q)}=x^{\lambda}\Delta^0_{(p+q+1)(p+q)}.\text{[6)]} \tag{5.3}$$

Hence, from (5.1), (5.2) and (5.3), we have

$$\pm x^{h+\mu-\lambda-q}\Delta(x)=\Delta^0_{(p+q+1)(p+q)}. \tag{5.4}$$

Consequently, the proof of the main theorem will be complete if only we prove the following

LEMMA 5.1. *$\Delta^0_{(p+q+1)(p+q)}$ has terms of the degrees $\sum_{i=0}^{m} w_i-m+d-2$ and $d-1$, where w_i denotes the number of the white regions in E_i, and where $d+1$ is the number of the positive domains.*

In fact, it will follow that $h+\mu-q-\lambda\leqq d-1$ and $h+\mu-q-\lambda+2t\geqq\sum_{i=0}^{m} w_i-m+d-2$, where $2t$ is the degree of $\Delta(x)$. Hence $2t\geqq\sum_{i=0}^{m} w_i-m+d-2-(h+\mu-q-\lambda)\geqq\sum_{i=0}^{m} w_i-m+d-2-(d-1)=\sum_{i=0}^{m} w_i-m-1$. On the other hand, we have $2t\leqq 2G(k)\leqq n-(\sum_{i=0}^{m} b_i+m)+1=(\sum_{i=0}^{m} w_i+\sum_{i=0}^{m} b_i-2)-(\sum_{i=0}^{m} b_i+m)+1=\sum_{i=0}^{m} w_i-m-1$, where $G(k)$ denotes the genus of k and b_i denotes the number of the black regions in E_i. Therefore we have $t=G(k)$.

6) See [3].

§ 6. Preparations for the proofs of theorems, continued.

Let us denote the white regions in E_i by $W_{i,1},\cdots,W_{i,h_i}$, and the black regions in E_i by $B_{i,1},\cdots,B_{i,l_i}$. Let $K_i=\bigcup_{\lambda=1}^{h_i} W_{i,\lambda}\bigcup_{\mu=1}^{l_i} B_{i,\mu}$.

DEFINITION 6.1. A crossing point such that at least two of four regions meeting at it are contained in E_i is called a crossing point which is *contained in* K_i (or simply *in* K_i).

Hereafter a side of K_i will mean a segment of K_i connecting two consecutive crossing points *in* K_i. Then K_i may be regarded as an image of the regular projection of a link[7] into S^2, and we have clearly

LEMMA 6.2. *K_i are alternating.*

Since there is no loop of the second kind in K_i, lemmas obtained in [3] hold for K_i with slight modifications. Consequently it follows in the same way as in Lemma 3.6 in [3]

LEMMA 6.3. *The corners of the black regions in E_i are either all dotted or all undotted. And the corners adjacent to the dotted (or undotted) corners of the white regions in E_i are undotted (or dotted). We shall say that the c-corner and c′-corner of a region are adjacent, if two crossing points c and c′ are connected by a side of K_i.*

Let c be a crossing point on C_i not contained in K_i and let a region r_j in E_i be one of the four regions meeting at c. Then it will be easily shown that

LEMMA 6.4. *The c-corner of r_j is either dotted or undotted according as E_i is positive or negative.*

LEMMA 6.5. *Let $\bar{s}$ be the number of the crossing points in K_i, $\bar{p}$ the number of the regions in E_i and let $\dot{E}_i$ consist of the $\bar{q}$ loops of the second kind. Then*

$$\bar{s}=\bar{p}+\bar{q}-2\,.$$

PROOF. The number of the sides of K_i is given by $2\bar{s}$. Since $\bar{s}$ crossing points and $2\bar{s}$ sides divide $\bar{E}_i$[8] into $\bar{s}$ points, $2\bar{s}$ segments and $\bar{p}$ faces, Euler's characteristic χ of $\bar{E}_i$ is given by $\chi=\bar{s}-2\bar{s}+\bar{p}=-\bar{s}+\bar{p}$. On the other hand, $\chi=-\bar{q}+2$, since $\bar{E}_i$ is homeomorphic to a 2-sphere with $\bar{q}$ holes. Thus we have $\bar{s}=\bar{p}+\bar{q}-2$, q. e. d.

LEMMA 6.6. *Let σ be an L^t-correspondence*[9] *such that each crossing point*

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7) A link means a figure composed of a finite number of the disjoint knots in S^3. We can define the standard loops of the first and of the second kind for an image of the regular projection of a link in the same way as for a knot.

8) A bar over the symbol denotes the closure of the set.

9) In the next section, we shall show that there exists such a σ. See [3] for the definition of an L^t-correspondence.

corresponds to one and only one of the $n+2$ *regions except for a pair of two adjacent regions* r_α *and* r_β *contained in* E_0 *and* E_1 *respectively. Then at least one region in* E_i *must correspond by* σ *to a crossing point on* C_i *not contained in* K_i *for* $i=2,\cdots,m$.

PROOF. If E_i is bounded by the outer boundary C_i alone, this lemma is true by Lemma 6.5. Now let us suppose that E_i is bounded by $l+1$ loops $C_{i_1},\cdots,C_{i_l}$ and C_i, and the lemma is true for domains $E_{i_1},\cdots,E_{i_{l-1}}$ and E_{i_l}. That is, let us suppose that $t_{i_h}(\geqq 1)$ regions in E_{i_h} correspond to crossing points not contained in K_{i_h}. Since the number of the crossing points in K_i is larger than the number of the regions in E_i by $l-1$, $\sum_{h=1}^{l} t_{i_h}-l+1$ $(\geqq 1)$ regions in E_i must correspond to the crossing points not contained in K_i. Thus at least one region in E_i must correspond to a crossing point on C_i not contained in K_i, q. e. d.

In the special case where $\bar{t}=\sum_{i=0}^{m} w_i-m+d-2$, it follows

LEMMA 6.7. w_i+b_i-1 *regions in* E_i *correspond to the crossing points in* K_i *for* $i=2,\cdots,m$.

PROOF. Let us suppose that t_i (>1) regions in E_i correspond to the crossing points on C_i not contained in K_i. If E_i is negative, t_i (white) regions in E_i correspond to the crossing points at which these regions have undotted corners. On the other hand, if E_i is positive, t_j (white) regions in E_j, which is separated from E_i by C_i, correspond to the crossing points at which these regions have undotted corners. Thus in all cases it is impossible that σ is an $L^{\Sigma w_i-m+d-2}$-correspondence, since at least one white region in every E_i for $i=d+1,\cdots,m$, corresponds to a crossing point on C_i not contained in K_i at which this region has undotted corner, q. e. d.

LEMMA 6.8. *Let* σ *be an* $L^{\bar{t}}$*-correspondence and let* τ *be another* $L^{\bar{t}}$*-correspondence,* $\bar{t}=\sum_{i=0}^{m} w_i-m+d-2$, *such that the following property* (P) *holds:*

(P) σ *and* τ *are defined on the same set of regions, and each of* σ, τ *assigns each region of this set to some crossing point, the correspondence between the regions and crossing points defined by* σ *and* τ *being allowed to be entirely different.*

Then denoting the terms in $\Delta^0_{(p+q+1)(p+q)}$ *corresponding to* σ *and* τ *by* $\varepsilon x^{\bar{t}}$ *and* $\bar{\varepsilon} x^{\bar{t}}$ *respectively, it follows*

$$\varepsilon=\bar{\varepsilon}.$$

PROOF. Let L_h be the closed and oriented L-chain corresponding to a cyclic permutation ζ_h as used in the proof of Lemma 4.2 in [3]. To show $\operatorname{sgn}\zeta_h=1$, it is sufficient to show that the number of the centers of regions on L_h is odd.

First we shall show that if L_h crosses over the outer boundary of a domain, then it will cross over the boundary in just two places. In fact, let us suppose that L_h crosses over C_i at least at four crossing points. If L_h goes over C_i into E_i through some two crossing points, we see from Lemma 6.6 that these crossing points are not contained in K_i and these correspond to some two regions in E_i, which contradicts Lemma 6.7. Moreover it follows from the above fact that L_h does not cross over C_1.

Next we shall show the following

LEMMA 6.9. *Let T_h be any L-chain and $T_h \cap E_j = T^1 \cup \cdots \cup T^p$ and*

$$T^i = c_{i,1}x_{i,1}c_{i,2} \cup c_{i,2}x_{i,2}c_{i,3} \cup \cdots \cup c_{i,\lambda_i}x_{i,\lambda_i}c_{i,\lambda_i+1}{}^{10)} \qquad \text{for } i=1,\cdots,p,$$

where $x_{i,1},\cdots,x_{i,\lambda_i}$ are the centers of the regions in T^i and $c_{i,1},\cdots,c_{i,\lambda_i+1}$ are the crossing points in T^i. Let t_i denote the number of the centers of the regions in T^i.

(a) *If all c_{i_μ} are contained in K_j, then it follows*

$$\sum t_i \equiv p+1 \qquad (\text{mod } 2).$$

(b) *If c_{11} and c_{p,λ_p+1} are not contained in K_j and others are all contained in K_j, then it follows*

$$\sum t_i \equiv p \qquad (\text{mod } 2).$$

(c) *If $x_{i,1},\cdots,x_{i,\lambda_i}$ are all the centers of the black regions for some i, then t_i is odd or even according as the $c_{i,1}$-corner of $r_{i,1}$ and the c_{i,λ_i+1}-corner of r_{i,λ_i} are either all dotted (or undotted) or not, where r_h denotes the black region in E_j with the center x_h.*

PROOF of (a). In the same way as in Lemma 4.2 in [3], we have $\sum t_i + p \equiv 1$ (mod 2), which is equivalent to (a).

PROOF of (b). Let us transform T^i into T^i_0 as constructed in the proof of Lemma 4.2 in [3]. Here, in particular, we transform $c_{11}x_{11}c_{12}$ and $c_{p,\lambda_p}x_{p,\lambda_p}c_{p,\lambda_p+1}$ into the chains $c'_{11}y_{11}c'_{12} \cup c'_{12}y_{12}c'_{13} \cup \cdots \cup c'_{1\mu}y_{1\mu}c_{12}$ and $c_{p,\lambda_p}z_{p1}c''_{p1} \cup c''_{p1}z_{p2}c''_{p2} \cup \cdots \cup c''_{p,\nu-1}z_p c''_{p\nu}$, respectively, where $c'_{1\xi}$ and $c''_{p\eta}$ are crossing points on the boundaries of the white regions r_{11} and r_{p,λ_p} respectively and c'_{11} and $c''_{p\nu}$ are contained in K_j and lie on C_h, and $y_{1\xi}$ and $z_{p\eta}$ are the centers of the black regions whose boundaries have the sides $c'_{1\xi}c'_{1,\xi+1}$ and $c''_{p,\eta-1}c''_{p,\eta}$ with r_{11} and r_{p,λ_p} in common, respectively, for $\xi=1,2,\cdots,\mu$, $\eta=1,2,\cdots,\nu$ and $c'_{1,\mu+1}=c_{12}$, $c''_{p,0}=c_{p,\lambda_p}$. Let $\bar{w}_1$ be the number of the white regions and $\bar{b}_1$ the number of the black regions, which are contained in a domain D in E_j bounded by $T^1,\cdots,T^p$ and the parts $C^0, C^1,\cdots,C^p$ of $C_j, C_{j_1},\cdots,C_{j_p}$, which are contained in $\dot{E}_j$. Let $\bar{s}_1$ be the number of the crossing points in $D \cap K_j$. Similarly let $\bar{w}_0$ and $\bar{b}_0$ be the numbers of the white and the black regions in D_0 respectively,

10) For the notation see the proof of Lemma 4.2 in [3].

which is bounded by $T_0^1, \cdots, T_0^p, C_0^0, C^1, \cdots, C^p$, where C_0^0 is the curve connecting c'_{11} with $c''_{p\nu}$ on C^0 or on the complement of C^0 with respect to C_i, and $\bar{s}_0$ be the number of the crossing points in $D_0 \cap K_j$. Then denoting the number of the centers of the white regions on T^i by u_i, we have $\bar{w}_0 = \bar{w}_1 + \sum_{i=1}^{p} u_i$. Let $\bar{b}_0 = \bar{b}_1 + \bar{k}$. Then, since $\bar{s}_1 = \bar{b}_1 + \bar{w}_1$ by the definition, it follows $\bar{s}_0 = \bar{b}_0 + \bar{w}_0 - 1 = \bar{s}_1 + \sum u_i + \bar{k} - 1$.[11] Moreover since one of μ and ν is odd and the other even, we can write $\mu + \nu - 2 = 2\gamma - 1$. Hence denoting the number of the centers of the regions in $\bigcup_{i=1}^{p} T_0{}^i$ by t_0, we have[12]

$$\begin{aligned} t_0 &= \sum_{i=1}^{p} t_i + \sum_{i=1}^{u_1-1} (2\lambda_{i1}-1) + \sum_{j=2}^{p-1} \sum_{i=1}^{u_j} (2\lambda_{ij}-1) + \sum_{i=1}^{u_p-1} (2\lambda_{ip}-1) + 2\gamma - 1 - (\bar{s}_0 - \bar{s}_1 + \bar{k}) \\ &\equiv \sum_{i=1}^{p} t_i - (u_1-1) - \sum_{j=2}^{p-1} u_j - (u_p-1) - 1 - (\sum_{i=1}^{p} u_i + 2\bar{k} - 1) \qquad \pmod 2 \\ &\equiv \sum_{i=1}^{p} t_i \qquad \pmod 2 \qquad (\lambda_{ij} \text{ intgers}). \end{aligned}$$

On the other hand, since $t_0 \equiv p \pmod 2$, we have $\sum_{i=1}^{p} t_i \equiv p \pmod 2$.

PROOF of (c). If the c-corner of the black region r_i is dotted, then the c-corner of the black region r_j which is opposite to r_i over c is undotted and conversely. From this, (c) is immediately proved.

Thus Lemma 6.9 is proved.

Now we shall prove Lemma 6.8.

Let L_h be divided into $L_h = \bigcup_{i=1}^{p_1} L_i^{(0)} \cup L^{(1)}$, where all $L_i^{(0)}$ are connected and contained in only one domain $\bar{E}_{h_1}$, and $\bigcup_{i=1}^{p_1} L_i^{(0)} \cap C_h = \phi$[13] and $L^{(1)} = L_h - \bigcup_{i=1}^{p_1} L_i^{(0)}$. Now denoting the number of the centers of regions in $L_j^{(i)}$ by $t_j^{(i)}$, we have, by Lemma 6.9 (a),

$$\sum_{i=1}^{p_1} t_i^{(0)} \equiv p_1 + 1 \qquad \pmod 2.$$

Next consider $L^{(1)}$. $L^{(1)}$ consists of p_1 L-chains $L_1^{(1)}, \cdots, L_{p_1}^{(1)}$, whose end points are on the outer boundaries $C_{l,1}, \cdots, C_{l,p_1}$ and are not contained in $K_{l,1}, \cdots, K_{l,p_1}$, respectively. Let $L_1^{(1)}$ be divided into $L_1^{(1)} = \bigcup_{i=1}^{p_{11}} L_i^{(11)} \cup L^{(110)}$, where all $L_i^{(11)}$ are contained in a domain $\bar{E}_{l,i}$ and $L^{(110)} = L_1^{(1)} - \bigcup_{i=1}^{p_{11}} L_i^{(11)}$. Then by Lemma 6.9 (b), we have

$$\sum t_i^{(11)} \equiv p_{11} \qquad \pmod 2.$$

11) See (4.3) in [3].

12) See (4.1) in [3].

13) ϕ denotes the empty set.

Defining $t_l^{(ij)}$ and p_{1j} in the same way as above, we have

$$\sum_l \sum_{j=1}^{p_1} t_l^{(1j)} \equiv \sum_{j=1}^{p_1} p_{1j} \qquad (\mathrm{mod}\ 2).$$

Moreover dividing $L^{(110)}$ into some L-chains and computing $t_j^{(11h)}$ and p_{11h} in the same way as above, we have

$$\sum_j \sum_h t_j^{(11h)} \equiv \sum_h p_{11h} \qquad (\mathrm{mod}\ 2).$$

Since the above decomposition will finish after a finite number of steps, the number t of the centers of the regions in L_h will finally be given by

$$t = \sum t_i^{(0)} + \sum_{j,l} t_l^{(1j)} + \sum t_j^{(1**)} + \cdots + \sum t_j^{(1**\cdots*)}$$

$$\equiv p_1 + 1 + \sum p_{1j} + \sum p_{1**} + \cdots + \sum p_{1*\cdots*}.$$

On the other hand, $p_1 + \sum p_{1j} + \cdots + \sum p_{1*\cdots*}$ is even by Lemma 6.9 (c). Hence we have $t \equiv 1$ (mod 2). Thus Lemma 6.8 is proved.

§ 7. Proof of Theorem 4.1.

In this section, we shall show that there exists an $L^{\bar{t}}$-correspondence, where $\bar{t} = \sum_{i=0}^{m} w_i - m + d - 2$.

Let G_j be the graph[14] of K_j. Denote the regions into which G_j divides S^2 by M_{ji}. Then, if we regard the complement of E_j as the black regions, then we see clearly that each M_{ji} contains one and only one black region. We can suppose that the indices i, j are so arranged that M_{j1} contains C_j for $j = 1, \cdots, m$, and M_{01} contains C_1, and $(\bigcup_{i=1}^{\lambda-1} \dot{M}_{ji}) \cap \dot{M}_{j\lambda}$ must contain at least one side of $\dot{M}_{j\lambda}$.

Let r_α and r_β be a pair of two adjacent white regions in E_0 and E_1 respectively. Then we can assign each one of the $w_0 + w_1$ white regions in E_0 and E_1 except for r_α and r_β to one and only one crossing point lying on its boundary by means of the graphs G_0 and G_1 in the same way as in [3], where the corner of the region at the corresponding crossing point is dotted. Let P_0 and P_1 denote the semi-graph of G_0 and G_1 with respect to the correspondences of the white regions in E_0 and E_1 respectively. Then P_0 and P_1 are disjoint and these are trees. Now let $\dot{E}_1 = C_1 \cup C_{i_1} \cup \cdots \cup C_{ij_1}$. Then we have

Lemma 7.1. *In each E_{i_λ}, there exists a region r_{i_λ}, say, whose center is on a*

14) The graph (or the dual graph) of K means the totality of the segments connecting the centers of the white (or the black) regions with the crossing points lying on their boundaries.

side m_{i_λ} in $M_{i_\lambda,0}$, and each $\dot{r}_{i_\lambda}$ contains at least one crossing point c_{i_λ} which is not contained in K_{i_λ}.

PROOF. If there does not exist such a region in E_{i_μ}, then P_1 would contain the boundary of $M_{1\lambda}$, in which E_{i_μ} would be contained.

Furthermore we have

LEMMA 7.2. *We can so choose these crossing points c_{i_λ} that they are different from each other.*

PROOF. If $c_{i_\mu}=c_{i_\nu}$ for some μ, ν, i.e. if there is only one crossing point which is not contained in K_{i_μ} and K_{i_ν}, there would be $M_{1\xi}$ and $M_{1\eta}$, in which E_{i_μ} and E_{i_ν} would be contained, and P_1 would contain a loop $\dot{M}_{1\xi}\cup\dot{M}_{1\eta}-(\dot{M}_{1\xi}\cap\dot{M}_{1\eta})$.

Now we can assign each one $r_{i_\lambda,j}$ of the w_{i_λ} white regions in E_{i_λ} except for the regions r_{i_λ}, whose existence is assured in Lemma 7.1, to only one crossing point contained in K_{i_λ} which lies on $\dot{r}_{i_\lambda,j}$ by means of the graphs G_{i_λ}, where the corners of the regions at the corresponding crossing points are dotted. Let P_{i_λ} denote the semi-graph of G_{i_λ} with respect to the correspondence of the white regions in E_{i_λ}. Then P_{i_λ} are the trees and these are mutually disjoint. In the same way, we obtain

LEMMA 7.3. *In each E_i, there is one white region r_i, say, whose center is on a side of M_{i0} and there exists on $\dot{r}_i$ at least one crossing point c_i, say, not contained in K_i. And these crossing points are different from each other.*

Let P_i be the semi-graph of G_i with respect to the correspondence of all the white regions except for r_i in E_i. P_i are mutually disjoint.

Now we shall prove the existence of an L^l-correspondence. This will be performed if we can assign each one of the $m-1$ white regions r_i and the $\sum_{i=0}^{m} b_i$ black regions to one and only one crossing point. To do this, we shall first assign r_i (in E_i) to a crossing point c_i obtained by Lemma 7.3. Next, to obtain a correspondence between the black regions in each E_i and the crossing points, we shall apply the proof of Lemma 5.3 in [3] to our case. We regard the region r_i and the connected component, which contains E_0, in the complement of E_i as r_α and r_β respectively and we consider the subset Q_i, disjoint to P_i, of the dual graph H_i of K_i. Then we can assign also black regions to the crossing points on its boundaries by means of Q_i. Thus we obtain the required correspondence. Thus we have

LEMMA 7.4. *There is an $L^{\Sigma w_i-m+d-2}$-correspondence σ as stated in Lemma 6.7.*

Similarly, it follows

LEMMA 7.5. *There is an L^{d-1}-correspondence.*

From Lemmas 7.4 and 7.5, we have Lemma 5.7. Thus the proof of Theorem 4.1 is completed.

§ 8. Proof of Theorem 4.4.

We can slightly extend Lemma 6.8 as follows.

LEMMA 8.1. *Let σ be an $L^{\bar{t}}$-correspondence and τ an $L^{\bar{s}}$-correspondence, $d-1\leqq\bar{t}$, $\bar{s}\leqq\sum_{i=0}^{m}w_i-m+d-2$, which have the property* (P) *as stated in Lemma* 6.8. *If the terms in $\Delta^0_{(p+q+1)(p+q)}$ corresponding to σ and τ are denoted by $\varepsilon x^{\bar{t}}$ and $\bar{\varepsilon}x^{\bar{s}}$, where $\varepsilon, \bar{\varepsilon}=\pm 1$, then $\varepsilon=\bar{\varepsilon}$ or $\varepsilon=-\bar{\varepsilon}$ according as $\bar{t}\equiv\bar{s}$* (mod 2) *or not.*

PROOF. We can assume without loss of generality that $\bar{t}=\sum_{i=0}^{m}w_i-m+d-2$. First we shall prove this lemma in the case where $m=0$ and $d=1$, i. e. $\bar{t}=w_0-1$. We may suppose that n crossing points $c_1, c_2, \cdots, c_n$ correspond to n regions $r_1, r_2, \cdots, r_n$ respectively, of which first w_0-1 regions are white, by σ. Let c_{ji} correspond to r_i by τ for $i=1,\cdots,n$ and let us assume that c_{jh}-corner of r_h are dotted for $h=1,\cdots,\bar{s}$ and c_{jl}-corner of r_l are undotted for $l=\bar{s}+1,\cdots,w_0-1$. Then, to prove Lemma 8.1, it is sufficient to show that

$$\operatorname{sgn}\zeta=\operatorname{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}. \tag{8.1}$$

Let ζ be represented as the product of some cyclic permutations $\zeta_1, \zeta_2, \cdots, \zeta_r$, which are mutually disjoint.

Let $\zeta_1=(y_1\cdots y_h)$, $1\leqq y_1,\cdots,y_h\leqq n$. Consider an oriented L-chain, L corresponding to ζ_1. Let us assume that L_1 contains t_1 centers of white regions, of which α_1 centers lie on the segments of L_1 oriented as proceeding from the dotted corner to the undotted corner. Then we shall transform L_1 into L_0 which does not contain the centers of white regions, in the same way as in the proof of Lemma 4.2 in [3]. Let p_1 be the number of the white regions, q_1 the number of the black regions and let s_1 the number of crossing points, which are contained in the interior[15] of L_1. Then we have $s_1=p_1+q_1$. On the other hand, the number of the white regions contained in the interior $\widetilde{L}_0$ of L_0 is given by p_1+t_1. Denoting the number of the black regions contained in $\widetilde{L}_0$ by $q_1+\bar{w}_1$, the number of the crossing points contained in $\widetilde{L}_0$ is given by $s_0=q_1+p_1+t_1+\bar{w}_1-1=s_1+\bar{w}_1+t_1-1$. If the number of the centers of the regions lying on L_0 is denoted by h_0, then it follows

$$h_0=h+\sum_{i=1}^{t_1-\alpha_1}(2\lambda_i-1)+\sum_{j=1}^{\alpha_1}2(\mu_j-1)-(s_0-s_1+\bar{w}_1)$$

15) The interior of L_1 means the parts in which L_0 is not contained, between two parts into which S^2 are divided by L_1.

$$=h+2\sum\lambda_i-(t_1-\alpha_1)+2\sum(\mu_j-1)-(2\bar{w}_1+t_1-1)$$
$$\equiv h+\alpha_1+1 \qquad (\text{mod } 2) \quad (\lambda_i, \mu_j \text{ being positive integers}).$$

Thus we have $h\equiv\alpha_1+1$, since $h_0\equiv 0$ (mod 2). Hence we have $\operatorname{sgn}\zeta_1=(-1)^{\alpha_1}$. In the same way, we have $\operatorname{sgn}\zeta_i=(-1)^{\alpha_i}$, where α_i are defined in the same way as α_1. Since $\sum\alpha_i=w_0-1-\bar{s}$, it follows $\operatorname{sgn}\zeta=\prod_{i=1}^{r}\operatorname{sgn}\zeta_i=\prod_{i=1}^{r}(-1)^{\alpha_i}=(-1)^{w_0-1-\bar{s}}$.

To prove this lemma in this case where $m>0$, we may compute the numbers of the centers on the chains, into which L_h is divided, in the same way as in the proof of Lemma 6.8. Since we can accomplish this computation in the same way as above, we shall omit the detail.

From this lemma and the fact that $\Delta(-1)$ is always odd, Theorem 4.4 is easily proved.

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