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# Class formations IV.

## (Infinite extension of the ground field)

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Let $K_0^*$ be an infinite algebraic extension of the rational number field (or of the $p$-adic number field). We may expect that a kind of analogy of class field theory would hold for finite abelian extensions over $K_0^*$. In 1936-37 M. Moriya has established in his papers [7], [8] such a theory over $K_0^*$ for those finite abelian extensions $K^*/K_0^*$ for which the degrees $[K^*:K_0^*]$ are relatively prime to each one of a certain set of prime numbers determined by $K_0^*$. Recently M. Mori [9] considered the same problem (in local case) from a different point of view and obtained similar results as in the theory of Moriya without any restriction on the degrees of abelian extensions $K^*/K_0^*$.

The purpose of the present paper is to consider an analogous problem in the framework of class formation theory. Let $\{A(K); K\in\mathfrak{K}\}$ be a given class formation over a ground field $k_0$ where $\mathfrak{K}$ is the set of all finite extensions of $k_0$ contained in a fixed infinite normal algebraic extension $\Omega/k_0$ and $A(K)$ is the abelian group attached to $K$ $(K\in\mathfrak{K})$. Furthermore, we assume that every $A(K)$ is a compact topological group. Now let $K_0^*$ be an arbitrary infinite extension of $k_0$ contained in $\Omega$. Then let us take $\mathfrak{K}^*=\{K^*; K_0^*\subset K^*\subset\Omega, [K^*:K_0^*]<\infty\}$ and let $A^*(K^*)$ be the inverse limit group of $\{A(k_\lambda); k_\lambda\subset K\}$. Our main result is that $\{A^*(K^*); K^*\in\mathfrak{K}^*\}$ is a class formation over the ground field $K_0^*$ (Theorem 1). We can apply this result also to the cases of local and global class field theory by taking a suitable class formation in each case (Theorem 5, 7). In particular, our results coincide with that of M. Mori in local case. In a previous paper [4] the author considered the same problem in a class formation after the method of M. Moriya. In § 6 we shall consider the relation between our present method and that used before in [4].

**1.** Let $k_0$ be a given ground field, $\Omega$ be a fixed infinite normal algebraic extension of $k_0$ and $\mathfrak{K}=\{K;\, k_0\subset K\subset\Omega,\, [K:k_0]<\infty\}$. Let us assume that to each $K\in\mathfrak{K}$ an additive group $A(K)$ is attached so that $\{A(K);\, K\in\mathfrak{K}\}$ satisfies the axioms (F1–4, C 1,2) of a class formation (see [4], §1 or [5] §1,1). We put a further axiom:

**T.** *Each* $A(K)$ *is a compact topological group such that* (i) *each* $\varphi_{K,k}: A(k)\to A(K)$ $(k\subset K;\, k, K\in\mathfrak{K})$ *is continuous and* (ii) *for each normal extension* $K/k$ $(k, K\in\mathfrak{K})$ $\sigma\in G=G(K/k)$ *operates continuously on* $A(K)$.

Then it is easy to see that for each $k\subset l$ $(k, l\in\mathfrak{K})$ $\mathrm{N}_{l/k}: A(l)\to A(k)$ is a continuous homomorphism. (For normal $K/k$, $\mathrm{N}_{K/k}a$ is defined by $\varphi_{K,k}{}^{-1}(\sum_{\sigma\in G}\sigma a)$).

Now let us take an arbitrary infinite extension $K_0^*$ of $k_0$ contained in $\Omega$ as the ground field and put

$$\mathfrak{K}^*=\{K^*;\, K_0^*\subset K^*\subset\Omega,\, [K^*:K_0^*]<+\infty\}\,. \tag{1}$$

Each $K^*\in\mathfrak{K}^*$ is the union of $\{K_\lambda;\, K_\lambda\in\mathfrak{K},\, K_\lambda\subset K^*\}: K^*=\bigcup_\lambda K_\lambda$. To each pair $K_\lambda\subset K_\mu$ the norm-mapping

$$\mathrm{N}_{\mu,\lambda}=\mathrm{N}_{K_\mu/K_\lambda}:\ A(K_\mu)\to A(K_\lambda)$$

is a continuous homomorphism. Hence we can define the inverse limit group of $\{A(K_\lambda),\, \mathrm{N}_{\mu,\lambda}\}$, where we define the partial order $\lambda<\mu$ by $K_\lambda\subset K_\mu$. Then we put

$$A^*(K^*)=\text{inv-lim}_\lambda\, A(K_\lambda)\,. \tag{2}$$

We denote the projection by $\mathrm{N}_\lambda: A^*(K^*)\to A(K_\lambda)$ so that $\mathrm{N}_{\mu,\lambda}\cdot\mathrm{N}_\mu=\mathrm{N}_\lambda$ for $\lambda<\mu$ holds.

THEOREM 1. *Let* $\{A(K);\, K\in\mathfrak{K}\}$ *be a class formation over the ground field* $k_0$ *which satisfies the axiom* T. *Let* $K_0^*$ *be an arbitrary infinite extension of* $k_0$ *contained in* $\Omega$. *Let us define* $\mathfrak{K}^*$ *and* $A^*(K^*)$ $(K^*\in\mathfrak{K}^*)$ *by* (1) *and* (2). *Then* $\{A^*(K^*);\, K^*\in\mathfrak{K}^*\}$ *is a class formation over the ground field* $K_0^*$ *with the axiom* T.

(PROOF) (i) F 1. Let $K^*\subset L^*$ $(K^*, L^*\in\mathfrak{K}^*)$ and $L^*=K^*(\theta_1,\cdots,\theta_r)$. Each $\theta_i$ satisfies an irreducible equation $f_i(X)=0$ $(i=1,\cdots,r)$ so that all the coefficients of $f_i$ $(i=1,\cdots,r)$ belong to some finite extension $K_{\lambda_0}$ of $k_0$. We denote then $\lambda_0=\lambda_0(L^*/K^*)$. Let $K^*=\bigcup_\lambda K_\lambda$ $(K_\lambda\in\mathfrak{K})$ and let us put $L_\lambda=K_\lambda(\theta_1,\cdots,\theta_r)$. Then we have $L^*=\bigcup_\lambda L_\lambda$ and $[L_\lambda: K_\lambda]=[L:K]$ for $\lambda>\lambda_0$. Since $\varphi_{L_\lambda,K_\lambda}\circ\mathrm{N}_{K_\mu/K_\lambda}=\mathrm{N}_{L_\mu/L_\lambda}\circ\varphi_{L_\mu,K_\mu}$ holds for $\lambda_0<\lambda<\mu$, we can define $\varphi^*_{L^*,K^*}: A^*(K^*)\to A^*(L^*)$ by the limit of $\varphi_{L_\lambda,K_\lambda}$. Since each $\varphi_{L_\lambda,K_\lambda}$ is a monomorphism, the limit $\varphi^*_{L^*,K^*}$ is also a monomorphism. F 2. Clearly the transitivity relation $\varphi^{*}_{M^*,K^*}=\varphi^{*}_{M^*,L^*}\circ\varphi^{*}_{L^*,K^*}$ for $K^*\subset L^*\subset M^*$ is satisfied. F 3. Suppose that $L^*/K^*$ is normal with the Galois group $G=G(L^*/K^*)$. Then we may take $\lambda_0=\lambda_0(L^*/K^*)$ such that $L_\lambda/K_\lambda$ is normal for any $\lambda>\lambda_0$ and the Galois group $G(L_\lambda/K_\lambda)$ is canonically isomorphic to $G$. We have $A^*(L^*)^G=\text{inv-lim } A(L_\lambda)^G=\varphi^{*}_{L^*,K^*}A^*(K^*)$. F 4. can be proved similarly.

(ii) We shall prove C 1:

$$H^1(G, A^*(L^*))=0 \tag{3}$$

for normal extension $L^*/K^*$ with the Galois group $G=G(L^*/K^*)$. Let us consider an arbitrary 1-cocycle $f^*(\sigma)\in A(L^*)$ $(\sigma\in G)$ of $G$ over $A(L^*)$, so that $\sigma f^*(\tau)-f^*(\sigma\tau)+f^*(\sigma)=0$ holds. We define $f_\lambda(\sigma)=\mathrm{N}_\lambda f^*(\sigma)$ which is a 1-cocycle of $G$ over $L_\lambda$ $(\lambda>\lambda_0)$. Since $H^1(G, A(L_\lambda))=0$ we can find $a_\lambda\in A(L_\lambda)$ such that $f_\lambda(\sigma)=(\sigma-1)a_\lambda$ $(\sigma\in G)$ holds. We denote the set of all such $a_\lambda$'s by $B_\lambda$. Clearly $B_\lambda$ is a closed set in $A(L_\lambda)$ and satisfies $\mathrm{N}_{\mu,\lambda}B_\mu\subset B_\lambda$ (for $\lambda<\mu$). Hence $\text{inv-lim } B_\mu=B^*$ is not empty (see e. g. Lefschetz [6], p. 32, Theorem (39.1)) and any element $a^*$ in $B^*$ satisfies $f^*(\sigma)=(\sigma-1)a^*$ $(\sigma\in G)$. This fact proves (3).

To prove C 2. we need the following Lemma:

LEMMA. *Let $\{A(K)\,;\,K\in\mathfrak{K}\}$ be a class formation. Let $k, l, K\in\mathfrak{K}$ be such that $K/k$ is normal with the Galois group $G$, $k\subset l$ and $l\cap K=k$. Let $L=Kl$ be the compositum. We denote by $\xi_{K/k}$ the fundamental 2-cohomology class of $H^2(G, A(K))$ and similarly for $\xi_{L/l}$. Then the norm-mapping $\mathrm{N}_{L/K}: H^2(G, A(L))\to H^2(G, A(K))$ is an isomorphism. Moreover, if $G$ is cyclic, we have*

$$\mathrm{N}_{L/K}\,\xi_{L/l}=\xi_{K/k}\,{}^{1)} \tag{4}$$

(PROOF) We shall prove (4) for cyclic group $G$. In a class formation the norm-residue symbol $(a, L/l)\in G$ is defined for $a\in A(l)$, and we have the formula

$$(a, L/l)=(\mathrm{N}_{l/k}a, K/k)\,. \tag{5}$$

Let $f_{L/l}(\sigma, \tau)$ be a 2-cocycle from the class $\xi_{L/l}$ then $\sigma=(a, L/l)$ is characterized by $a+\mathrm{N}_{L/l}A(L)=f_{L/l}(G, \sigma)+\mathrm{N}_{L/l}A(L)$ (where $f(G, \sigma)$ means $\sum_{\tau\in G}f(\tau, \sigma)$). Similarly $\sigma=(b, K/k)$ is characterized by $b+\mathrm{N}_{K/k}A(K)=f_{K/k}(G, \sigma)+\mathrm{N}_{K/k}A(K)$. Now let $G$ be cyclic and $\mathrm{N}_{L/K}\,\xi_{L/l}=m\xi_{K/k}$. This implies $\mathrm{N}_{L/K}f_{L/l}(G, \sigma)+\mathrm{N}_{K/k}A(K)=mf_{K/k}(G, \sigma)+\mathrm{N}_{K/k}A(K)$. Let us choose

1) Added in Proof. The formula (4) holds for any group $G$. See, Algebraic Theory of Numbers (in Japanese), Kyoritu-Syuppan, (1957), Appendix.

a generator of $G$ as $\sigma$, then we have $m\sigma=\sigma$ from (5) and hence $m=1$. This proves the relation (4).

For general $G$ let $\mathrm{N}_{L/K}\,\xi_{L/l}=m\xi_{K/k}$. Since $H^2(G, A(L))\cong H^2(G, A(K))\cong Z/nZ$ ($n=[G:1]$) it suffices to prove $(m, n)=1$. Suppose that there exists a prime number $p$ such that $p$ divides $(m, n)$. Then take a subgroup $H$ of $G$ of order $p$. Let $L'$ and $K'$ be the subfields of $L$ and $K$ respectively which correspond to $H$ in the sense of the Galois theory. Then $L=KL'$ and $K'=K\cap L'$ hold. Since $\mathrm{Res}_{G/H}\,\xi_{L/l}=\xi_{L/L'}$ and $\mathrm{Res}_{G/H}\,\xi_{K/k}=\xi_{K/K'}$ we would have $\mathrm{N}_{L/K}\,\xi_{L/L'}=\mathrm{N}_{L/K}\circ\mathrm{Res}_{G/H}\,\xi_{L/l}=\mathrm{Res}_{G/H}\circ\mathrm{N}_{L/K}\,\xi_{L/l}=m\xi_{K/K'}=0$ by the condition $p|m$. This is a contradiction to (4). Hence we have $(m, n)=1$, q. e. d.

Now let $f^*(\sigma, \tau)$ be a standard 2-cocycle of $G$ over $A^*(L^*)$: $\rho f^*(\sigma, \tau)-f^*(\rho\sigma, \tau)+f^*(\rho, \sigma\tau)-f^*(\rho, \sigma)=0$. We denote $\mathrm{N}_\lambda f^*(\sigma, \tau)=f_\lambda(\sigma, \tau)\in A(L_\lambda)$ which is a 2-cocycle of $G$ over $A(L_\lambda)$ $(\lambda>\lambda_0)$.

($\alpha$) We shall prove first that if $f_\lambda\sim 0$ in $H^2(G, A(L_\lambda))$ holds for some $\lambda$ $(>\lambda_0)$ then $f^*\sim 0$ holds in $H^2(G, A^*(L^*))$. This can be done quite similarly as in the case of $H^1(G^*, A^*(L^*))$. Namely, from Lemma follows that $f_\mu\sim 0$ for all $\mu$ $(>\lambda_0)$. Let $B_\mu$ be the subset of $A(L_\mu)\times\cdots\times A(L_\mu)$ ($n$-times) consisting of $g_\mu(\sigma)\in A(L_\mu)$ $(\sigma\in G)$; $f_\mu(\sigma, \tau)=\sigma g_\mu(\tau)-g_\mu(\sigma\tau)+g_\mu(\sigma)$. Then $\mathrm{N}_{\mu,\lambda}B_\mu\subset B_\lambda$ $(\mu>\lambda)$ holds. Since each $B_\mu$ is a closed subset and hence compact, the inverse limit of $\{B_\mu\}$ is not empty. Take $\{g^*(\sigma)\}$ arbitrarily in this limit set, then we have $f^*(\sigma, \tau)=\sigma g^*(\tau)-g^*(\sigma\tau)+g^*(\sigma)$ $(\sigma, \tau\in G)$, i. e. $f^*\sim 0$.

($\beta$) Let us fix $\lambda$ $(>\lambda_0)$ arbitrarily and consider the homomorphism $\mathrm{N}_\lambda: H^2(G, A^*(L^*))\to H^2(G, A(L_\lambda))$. Firstly, $\mathrm{N}_\lambda$ is a monomorphism by ($\alpha$). Secondly, let $\alpha_\lambda\in H^2(G, A(L_\lambda))$, then from Lemma follows that there exists $\alpha_\mu\in H^2(G, A(L_\mu))$ $(\lambda<\mu)$ such that $\mathrm{N}_{\mu,\lambda}\alpha_\mu=\alpha_\lambda$ holds. Let $B_\mu$ be the subset of $A(L_\mu)\times\cdots\times A(L_\mu)$ ($n^2$-times) consisting of standard 2-cocycles $\{f_\mu(\sigma, \tau)\}$ of $G$ over $A(L_\mu)$ which belong to $\alpha_\mu$. Then we have $\mathrm{N}_{\mu,\lambda}B_\mu\subset B_\lambda$ and we can prove the existence of a class $\alpha^*$ in $H^2(G, A^*(L^*))$ such that $\mathrm{N}_\lambda\alpha^*=\alpha_\lambda$ holds. This proves that $\mathrm{N}_\lambda$ is an epimorphism. Therefore, $\mathrm{N}_\lambda$ gives the isomorphism $H^2(G, A^*(L^*))\cong H^2(G, A(L_\lambda))\cong Z/nZ$, which proves C 2. for $\{A^*(L^*);\ L^*\in\mathfrak{K}^*\}$.

(iii) T. Since each $A(K)$ is compact the inverse limit group $A^*(L^*)$ is also compact. The continuity of $\varphi^*_{L^*,K^*}$ and of the automorphism $\sigma\in G=G(L^*/K^*)$ for normal $L^*/K^*$ follows easily from the general properties on inverse limit groups, q. e. d.

**2.** Let $\{A(K); K\in\mathfrak{K}\}$ be a class formation with axiom T. Let $(a, K)$ $(K\in\mathfrak{K}, a\in A(K))$ be the generalized norm-residue symbol (see [4], (15)) which takes value in the Galois group $\Gamma(K)$ of the maximal abelian extension $A_K$ of $K$ (in $\Omega$). We denote by $\mathfrak{J}(K)$ $(\subset\Gamma(K))$ the image, and by $\mathfrak{K}(K)$ $(\subset A(K))$ the kernel of the generalized norm-residue mapping:

$$\Psi_K:\ a\,(\in A(K))\rightarrow(a, K)\ \ (\in\Gamma(K))\,. \tag{6}$$

THEOREM 2. *Let us assume the axiom* T. *Then the generalized norm-residue mapping* $\Psi_K: A(K)\rightarrow\Gamma(K)$ *is continuous. Moreover, we have* $\mathfrak{J}(K)=\Gamma(K)$ *and*

$$A(K)/\mathfrak{K}(K)\cong\Gamma(K) \tag{7}$$

*algebraically and topologically.*

(PROOF) Let $U(L)$ be the subgroup of $\Gamma(K)$ corresonding to a finite abelian extension $L/K$ in the sense of (infinite) Galois theory. Then to prove the continuity of $\Psi_K$ it is sufficient to see that $\Psi_K^{-1}(U(L))$ is open for every $L$ with $K\subset L\subset A_K$, $[L:K]<+\infty$. By definition $\Psi_K^{-1}(U(L))=\mathrm{N}_{L/K}A(L)$ which is a subgroup of $A(K)$ of finite index. Since $\mathrm{N}_{L/K}$ is continuous and $A(L)$ is compact, $\Psi_K^{-1}(U(L))$ is compact and open.

Since $\mathfrak{J}(K)$ is dense in $\Gamma(K)$, $\mathfrak{J}(K)=\Gamma(K)$ follows from the continuity of $\Psi_K$ and the compactness of $A(L)$. Finally (7) follows easily from these properties, q. e. d.

**3.** We come back to the case of the infinite extension of the ground field, and we use the same notations as in § 1. Let $L^*/K^*$ be normal and $K^*=\bigcup_\lambda K_\lambda$, $L^*=\bigcup_\lambda L_\lambda$ such that $L_\lambda/K_\lambda$ $(\lambda>\lambda_0)$ is normal whose Galois group is canonically isomorphic to $G=G(L^*/K^*)$. Since $A^*(K^*)=\text{inv-lim}_\lambda A(K_\lambda)$ by means of the mappings $\mathrm{N}_{\mu,\lambda}: A(K_\mu)\rightarrow A(K_\lambda)$ $(\lambda<\mu)$, $a^*\in A^*(K^*)$ can be represented by $a^*=\text{inv-lim}_\lambda a_\lambda$ $(a_\lambda\in A(K_\lambda))$ such that $\mathrm{N}_{\mu,\lambda}a_\mu=a_\lambda$ holds. By the translation theorem we have $(a_\lambda, L_\lambda/K_\lambda)=(a_\mu, L_\mu/K_\mu)$. Then we define the norm-residue symbol by

$$(a^*, L^*/K^*)=(a_\lambda, L_\lambda/K_\lambda)\quad(\lambda>\lambda_0)\,. \tag{8}$$

THEOREM 3. *In case of Theorem* 1 *we can define the norm-residue symbol* $(a^*, L/K^*)$ *for normal extension* $L^*/K^*$ $(K^*, L^*\in\mathfrak{K}^*, a^*\in A^*(K^*))$ *by* (8) *which takes value in the Galois group* $G$ *of* $L^*/K^*$. *The mapping* $\Psi_{K^*}: a^*\rightarrow(a^*, K^*)$ *induces the* (*algebraic and topological*) *isomorphism*

$$A^*(K^*)/\mathfrak{K}^*(K^*) \cong \Gamma(K^*) \tag{9}$$

*where* $\mathfrak{K}^*(K^*) = \text{inv-lim}_\lambda\, \mathfrak{K}(K_\lambda)$.

The proof is easy and we omit it here.

4. We shall give here two simple applications of Theorem 1.

(I) Kummer extensions. Let $\{A(K)\,;\, K \in \mathfrak{K}\}$ be the class formation for Kummer extensions (see [4], § 3). Namely, we assume K 1: the characteristic of the ground field $k_0$ is 0. K 2: $k_0$ contains all the roots of unity, K 3: for every $K, k$ such that $k, K \in \mathfrak{K}$ and $K/k$ is normal, $\mathrm{N}_{K/k} K = k$ holds. Then we can take $A(K) = (K^\times \otimes (Q/Z))^\wedge$, where $K^\times$ means the multiplicative group of $K$ and $\wedge$ means the compact character group. Let $\psi_{\lambda,\mu}$ for $\lambda < \mu$ be the injection: $K_\lambda^\times \otimes (Q/Z) \to K_\mu^\times \otimes (Q/Z)$, then as in the proof of Theorem 3.2 in [4], we have

$$\mathrm{N}_{\mu,\lambda}\, \chi = \chi \circ \psi_{\lambda,\mu}$$

for $\chi \in A(K_\mu)$. Hence we have

$$A^*(K^*) = \text{inv-lim}_\lambda\, A(K_\lambda) \cong ((K^*)^\times \otimes (Q/Z))^\wedge \tag{10}$$

canonically. On the other hand, it is easy to see that if $\mathfrak{K}$ satisfies the conditions K 1, 2, 3 then $\mathfrak{K}^*$ over the ground field $K_0^*$ (which is an infinite algebraic extension of $k_0$) also satisfies the conditions K 1, 2, 3. Hence the class formation over $K_0^*$ which is constructed by Theorem 1 is nothing new and is only a special case of the general class formation theory for Kummer extensions.

(II) $p$-extensions over a field of characteristic $p$. Let $k_0$ be a field of characteristic $p$ and $\Omega$ be the maximal $p$-extension of $k_0$. Let

$$A(K) = ((V_K/\wp V_K) \otimes (Q^{(p)}/Z))^\wedge$$

where $V_K$ means the additive group of all Witt's vectors over $K$. Then $\{A(K)\,;\, K \in \mathfrak{K}\}$ is a class formation with the Axiom T (see [4], § 5). In case $K_\lambda \subset K_\mu$ we have for $\chi \in A(K_\mu)$

$$\mathrm{N}_{\mu,\nu}(\chi) = \chi \circ \psi_{\lambda,\mu}$$

by the natural injection $\psi_{\lambda,\mu} : (V_{K_\lambda}/\wp V_{K_\lambda}) \otimes (Q^{(p)}/Z) \to (V_{K_\mu}/\wp V_{K_\mu}) \otimes (Q^{(p)}/Z)$. It follows then for an infinite extension $K^* = \bigcup_\lambda K_\lambda$

$$A^*(K^*) = \text{inv-lim}_\lambda\, A(K_\lambda) \cong ((V_{K^*}/\wp V_{K^*}) \otimes (Q^{(p)}/Z))^\wedge .$$

Therefore, the class formation over the ground field $K_0^*$ which is an infinite extension of $k_0$ gives us nothing new other than the general theory of $p$-extensions.

**5.** We shall apply Theorem 1 to local and global class field theory.

(III) Let $k_0$ be a $\mathfrak{p}$-adic number field, $\Omega$ be its algebraic closure and $\mathfrak{K}$ be the set of all finite extensions of $k_0$ contained in $\Omega$. If we put $A(K)=K^\times$ (the multiplicative group of $K$) and $\varphi_{K,k}: k\to K$ as the injection, then by the local class field theory $\{A(K)\,;\,K\in\mathfrak{K}\}$ is a class formation. Here $A(K)$ is locally compact and is not compact with respect to the natural topology.

We shall introduce another topology (see Artin [1], Chapter 9). Namely, let $\mathfrak{U}$ be the set of all (closed) subgroups of finite indices and we take $\mathfrak{U}$ as the fundamental system of neighborhoods of the unity in $K^\times$. This new topology is weaker than the natural one, and $K^\times$ is totally bounded. Let $K^\#$ be the completion of $K^\times$, then $K^\#$ is a compact group. Let $Z^\#=\prod_p Z_p$ be the direct pruduct of all $p$-adic integers which is a compact additive group. The additive group $Z$ of integers can be naturally imbedded in $Z^\#$ so that $Z^\#$ is the completion of $Z$. It is easy to see that $Z^\#/Z$ has the property of unique divisibility by any integer $n$. Now let $U$ be the unit group of $K^\times$, then $K^\times/U\cong Z$ (additive group) and $K^\#/U\cong Z^\#$ (additive group). Hence we have

$$K^\#/K^\times\cong Z^\#/Z \qquad \text{(additive group)}. \tag{11}$$

Let us take for each $K\in\mathfrak{K}$

$$A^\#(K)=K^\#. \tag{12}$$

THEOREM 4. *Let $k_0$ be a $\mathfrak{p}$-adic number field. Then $\{A^\#(K): K\in\mathfrak{K}\}$ defined by* (12) *is a class formation with the axiom* T. *Moreover, $A^\#(K)$ is isomorphic* (*both algebraically and topologically*) *to the compact Galois group $\Gamma(K)$.*

(PROOF) It is easy to see that F 1–4 are satisfied in this case. Next, let $K/k$ be a normal extension with the Galois group $G$. Let us consider the exact sequence of $G$-homomorphisms:

$$1\to K^\times\to K^\#\to K^\#/K^\times\to 1\,.$$

By (11) the group $K^\#/K^\times$ has the property of unique divisibility and hence $H^r(G, K^\#/K^\times)=1$ for all $r\in Z$. Therefore, from the fundamental exact sequence of cohomology groups we have $H^r(G, K^\#)\cong H^r(G, K^\times)$

$(r \in Z)$. Hence $A^{\sharp}(K) = K^{\sharp}$ satisfies the axioms C **1, 2**. It is easy to verify T, q. e. d.

Hence we can apply Theorem **1** and we get

THEOREM **5**. *Let $K_0^*$ be an infinite algebraic extension of a $\mathfrak{p}$-adic number field and let $\mathfrak{K}^*$ be defined by* (1). *Let $K^* \in \mathfrak{K}^*$ and $K^* = \bigcup_\lambda K_\lambda$ $(K_\lambda \in \mathfrak{K})$, and*

$$A^{\sharp}(K^*) = \text{inv-lim}\, K_\lambda^{\sharp} \tag{13}$$

*with respect to $\mathrm{N}_{\mu,\lambda}: K_\mu^{\sharp} \to K_\lambda^{\sharp}$ for $\lambda < \mu$. Then $\{A^{\sharp}(K^*) ; K^* \in \mathfrak{K}^*\}$ is a class formation with the axiom* T. *Moreover, $A^{\sharp}(K^*)$ is isomorphic to the compact Galois group $\Gamma(K^*)$.*

REMARK. (i) If we take simply $A^*(K^*) = \text{inv-lim}_\lambda K_\lambda^{\times}$ instead of $A^{\sharp}(K^*)$, then $\{A^*(K^*) ; K^* \in \mathfrak{K}^*\}$ does not satisfy the axioms C **1, 2**.

(ii) If we apply the general results in class formation theory we obtain the theorems in M. Mori [9].

(IV) Let $k_0$ be an algebraic number field of a finite degree, $\Omega$ its algebraic closure and $\mathfrak{K}$ be the set of all finite extensions of $k_0$ contained in $\Omega$. If we put $A(K) = C(K)$ (the idèle class group of $K$) and $\varphi_{K,k}: C(k) \to C(K)$ be the natural injection, then by class field theory $\{A(K) ; K \in \mathfrak{K}\}$ is a class formation. In this case $C(K)$ is locally compact and we cannot apply Theorem **1** directly.

Let the volume of an idèle $\mathfrak{a} = \{\alpha_\mathfrak{p}\}$ of $K$ be defined by $V_k(\mathfrak{a}) = \prod_\mathfrak{p} w_\mathfrak{p}(\alpha_\mathfrak{p})$ where $w_\mathfrak{p}$ means the normal valuation. By the product formula $V_k((\alpha)) = 1$ for a principal idèle $(\alpha) \in P(K)$, so that the volume of an idèle class $\tilde{\mathfrak{a}}$ can be defined. Let $C^0(K)$ be the subgroup of all idèle classes of $K$ with volume **1**. Then it is known that $C^0(K)$ is a compact group.

Let $K/k$ be a normal extension with the Galois group $G$. Consider the exact sequence of $G$-homomorphisms

$$1 \longrightarrow C^0(K) \xrightarrow{\iota} C(K) \xrightarrow{V} R^+ \longrightarrow 1 .$$

Here $V$ means the mapping $\tilde{\mathfrak{a}} \to V_K(\tilde{\mathfrak{a}})$ and $R^+$ means the multiplicative group of positive real numbers. Since $R^+$ has the property of unique divisibility by any integer $n$ we have $H^r(G, C^0(K)) \cong H^r(G, C(K))$ $(r \in Z)$.

For any extension $k \subset l$ we see immediately $V_k(N_{l/k}\tilde{\mathfrak{a}}) = V_l(\tilde{\mathfrak{a}})$ for $\tilde{\mathfrak{a}} \in C(l)$ and $V_l(\varphi_{l,k}\tilde{\mathfrak{a}}) = V_k(\tilde{\mathfrak{a}})^{[l:k]}$ for $\tilde{\mathfrak{a}} \in C(k)$. Using these properties it is easy to prove the following theorem.

THEOREM 6. *Let $k_0$ be an algebraic number field of a finite degree. Let us take*

$$A^{\sharp}(K)=C^{0}(K) \tag{14}$$

*then* $\{A^{\sharp}(K)\,;\,K\in\mathfrak{K}\}$ *is a class formation with the axiom* T.

Since $A^{\sharp}(K)$ is compact we can apply Theorem 1 to our class formation. Let $K_0^*$ be an infinite algebraic extension of $k_0$, and $\mathfrak{K}^*$ be defined by (1). Let $K^*\in\mathfrak{K}$ and $K^*=\bigcup_\lambda K_\lambda$ $(K_\lambda\in\mathfrak{K})$. Let us put

$$A^{\sharp}(K^*)=\text{inv-lim}_\lambda\, C^{0}(K) \tag{15}$$

with respect to $\mathrm{N}_{\mu,\lambda}: C^0(K_\mu)\rightarrow C^0(K_\lambda)$ $(\lambda<\mu)$. Then from Theorem 1 follows that $\{A^{\sharp}(K^*)\,;\,K^*\in\mathfrak{K}^*\}$ is a class formation with the axiom T. Let us put

$$A^{*}(K^*)=\text{inv-lim}_\lambda\, C(K_\lambda) \tag{16}$$

with respect to $\mathrm{N}_{\mu,\lambda}: C(K_\mu)\rightarrow C(K_\lambda)$ $(\lambda<\mu)$.

THEOREM 7. *Let $K_0^*$ be an infinite algebraic number field. Let $\mathfrak{K}^*$ be defined by* (1) *and* $A^*(K^*)$ *by* (16). *Then* $\{A^*(K^*)\,;\,K^*\in\mathfrak{K}^*\}$ *is a class formation.*

(PROOF) It is easy to prove F 1–4. To prove C 1, 2 it suffices to see that for a normal extension $L^*/K^*$ with the Galois group $G$ the sequence

$$1\longrightarrow A^{\sharp}(L^*)\xrightarrow{\ \iota\ }A^*(L^*)\xrightarrow{\ V\ }R^+\longrightarrow 1$$

is exact. Here $\iota$ means the injection and $V$ is defined as follows. Let $\tilde{\mathfrak{a}}^*\in A^*(L^*)$, $\tilde{\mathfrak{a}}^*=\text{inv-lim}_\lambda\,\tilde{\mathfrak{a}}_\lambda$, $\tilde{\mathfrak{a}}_\lambda\in C(L_\lambda)$ such that $\mathrm{N}_{\mu,\lambda}\,\tilde{\mathfrak{a}}_\mu=\tilde{\mathfrak{a}}_\lambda$ for $\lambda<\mu$. Since $V_{L_\lambda}(\tilde{\mathfrak{a}}_\lambda)=V_{L_\mu}(\tilde{\mathfrak{a}}_\mu)$ we can define $V(\tilde{\mathfrak{a}}^*)=V_{L_\lambda}(\tilde{\mathfrak{a}}_\lambda)$. Then $\tilde{\mathfrak{a}}^*\in C^0(L^*)$ is equivalent to $V(\tilde{\mathfrak{a}}^*)=1$, so that $A^{\sharp}(K^*)$ is the kernel of the homomorphism $V$.

It remains to prove that the image of $V$ is $R^+$. Let us choose a cofinal increasing sequence $\{L_n\,;\,n=1,2,\cdots\}$ such that $L^*=\bigcup_n L_n$ holds. Let $\mathfrak{p}_n$ be an infinite prime divisor of $L_n$ such that $\mathfrak{p}_{n+1}$ is an extension of $\mathfrak{p}_n$. Let $E_n$ be the set of all idèles $\mathfrak{a}=\{\alpha_\mathfrak{p}\}$ of $L_n$ such that $\alpha_\mathfrak{p}=1$ except $\mathfrak{p}_n$ and $\alpha_{\mathfrak{p}_n}\in R^+$, and let $D_n$ be the set of all idèle classes of $L_n$ containing some $\mathfrak{a}\in E_n$. Then $D_n$ is a group isomorphic to $R^+$ and we have a decomposition $C(L_n)=C^0(L_n)\times D_n$. Moreover, $\mathrm{N}_{n+1,n}C^0(L_{n+1})\subset C^0(L_n)$, $\mathrm{N}_{n+1,n}(D_{n+1})=D_n$ hold. Hence for any given number $\lambda\in R^+$ we can take $\mathfrak{a}_n\in C(L_n)$ successively such that

$V_{L_n}(\mathfrak{a}_n)=\lambda$ and $\mathrm{N}_{n+1,n}(\mathfrak{a}_{n+1})=\mathfrak{a}_n$ hold. This shows the existence of $\mathfrak{a}^*=\text{inv-lim}\,\mathfrak{a}_n\in A^*(K^*)$ such that $V(\mathfrak{a}^*)=\lambda$. Hence the mapping $V$ is onto, q. e. d.

(V) Let $k_0$ be a formal power series field of one variable over a finite field, or an algebraic function field of one variable with a finite constant field. In these cases the situation is quite similar to the case (III). However we shall not treat these cases here in detail.

**6.** Finally we shall consider the relation between our present method (projective method) and the former method (inductive method) used in [4]. Let us use the same notations as in § 1, namely, let $K_0^*=\bigcup_\lambda k_\lambda$ $(k_\lambda\in\mathfrak{K})$ and let

$$A_0^*=\text{inv-lim}\,A(k_\lambda)\qquad(=A^*(K_0^*))$$

be the inverse limit group with respect to the homomorphisms $\mathrm{N}_{\mu,\lambda}: A(k_\mu)\to A(k_\lambda)$ $(\lambda<\mu)$. Let

$$B_0^*=\text{dir-lim}\,A(k_\lambda)$$

be the direct limit group with respect to the homomorphisms $\varphi_{\mu,\lambda}(=\varphi_{k_\mu,k_\lambda}): A(k_\lambda)\to A(k_\mu)$ $(\lambda<\mu)$. We denote by $\varphi_\lambda: A(k_\lambda)\to B_0^*$ the canonical injection such that $\varphi_\mu\circ\varphi_{\mu,\lambda}=\varphi_\lambda$ $(\lambda<\mu)$ holds.

Now let $[k_\lambda: k_0]=N_\lambda$, $N_\lambda=p^{r(\lambda)}M_\lambda$, $(p, M_\lambda)=1$ and we put $N(K_0/k_0)=\prod_p p^r$ $(r=\sup r(\lambda)\leqq\infty)$. We decompose $N(K_0^*/k_0)=N^*N_0$ where $N^*=\prod' p^\infty$ and $N_0=\prod'' p^r$ $(r<\infty)$. Then we call $N^*$ and $N_0$ the infinite and finite part of $N(K_0^*/k_0)$ respectively. Let $m$ be any positive integer. Then we can decompose $m=m^*m_0$ such that $(m_0, N^*)=1$ and every prime factor of $m^*$ divides $N^*$. We call then $m_0$ the $F$-part of $m$.

Next, we remind the definition of a characteristic subgroup $H$ of $B_0^*$ in [4]. A subgroup $H$ of $B_0^*$ is called a characteristic subgroup if (i) $[B_0^*: H]=n<\infty$, (ii) $(n, N^*)=1$, (iii) $H_\mu=N_{\mu,\lambda}^{-1}(H_\lambda)$ $(\mu>\lambda>\nu$ for a suitable $\nu=\nu(H))$ where $H_\lambda$ $(\subset A(k_\lambda))$ is defined by $\varphi_\lambda(H_\lambda)=\varphi_\lambda(A(k_\lambda))\cap H$ and (iv) $H_\lambda$ corresponds to an abelian extension of $k_\lambda$.

Now let $\mathfrak{A}(K_0^*)$ be the lattice consisting of all closed subgroups of $A_0^*$ of finite indices which contain $\mathfrak{R}(K)$ and let $\mathfrak{B}(K_0^*)$ be the lattice consisting of all characteristic subgroups of $B_0^*$. To each $H\in\mathfrak{A}(K_0^*)$ we assign

$$\Psi(H)=\bigcup_{\lambda>\nu}\varphi_\lambda\circ N_\lambda(H)$$

where $\nu=\nu(H)$ is taken sufficiently large as in (i) of the proof of Theorem 1.

THEOREM 8. *Let* $H\in\mathfrak{A}(K_0^*)$ *then* $\Psi(H)\in\mathfrak{B}(K_0^*)$. *The mapping* $\Psi:\mathfrak{A}(K_0)\to\mathfrak{B}(K_0)$ *is a lattice-homomorphism of* $\mathfrak{A}(K_0^*)$ *onto* $\mathfrak{B}(K_0^*)$ *such that* $[B_0^*:\Psi(H)]$ *is the F-part of* $[A_0^*:H]$.

(PROOF) (i) Let $H\in\mathfrak{A}(K_0)$. Then there exists an abelian extension $K^*$ of $K_0^*$ such that the Galois group $G(K^*/K_0^*)$is isomorphic to $A_0^*/H$. The group $\Psi(H)$ is then just the subgroup corresponding to $K^*$ in our former paper [3] (see, p. 109, (34)). Hence $[B_0^*:\Psi(H)]$ is the $F$-part of $[K^*:K_0^*]=[A_0^*:H]$.

(ii) That $\Psi$ is a lattice-homomorphism of $\mathfrak{A}(K_0^*)$ onto $\mathfrak{B}(K_0^*)$ follows from the known results both in projective and inductive theory (see [3], p. 111, (4.5) and [4], p. 168), q. e. d.

## References


[1] E. Artin, Algebraic numbers and algebraic functions I. Mimeographed Notes, New York University (1951).

[2] E. Artin and J. Tate, Algebraic numbers and algebraic functions II. to appear.

[3] Y. Kawada, On the class field theory on algebraic number fields with infinite degree, Journ. Math. Soc. Japan, 3 (1951), 104–115.

[4] ———, Class formations, Duke Math. J., 22 (1955), 165–178.

[5] ———, Class formations III, Journ. Math. Soc. Jaran, 7 (1955), 453–490.

[6] S. Lefschetz, Algebraic topology. Princeton Univ. Press, (1941).

[7] M. Moriya, Klassenkörpertheorie im Kleinen für die unendlichen algebraischen Zahlkörper, Journ. Fac. Sci. Hokkaido Univ., 5 (1937), 9–66.

[8] ———, Klassenkörpertheorie im Grossen für unendliche Zahlkörper, Journ. Fac. Sci. Hokkaido Univ., 6 (1937), 63–101.

[9] M. Mori, Über die Klassenkörpertheorie für unendliche Erweiterungen von einem $\mathfrak{p}$-adischen Zahlkörper, Proc. Japan Acad., 33 (1957), 376–379.