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Geodesic correspondence of Riemann spaces.

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It is well known that the central projection of a sphere on a plane induces a geodesic correspondence between the sphere and the plane, while the stereographic projection induces a conformal correspondence between them. In the present paper we define characteristic roots of a conformal correspondence between Riemann spaces and show that when all the characteristic roots are equal and some additional conditions are satisfied, two spaces have some special properties concerning geodesic and conformal correspondences, which can be regarded as a generalization of the case of sphere and plane. Throughout the paper our treatment is local and we follow the convention that the repeated indices imply summation and assume that the indices i, j, k, h run from 1 to n unless otherwise stated.

§ 1. Characteristic roots of a geodesic correspondence.

1. We consider two n-dimensional affinely connected spaces S and $\bar{S}$, which are both without torsion and are locally homeomorphic in such a way that the geodesics in the two spaces correspond to each other. Let the connections of S and $\bar{S}$ be respectively defined by

$$dA=\omega^i e_i\,, \qquad de_i=\omega_i^j e_j\,,$$

$$dA=\bar{\omega}^i e_i\,, \qquad de_i=\bar{\omega}_i^j e_j\,.$$

Since there are no torsions we have

$$d\omega^i=\omega^j\wedge\omega_j^i\,, \qquad d\bar{\omega}^i=\bar{\omega}^j\wedge\bar{\omega}_j^i\,. \tag{1}$$

On account of the local homeomorphism we can take frames in the tangent spaces in such a way that we have

(2) $$\omega^i = \bar{\omega}^i .$$

As the geodesics in S and $\bar{S}$ correspond to each other we have, as is well known,

(3) $$\bar{\omega}_i^j = \omega_i^j + \delta_i^j b_k \omega^k + b_i \omega^j$$

with a vector (b_i). Putting

$$\Omega_i^j = d\omega_i^j - \omega_i^k \wedge \omega_k^j, \qquad \bar{\Omega}_i^j = d\bar{\omega}_i^j - \bar{\omega}_i^k \wedge \bar{\omega}_k^j$$

we get, by virtue of (3),

(4) $$\begin{aligned}\bar{\Omega}_i^j &= d\omega_i^j + \delta_i^j d(b_k \omega^k) + db_i \wedge \omega^j + b_i d\omega^j \\ &\quad - (\omega_i^k + \delta_i^k b_h \omega^h + b_i \omega^k) \wedge (\omega_k^j + \delta_k^j b_h \omega^h + b_k \omega^j) \\ &= \Omega_i^j + \delta_i^j d(b_h \omega^h) + (db_i - b_k \omega_i^k - b_i b_k \omega^k) \wedge \omega^j .\end{aligned}$$

Now we assume that the groups of holonomy of S and $\bar{S}$ preserve the volume in the tangent spaces. Then we have, for contracted curvature forms,

(5) $$\Omega_i^i = 0 , \qquad \bar{\Omega}_i^i = 0$$

and hence, by (4) and (5),

(6) $$(n+1)\, d(b_h \omega^h) = 0 ,$$

thus $b_h \omega^h$ is locally a total differential. We put

(7) $$db_i - b_k \omega_i^k - b_i b_k \omega^k = p_{ik} \omega^k ,$$

then by (6) and (7) we get

(8) $$p_{ik} = p_{ki} .$$

For the curvature form we get from (4) and (6),

(9) $$\bar{\Omega}_i^j = \Omega_i^j + p_{ik} \omega^k \wedge \omega^j .$$

The rank of the matrix (p_{ik}) has an intrinsic meaning and we can classify the geodesic correspondence of affinely connected spaces by this rank, but we do not treat this here.

2. Hereafter we consider a geodesic correspondence between two Riemann spaces S and $\bar{S}$, which has already been investigated by Levi-Civita and L. P. Eisenhardt. Here we define characteristic roots of the correspondence. We take orthogonal frames in the tangent space of S and denote by $\omega^i, \omega_j^i=-\omega_i^j$ the corresponding Pfaffian forms of Riemannian connection. We then have $d\omega^i=\omega^j\wedge\omega_j^i$.

The Riemannian metric of S is given by

$$ds^2=\sum_i(\omega^i)^2, \tag{10}$$

while that of $\bar{S}$ is given by the equation of the form

$$d\bar{s}^2=a_{ij}\omega^i\omega^j \qquad (a_{ij}=a_{ji}), \tag{11}$$

and the corresponding connection forms $\bar{\omega}_i^j$ are not always skew symmetric in i, j. As (a_{ij}) is a parallel tensor field in the affinely connected space defined by $\omega^i, \bar{\omega}_i^j$, we have

$$da_{ij}=a_{ik}\bar{\omega}_j^k+a_{kj}\bar{\omega}_i^k. \tag{12}$$

As S and $\bar{S}$ correspond geodesically, we have (3), and as (5) holds good in our case we have (8) and (9). Conversely (3) and (12) with positive definite a_{ij} are sufficient for the geodesic correspondence of (10) with (11). By taking a suitable orthogonal frames in S we can transform the symmetric covariant tensor (p_{ij}) into a diagonal form. Denoting the diagonal elements by p_i we get

$$\bar{\Omega}_i^j=\Omega_i^j+p_i\omega^i\wedge\omega^j \qquad \text{(not summed for } i\text{)}. \tag{13}$$

We call $p_i(i=1,\cdots,n)$ characteristic roots of the geodesic correspondence of S with $\bar{S}$.

3. As an application of (13) we give a new proof of the classical theorem which states that a projectively flat Riemann space is of constant curvature. We take $\bar{S}$ as Euclidean space. Then we get $\bar{\Omega}_i^j=0$ and by (13)

$$\Omega_i^j=-p_i\omega^i\wedge\omega^j \qquad \text{(not summed for } i\text{)}.$$

As Ω_i^j is skew symmetric in i, j we have $p_i=p_j$, and consequently, putting $p_1=p_2=\cdots=p_n=K$, we get $\Omega_i^j=-K\omega^i\wedge\omega^j$, which shows that S is of constant curvature.

4. Here we investigate the case in which all b's are zero. In this case we have $\bar{\omega}_i^j = \omega_i^j$ and by (12)

$$da_{ij} = a_{ik}\,\omega_j^k + a_{kj}\,\omega_i^k$$

and (a_{ij}) is a parallel symmetric covariant tensor field. As is well known, the space S then decomposes into the direct product of Riemann spaces, namely

$$ds^2 = d\sigma_1^2 + \cdots + d\sigma_k^2$$

where each $d\sigma_i^2 (i=1,\cdots,k)$ is an irreducible Riemannian metric. This can be verified by taking orthogonal frames such that (a_{ij}) is of diagonal form and by using E. Cartan's lemma. Then the metric of $\bar{S}$ is given by

$$d\bar{s}^2 = a_1 d\sigma_1^2 + \cdots + a_k d\sigma_k^2,$$

a's being constant.

§ 2. The case of equal characteristic roots.

5. Now we consider Riemann spaces for which the characteristic roots of the geodesic correspondence of $\bar{S}$ with S are all equal. We denote these roots by p. Then (p_{ij}) is of diagonal form for any orthogonal frame in S, the diagonal elements being equal to p. We take orthogonal frames in S in such a way that

$$b_1 = b, \qquad b_2 = b_3 = \cdots = b_n = 0. \tag{14}$$

Then, by virtue of (7), we have

$$db - b^2\omega^1 = p\omega^1, \qquad -b\omega_\alpha^1 = p\omega^\alpha \quad (\alpha = 2,\cdots,n). \tag{15}$$

By the assumption $b \neq 0$ (the case $b=0$ has been treated in **4**)

$$d\omega^1 = \omega^\alpha \wedge \omega_\alpha^1 = \omega^\alpha \wedge (-p/b\ \omega^\alpha) = 0$$

and so we can take local coordinates $x^1,\cdots,x^n$ such that

$$\omega^1 = dx^1$$

holds good. Then we have

$$db=(b^2+p)\,dx^1\,. \tag{16}$$

Thus b and p are functions of only one variable x^1. Now let the indices α,β run from 2 to n. Then we have by virtue of (15)

$$d\omega^\alpha=\omega^1\wedge\omega_1^\alpha+\omega^\beta\wedge\omega_\beta^\alpha=dx^1\wedge(p/b\;\omega^\alpha)+\omega^\beta\wedge\omega_\beta^\alpha\,.$$

Hence putting

$$c=c(x^1)=\exp\int p/b\;dx^1\,,\qquad \pi^\alpha=\omega^\alpha/c \tag{17}$$

we get the relations

$$d\pi^\alpha=\pi^\beta\wedge\omega_\beta^\alpha\,,\qquad \omega_\alpha^\beta=-\omega_\beta^\alpha$$

and by E. Cartan's lemma $\sum_\alpha(\pi^\alpha)^2$ does not contain x^1 provided that we choose $x^2,\cdots,x^n$ suitably, namely we have

$$d\sigma^2=\sum(\pi^\alpha)^2=g_{\alpha\beta}(x^2,\cdots,x^n)\,dx^\alpha\,dx^\beta\,,$$

and the metric of S can be written in the form

$$ds^2=(dx^1)^2+c(x^1)^2\,d\sigma^2\,.$$

Here $c=c(x^1)$ is not arbitrary. It will be found as follows. We take orthogonal frames such that $\omega^1=dx^1$ and that π^α do not contain x^1 and dx^1. Then the same is true for ω_α^β. Putting $da=b_i\omega^i$ we get by (14)

$$da=b_1\omega^1=b\;dx^1 \tag{18}$$

and by (3)

$$\varpi_1^1=da+b\omega^1=2da\,,\qquad \varpi_\alpha^\beta=\omega_\alpha^\beta\quad(\alpha\neq\beta)$$

$$\varpi_\alpha^\alpha=da\quad(\text{not summed for }\alpha)$$

$$\varpi_\alpha^1=\omega_\alpha^1=-p/b\;\omega^\alpha\,,\qquad \varpi_1^\alpha=\omega_1^\alpha+b\omega^\alpha=(b+p/b)\omega^\alpha$$

and (12) can be written as

$$da_{11}=2(a_{\alpha1}\varpi_1^\alpha+a_{11}\varpi_1^1)=2a_{\alpha1}(b+p/b)\omega^\alpha+4a_{11}da\,,$$

$$da_{\alpha1}=a_{\alpha\beta}\varpi_1^\beta+a_{\alpha1}\varpi_1^1+a_{\beta1}\varpi_\alpha^\beta+a_{11}\varpi_\alpha^1$$

$$=a_{\alpha\beta}(b+p/b)\omega^\beta+3a_{\alpha1}da+a_{\beta1}\omega^\beta_\alpha-p/b\,a_{11}\omega^\alpha$$

$$da_{\alpha\beta}=a_{\alpha\gamma}\overline{\omega}^\gamma_\beta+a_{\gamma\beta}\overline{\omega}^\gamma_\alpha+a_{\alpha1}\overline{\omega}^1_\beta+a_{\beta1}\overline{\omega}^1_\alpha$$

$$=a_{\alpha\gamma}\omega^\gamma_\beta+a_{\gamma\beta}\omega^\gamma_\alpha+2a_{\alpha\beta}da-a_{\alpha1}\,p/b\;\omega^\beta-a_{\beta1}\,p/b\;\omega^\alpha\,.$$

We put

$$a_{11}e^{-4a}=b_{11}\,,\qquad a_{\alpha1}\,e^{-3a}=b_{\alpha1}\,,\qquad a_{\alpha\beta}\,e^{-2a}=b_{\alpha\beta} \tag{19}$$

and get

$$db_{11}=2b_{\alpha1}\,ce^{-a}(b+p/b)\;\pi^\alpha\,, \tag{20}$$

$$db_{\alpha1}=b_{\alpha\beta}\,ce^{-a}(b+p/b)\pi^\beta+b_{\beta1}\,\omega^\beta_\alpha-b_{11}\,ce^a\,p/b\;\pi^\alpha\,, \tag{21}$$

$$db_{\alpha\beta}=b_{\alpha\gamma}\,\omega^\gamma_\beta+b_{\gamma\beta}\omega^\gamma_\alpha-b_{\alpha1}\,ce^a\,p/b\;\pi^\beta-b_{\beta1}\,ce^a\,p/b\;\pi^\alpha\,. \tag{22}$$

As π^α and ω^β_α do not contain x^1 and dx^1, (b_{ij}) do not contain x^1, and so if one of $b_{\alpha1}$ is not zero, $cp/b\,e^a$ which is a function of x^1 is constant on account of (22). By taking its differential and taking the relations

$$\frac{1}{c}\frac{dc}{dx^1}=\frac{p}{b}\,,\qquad b=\frac{da}{dx^1}\,,\qquad \frac{db}{dx^1}=b^2+p \tag{23}$$

into consideration we get

$$0=\frac{b}{p}\,d\left(\frac{p}{b}\right)+\frac{dc}{c}+da=\frac{b}{p}\left(\frac{dp}{b}-\frac{pdb}{b^2}\right)+\frac{p}{b}\,dx^1+bdx^1=\frac{dp}{p}$$

thus p is a constant.

If $b_{\alpha1}=0$ $(\alpha=2,\cdots,n)$, b_{11} is non zero constant by (20) and by (21)

$$b_{\alpha\beta}\,ce^{-a}(b+p/b)=0\quad(\alpha\neq\beta)\,, \tag{24}$$

$$b_{\alpha\alpha}\,ce^{-a}(b+p/b)-b_{11}\,ce^a\,p/b=0\qquad\text{(not summed for } \alpha)\,. \tag{25}$$

If one of $b_{\alpha\beta}$ $(\alpha\neq\beta)$ is not zero, we have $b+p/b=0$ and $p/b=0$ by (25) and so $b=0$. Thus we get a contradiction. Hence $b_{\alpha\beta}=0$ $(\alpha\neq\beta)$ and by (22) $db_{\alpha\alpha}=0$ (not summed for α) and by (25)

$$e^{2a}\,p/(b^2+p)=b_{\alpha\alpha}/b_{11}=\text{const.}.$$

Hence by differentiation

$$0=\frac{dp}{p}-\frac{2bdb+dp}{b^2+p}+2da=\frac{b^2dp}{p(b^2+p)},$$

and p is a constant in this case too. Thus we get

THEOREM 1. *If the characteristic roots of the geodesic correspondence of $\overline{S}$ with S are all equal, then the roots are constant.*

6. We consider first the case in which p is a positive constant. Putting $p=A^2$ we get from (16) $b=A\tan Ax^1$ by a suitable choice of additive constant in the variable x^1. By (17) $c=B\sin Ax^1$ (B const.) and we have for S

$$ds^2=(dx)^2+B^2\sin^2 Ax^1\, d\sigma^2 .$$

Denoting Ax^1, A^2ds^2 and $A^2B^2\, d\sigma^2$ by x^1, ds^2 and $d\sigma^2$ respectively, we get

$$ds^2=(dx^1)^2+\sin^2 x^1\, d\sigma^2 . \tag{26}$$

We consider the space with this metric. Then we have

$$p=1 , \qquad b=\tan x^1 , \qquad c=\sin x^1$$

and by (18) $a=-\log\cos x^1$, as we can put an additive constant equal to zero without loss of generality. Hence we get

$$e^a=\sec x^1 , \tag{27}$$

$$ce^a\, p/b=1 , \qquad (b+p/b)\, ce^{-a}=1$$

and (20) (21) (22) can be written respectively in the form

$$db_{11}=2b_{a1}\, \pi^a \tag{28}$$

$$db_{a1}=b_{a\beta}\pi^\beta+b_{\beta 1}\omega^\beta_a-b_{11}\pi^a \tag{29}$$

$$db_{a\beta}=b_{a\gamma}\, \omega^\gamma_\beta+b_{\beta\gamma}\, \omega^\gamma_a-b_{a1}\, \pi^\beta-b_{\beta 1}\, \pi^a . \tag{30}$$

Now we consider one solution of these equations, namely

$$b_{11}=1 , \qquad b_{a1}=0 , \qquad b_{a\beta}=\delta_{a\beta} . \tag{31}$$

For the solution we have by (11), (19) and (27)

$$ds^2=\sec^4 x^1\, (dx^1)^2+\tan^2 x^1\, d\sigma^2 , \tag{32}$$

This can be transformed into the form

$$ds^2 = 1/4 \sec^4 x^1 \, (d(2x^1)^2 + \sin^2 (2x^1)\, d\sigma^2)$$

and a correspondence of the point $(x^1, x^2, \cdots, x^n)$ of S with the point $(2x^1, x^2, \cdots, x^n)$ of $\bar{S}$ is conformal. Thus we get the following theorem

THEOREM 2. *Let $\bar{S}$ be a Riemann space which is in geodesic correspondence with another Riemann space S and let the characteristic roots of the correspondence of $\bar{S}$ with S be all equal and positive. Then by a suitable choice of coordinates we have for the metric of S*

$$ds^2 = (dx^1)^2 + \sin^2 x^1 \cdot g_{\alpha\beta}(x^2, \cdots, x^n)\, dx^\alpha \, dx^\beta$$

except for a similar transformation. Among the space $\bar{S}$ with this property there is one with the metric

$$ds^2 = \sec^4 x^1 \, (dx^1)^2 + \tan^2 x^1 \cdot g_{\alpha\beta}(x^2, \cdots, x^n)\, dx^\alpha \, dx^\beta$$

This is conformal to S by the correspondence from the point $(2x^1, x^2, \cdots, x^n)$ of $\bar{S}$ to $(x^1, x^2, \cdots x^n)$ of S.

Here we mean by a similar transformation a multiplication of a Riemannian metric by a constant. The latter half of the theorem gives a generalization of the well known theorem on the correspondence between a sphere and a plane. In fact if we denote by P a point on the sphere $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ in the Euclidean space with spherical coordinates $(1, \theta, \varphi)$ we have for the induced Riemannian metric $ds^2 = d\theta^2 + \sin^2 \theta \, d\varphi^2$ and by a central projection of the sphere on the plane $x^3 = 1$ the point is mapped on the point with the polar coordinates $(\tan \theta, \varphi)$. The Euclidean metric on the plane $x^3 = 1$ is give by

$$ds^2 = d\,(\tan \theta)^2 + \tan^2 \theta \, d\varphi^2 = \sec^4 \theta \, d\theta^2 + \tan^2 \theta \, d\varphi^2 \,,$$

while by a stereographic projection from the point $(0,\ 0,\ -1)$ of the sphere on the plane $x^3 = 0$ the point $(1, 2\theta, \varphi)$ is mapped on the point with the polar coordinates $(\tan \theta, \varphi)$ and the metric of the Euclidean plane is given by

$$ds^2 = d(\tan \theta)^2 + \tan^2 \theta \, d\varphi^2 = 1/4 \sec^4 \theta (d(2\theta)^2 + \sin^2 2\theta \, d\varphi^2) \,.$$

The case $p<0$ can be treated in an analogous way by putting $p=-A^2$. Then we have for the Riemannian metric of S

$$ds^2=(dx^1)^2+\sinh^2 x^1\, g_{\alpha\beta}(x^2,\cdots,x^n)\, dx^\alpha\, dx^\beta$$

and instead of (32) we get

$$ds^2=\operatorname{sech}^4 x^1(dx^1)^2+\tanh^2 x^1\, g_{\alpha\beta}(x^2,\cdots,x^n)\, dx^\alpha\, dx^\beta\, .$$

The case $p=0$ will be treated later.

7. Here we consider solutions of (28), (29) and (30) other than (31). If $b_{11}=0$, we have by (28) $b_{\alpha 1}=0$ $(\alpha=2,3,\cdots,n)$ and this is a contradiction. Hence we can assume $b_{11}\neq 0$. Putting

$$h_\alpha=-b_{\alpha 1}/b_{11} \tag{33}$$

we have by (28)

$$h_\alpha\pi^\alpha=b_{\alpha 1}\pi^\alpha/b_{11}=-db_{11}/(2b_{11})\, .$$

Hence we can put $dh=h_\alpha\pi^\alpha$ and we have

$$db_{11}=-2b_{11}\, dh\, . \tag{34}$$

Now we put

$$c_{\alpha\beta}=\frac{b_{\alpha\beta}}{b_{11}}-\frac{b_{\alpha 1}}{b_{11}}\cdot\frac{b_{\beta 1}}{b_{11}} \tag{35}$$

$$\pi_\alpha^\beta=\omega_\alpha^\beta+\delta_\alpha^\beta\, dh+h_\alpha\pi^\beta\, . \tag{36}$$

Then by (28), (29), (30), (35) and (36)

$$\begin{aligned} dc_{\alpha\beta}&=\frac{db_{\alpha\beta}}{b_{11}}-\frac{db_{\alpha 1}}{b_{11}}\frac{b_{\beta 1}}{b_{11}}-\frac{b_{\alpha 1}}{b_{11}}\frac{db_{\beta 1}}{b_{11}}-\left(\frac{b_{\alpha\beta}}{b_{11}}-2\frac{b_{\alpha 1}}{b_{11}}\frac{b_{\beta 1}}{b_{11}}\right)\frac{db_{11}}{b_{11}} \\ &=\frac{b_{\alpha\gamma}}{b_{11}}\omega_\beta^\gamma+\frac{b_{\beta\gamma}}{b_{11}}\omega_\alpha^\gamma-\frac{b_{\alpha 1}}{b_{11}}\pi^\beta-\frac{b_{\beta 1}}{b_{11}}\pi^\alpha-\left(\frac{b_{\alpha\gamma}}{b_{11}}\pi^\gamma+\frac{b_{\gamma 1}}{b_{11}}\omega_\alpha^\gamma-\pi^\alpha\right)\frac{b_{\beta 1}}{b_{11}} \\ &\quad-\left(\frac{b_{\beta\gamma}}{b_{11}}\pi^\gamma+\frac{b_{\gamma 1}}{b_{11}}\omega_\beta^\gamma-\pi^\beta\right)\frac{b_{\alpha 1}}{b_{11}}-\left(c_{\alpha\beta}-\frac{b_{\alpha 1}}{b_{11}}\frac{b_{\beta 1}}{b_{11}}\right)(-2dh) \\ &=c_{\alpha\gamma}\,\omega_\beta^\gamma+c_{\gamma\beta}\,\omega_\alpha^\gamma+2c_{\alpha\beta}\,dh+h_\alpha\, c_{\gamma\beta}\,\pi^\gamma+h_\beta\, c_{\alpha\gamma}\,\pi^\gamma\, . \end{aligned}$$

Hence we have by (36)

(37) $$dc_{\alpha\beta}=c_{\alpha\gamma}\,\pi^{\gamma}_{\beta}+c_{\gamma\beta}\,\pi^{\gamma}_{\alpha}\,.$$

Next we get

$$dh_{\alpha}=-d\left(\frac{b_{\alpha 1}}{b_{11}}\right)=-\frac{db_{\alpha 1}}{b_{11}}+\frac{b_{\alpha 1}}{b_{11}}\frac{db_{11}}{b_{11}}$$

$$=-\frac{b_{\alpha\beta}}{b_{11}}\pi^{\beta}-\frac{b_{\beta 1}}{b_{11}}\omega^{\beta}_{\alpha}+\pi^{\alpha}-\frac{b_{\alpha 1}}{b_{11}}dh$$

$$=-(c_{\alpha\beta}+h_{\alpha}h_{\beta})\,\pi^{\beta}+h_{\beta}\,\omega^{\beta}_{\alpha}+\pi^{\alpha}+2h_{\alpha}dh$$

(38) $$dh_{\alpha}-h_{\beta}\omega^{\beta}_{\alpha}-h_{\alpha}dh=-c_{\alpha\beta}\pi^{\beta}+\pi^{\alpha}\,.$$

If we put

(39) $$dh_{\alpha}-h_{\beta}\omega^{\beta}_{\alpha}-h_{\alpha}dh=q_{\alpha\beta}\pi^{\beta}\,,$$

we get

(40) $$q_{\alpha\beta}=-c_{\alpha\beta}+\delta_{\alpha\beta}\,.$$

Now by (35) and by a straight forward calculation, we get

(41) $$\det\,(c_{\alpha\beta})=b_{11}^{-n}\det\,(b_{ij})>0$$

and, for variables λ^{α},

(42) $$c_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}=b_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}/b_{11}-(b_{\alpha 1}\lambda^{\alpha}/b_{11})^{2}\,.$$

This quadratic form is positive definite. The reason is as follows. If $c_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}$ were not positive definite, it would be of the form $c_{\alpha}(\Lambda^{\alpha})^{2}$ where Λ's are linear combinations of λ^{α} and $c_{2}<0$, $c_{3}<0$ by (41). Now for non zero vector $(\lambda^{2},\cdots,\lambda^{n})$ such that $\Lambda^{4}=\Lambda^{5}=\cdots=\Lambda^{n}=0$ and $b_{\alpha 1}\lambda^{\alpha}=0$, $c_{\alpha\beta}\lambda^{\alpha}\lambda^{\beta}$ is negative, while by (42) it is positive. Thus we have

THEOREM 3. *If the Riemann space with the metric* (26) *is in geodesic correspondence with the space whose metric cannot be reduced to* (32) *by a similar mapping, the Riemann space with the metric* $d\sigma^{2}=\sum_{\alpha}(\pi^{\alpha})^{2}=g_{\alpha\beta}(x^{2},\cdots,x^{n})\,dx^{\alpha}\,dx^{\beta}$ *admits a geodesic correspondence to the* *space with the metric* $d\sigma^2 = c_{\alpha\beta}\pi^\alpha\pi^\beta$ *and* (40) *holds good for* $q_{\alpha\beta}$ *determined by* (39).

Conversely if $d\sigma^2$ *admits a geodesic correspondence satisfying* (40) *then* ds^2 *determined by* (26) *is in a geodesic correspondence with the space with the metric* $ds^2 = \sec^4 x^1\,(dx^1)^2 + \tan^2 x^1\, c_{\alpha\beta}\pi^\alpha\pi^\beta$ *and the characteristic roots are all equal.*

The converse can be proved as follows. We determine b_{11} by (34) and put $b_{\alpha 1} = -b_{11}h_\alpha$, $b_{\alpha\beta} = b_{11}(c_{\alpha\beta} + h_\alpha h_\beta)$. Then (28), (29) and (30) are satisfied and we get (12).

By (40) and (37) we have

$$-dq_{\alpha\beta} = dc_{\alpha\beta} = c_{\alpha\gamma}\pi^\gamma_\beta + c_{\gamma\beta}\pi^\gamma_\alpha = c_{\alpha\gamma}\omega^\gamma_\beta + c_{\gamma\beta}\omega^\gamma_\alpha + 2c_{\alpha\beta}dh + c_{\alpha\gamma}h_\beta\pi^\gamma + c_{\gamma\beta}h_\alpha\pi^\gamma$$

$$dq_{\alpha\beta} \wedge \pi^\beta = c_{\alpha\gamma}d\pi^\gamma - c_{\gamma\beta}\omega^\gamma_\alpha \wedge \pi^\beta - c_{\alpha\beta}dh \wedge \pi^\beta\,. \tag{43}$$

By taking exterior differential of (39) and eliminating dh_α and $dq_{\alpha\beta}$ by (38) and (43) we get

$$-h_\beta d\omega^\beta_\alpha - (h_\gamma\omega^\gamma_\beta + h_\beta dh - c_{\beta\gamma}\pi^\gamma + \pi^\beta) \wedge \omega^\beta_\alpha - (h_\beta\omega^\beta_\alpha + h_\alpha dh - c_{\alpha\beta}\pi^\beta + \pi^\alpha) \wedge dh$$
$$= c_{\alpha\gamma}d\pi^\gamma - c_{\gamma\beta}\omega^\gamma_\alpha \wedge \pi^\beta - c_{\alpha\beta}dh \wedge \pi^\beta + d\pi^\alpha - c_{\alpha\beta}d\pi^\beta$$

$$h_\beta(\Omega^\beta_\alpha + \pi^\alpha \wedge \pi^\beta) = 0 \tag{44}$$

We put $\Omega^\beta_\alpha = \dfrac{1}{2}\, R^\beta_{\alpha\gamma\varepsilon}\pi^\gamma \wedge \pi^\varepsilon$ and call the number of the independent solutions of the linear equations in h_β

$$h_\beta(R^\beta_{\alpha\gamma\varepsilon} + \delta^\gamma_\alpha\delta^\varepsilon_\beta - \delta^\varepsilon_\alpha\delta^\gamma_\beta) = 0 \qquad (\alpha, \beta, \gamma, \varepsilon = 2, \cdots, n)$$

the relative index of nullity at the point of the Riemann space with the metric $d\sigma^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. If the relative index of nullity is zero at every point, we have by (44) $h_\beta = 0$ and by (33) $b_{\alpha 1} = 0$. Thus by (28) b_{11} is constant and by (29) $b_{\alpha\beta} = b_{11}\delta_{\alpha\beta}$ and the solution coincides with (31) except for a similar mapping. Thus we get

THEOREM 4. *Let a metric of the space S be given by*

$$ds^2 = (dx^1)^2 + \sin^2 x^1\, d\sigma^2$$

where the space S_0 *with the metric* $d\sigma^2 = g_{\alpha\beta}(x^2, \cdots, x^n)\, dx^\alpha dx^\beta$ *has one of the following properties.*

(*a*) *S_0 does not admit non similar geodesic correspondence with another space;*

(*b*) *Relative index of nullity of S_0 is zero at every point.*

Then the space $\bar{S}$ whose characteristic roots of the geodesic correspondence with S are all equal has the metric

$$ds^2 = \sec^4 x^1 (dx^1)^2 + \tan^2 x^1 d\sigma^2$$

except for a similar mapping.

8. Next we consider the case $p=0$. By (17) c is a constant and we can assume it to be equal to unity without loss of generality. Then the metric of S is given by

$$ds^2 = (dx^1)^2 + d\sigma^2 \tag{45}$$

where $d\sigma^2 = g_{\alpha\beta}(x^2,\cdots, x^n)\, dx^\alpha dx^\beta$. By (16) we have $db/dx^1 = b^2$, and we get $b = -1/x^1$ by a suitable choice of additive constant in x^1. By (18) we get $a = \int b\, dx^1 = -\log x^1 + \text{const.}$ Here an additive constant may be taken as zero. Then

$$e^a = 1/x^1\,, \qquad be^{-a} = -1\,.$$

Then (20), (21) and (22) take respectively the following forms

$$db_{11} = -2b_{\alpha 1}\pi^\alpha \tag{46}$$

$$db_{\alpha 1} = -b_{\alpha\beta}\pi^\beta + b_{\beta 1}\omega_\alpha^\beta \tag{47}$$

$$db_{\alpha\beta} = b_{\alpha\gamma}\omega_\beta^\gamma + b_{\gamma\beta}\omega_\alpha^\gamma\,. \tag{48}$$

As we have taken frames such that π^α and ω_α^β do not contain x^1, $b_{\alpha\beta}$ do not contain x^1. In this case we have no solution such that $b_{\alpha 1}=0$ $(\alpha=2,\cdots, n)$. We take orthogonal frames in the space with the metric $d\sigma^2$ for which $(b_{\alpha\beta})$ reduces to a diagonal form. We put $b_\alpha = b_{\alpha\alpha}$ (not summed for α) and assume

$$b_2 = \cdots = b_{k_1} \neq b_{k_1+1} = \cdots = b_{k_1+k_2} \neq \cdots = b_k \neq b_{k+1} \neq b_{k+2} \neq \cdots \neq b_n\,.$$

Let the indices α', β' run from 2 to k_1. By (48) b's are constant and $\omega_{\alpha'}^\beta = 0$ $(\beta = k_1+1,\cdots, n)$. Hence

$$d\pi^{\alpha'} = \omega^{\beta} \wedge \omega_{\beta}^{\alpha'} = \pi^{\beta'} \wedge \omega_{\beta'}^{\alpha'} .$$

By E. Cartan's lemma, $\sum(\pi^{\alpha'})^2$ contains $k_1 - 1$ variables $x^2, \cdots, x^{k_1}$ by a suitable choice of coordinates. By an analogous argument on the other part of indices we get

$$d\sigma^2 = d\sigma_1^2 + \cdots + d\sigma_l^2 + (dx^{k+1})^2 + \cdots + (dx^n)^2$$

where $d\sigma_i^2$ $(i = 1, \cdots, l)$ are metrics of Riemann spaces. We take orthogonal frames in the tangent spaces of the space with the metric $d\sigma_1^2$ in such a way that $\pi^{\alpha'}$ contains only $x^2, \cdots, x^{k_1}$, and etc. Then $\omega_{\alpha'}^{\beta'}$ contain $x^2, \cdots, x^{k_1}$ only and we get from (47)

$$db_{\alpha' 1} - b_{\beta' 1}\, \omega_{\alpha'}^{\beta'} = -b' \pi^{\alpha'} , \qquad b' = b_2 .$$

As $b_{\alpha' 1}$ $(\alpha' = 2, \cdots, k_1)$ can be considered as components of a vector in the space with the metric $d\sigma_1^2$, we take orthogonal frames such that $(b_{21}, \cdots, b_{k_1 1})$ reduces to $(b'c', \cdots, 0)$. Then we get

$$dc' = -\pi^2 , \qquad c' \omega_{\alpha'}^2 = \pi^{\alpha 1} \qquad (\alpha' = 3, \cdots, k_1) .$$

If we put $\pi^{\alpha'}/c' = \rho^{\alpha'}$ $(\alpha' = 3, \cdots, k_1)$ we get

$$\begin{aligned} d\rho^{\alpha'} &= -dc'/(c')^2 \wedge \pi^{\alpha} + 1/c'\, d\pi^{\alpha'} \\ &= -dc'/(c')^2 \wedge \pi^{\alpha} + 1/c'\, \pi^2 \wedge \omega_{?}^{\alpha'} + 1/c'\, \pi^{\beta'} \wedge \omega_{\beta'}^{\alpha'} \\ &= \rho^{\beta'} \wedge \omega_{\beta'}^{\alpha'} \qquad (\alpha', \beta' = 3, \cdots, k_1) \end{aligned}$$

and by a suitable choice of coordinates $x^2 = c'$, $x^3, \cdots, x^{k_1}$ $d\tau_1^2 = \sum(\pi^{\alpha'})^2$ contains only $x^3, \cdots, x^{k_1}$ and $d\sigma_1^2 = (dx^2) + (x^2)^2\, d\tau_1^2$. Treating $d\sigma_2^2, \cdots, d\sigma_l^2$ analogously we get by (46)

$$db_{11} = 2b' x^2 dx^2 + \cdots + 2b^{(l)} x^{k-k_l+1}\, dx^{k-k_l+1} + 2b_{k+1} x^{k+1}\, dx^{k+1} + \cdots + 2b_n x^n dx^n .$$

Hence

$$b_{11} = b'(x^2)^2 + \cdots + b^{(l)}(x^{k-k_l+1})^2 + b_{k+1}(x^{k+1})^2 + \cdots + b_n(x^n)^2 + \text{const.} .$$

Thus we have completely determined the Riemann spaces for which the characteristic roots of the geodesic correspondence are all zero, namely

$$S: \quad d\bar{s}^2 = (dx^1)^2 + d\sigma_1^2 + \cdots + d\sigma_l^2 + (dx^{k+1})^2 + \cdots + (dx^n)^2 ,$$

$$\bar{S}: \quad ds^2 = \frac{1}{(x^1)^4} b_{11}(dx^1)^2 + \frac{1}{(x^1)^2} [b' d\sigma_1^2 + \cdots + b^{(l)} d\sigma_l^2 + b_{k+1}(dx^{k+1})^2$$

$$+ \cdots + b_n(dx^n)^2] + \frac{1}{(x^1)^3} dx^1 db_{11} .$$

§ 3. The case in which the characteristic roots are equal except one.

9. It seems rather complicated to find all the Riemann spaces which are in geodesical correspondence with another spaces with the characteristic roots equal except one. We will give some examples of such spaces. The space with the metric (26) is a special case of the one with the metric

$$ds^2 = (dx^1)^2 + c(x^1)^2 \, d\sigma^2 \tag{53}$$

where $d\sigma^2 = g_{\alpha\beta}(x^2, \cdots, x^n)\, dx^\alpha \, dx^\beta$ $(\alpha, \beta = 2, \cdots, n)$. This space has been treated by A. Fialkow [1] and K. Yano [2]. In my terminology [3] it is the space for which another Riemann space is conformal and the characteristic roots of the conformal correspondence are all equal. In the following we investigate the geodesic correspondence of (53) with another space. We denote by S the space with the metric (53). We put

$$d\sigma^2 = \sum_\alpha (\pi^\alpha)^2 \qquad (\alpha = 2, \cdots, n) \tag{54}$$

where π^α are Pfaffian forms in $x^2, \cdots, x^n$. We take ω_α^β such that

$$d\pi^\alpha = \pi^\beta \wedge \omega_\beta^\alpha , \qquad \omega_\beta^\alpha = -\omega_\alpha^\beta \qquad (\alpha, \beta = 2, \cdots, n)$$

and put

$$\omega^1 = dx^1 , \qquad \omega^\alpha = c(x^1)\pi^\alpha \tag{55}$$

$$\omega_1^\alpha = -\omega_\alpha^1 = c'\pi^\alpha , \qquad \omega_1^1 = 0 \qquad (c' = dc/dx^1) , \tag{56}$$

then we get

$$d\omega^1 = 0 = \omega^\alpha \wedge \omega_\alpha^1 , \qquad d\omega^\alpha = dc \wedge \pi^\alpha + c\pi^\beta \wedge \omega_\beta^\alpha = \omega^1 \wedge \omega_1^\alpha + \omega^\beta \wedge \omega_\beta^\alpha$$

and $\omega^1, \omega^\alpha, \omega_\alpha^1$ and ω_α^β are Pfaffian forms of Riemannian connection of the space (53).

Now we assume that the space $\bar{S}$ with the metric

$$(57) \qquad d\bar{s}^2 = a_{ij}\, \omega^i \omega^j \qquad (a_{ij} = a_{ji})$$

is in geodesic correspondence with S by the mapping of points with the same coordinates. Then the parameters of connection $\bar{\omega}_i^j$ satisfy the following equations

$$(58) \qquad \bar{\omega}_i^j = \omega_i^j + \delta_i^j db + b_i \omega^j, \qquad db = b_i \omega^i$$

$$(59) \qquad da_{ij} = a_{ik} \bar{\omega}_j^k + a_{kj} \bar{\omega}_i^k .$$

It seems complicated to find b_i, a_i satisfying (58) and (59) for given (53). Here we consider the case

$$(60) \qquad a_{1\alpha} = a_{\alpha 1} = 0, \qquad a_{\alpha\beta} = 0 \quad (\alpha \neq \beta)$$

$$(61) \qquad b_\alpha = 0 \qquad (\alpha, \beta = 2, \cdots, n).$$

By (56) and (58) we have

$$\bar{\omega}_1^1 = db + b_1 dx^1, \qquad \bar{\omega}_1^\alpha = c' \pi^\alpha + b_1 \omega^\alpha,$$

$$\bar{\omega}_\alpha^1 = -c' \pi^\alpha, \qquad \bar{\omega}_\alpha^\beta = \omega_\alpha^\beta + \delta_\alpha^\beta db .$$

Putting $a_{ii} = a_i$ (not summed for i) we have by (56) and (60)

$$0 = da_{\alpha 1} = a_\alpha \bar{\omega}_1^\alpha + a_{\alpha 1} \bar{\omega}_1^1 + a_{\beta 1} \bar{\omega}_\alpha^\beta + a_{11} \bar{\omega}_\alpha^1$$

$$= a_\alpha (c' \pi^\alpha + b_1 \omega^\alpha) - a_1 c' \pi^\alpha .$$

Hence we have

$$(62) \qquad a_\alpha (c'/c + b_1) = a_1 c'/c .$$

We exclude the case $b_1 = 0$, which has been investigated in **4.** Then c' cannot be zero on account of (62), and we have

$$(63) \qquad a_2 = a_3 = \cdots = a_n .$$

By (59) we have

$$da_{\alpha\beta}=a_\alpha\bar{\omega}^\alpha_\beta+a_\beta\bar{\omega}^\beta_\alpha+a_{\alpha 1}\bar{\omega}^1_\beta+a_{\beta 1}\bar{\omega}^1_\alpha$$
$$=a_\alpha\omega^\alpha_\beta+a_\beta\omega^\beta_\alpha=0 \qquad \text{(not summed for } \alpha, \beta)$$

and (59) is satisfied for $a_{\alpha\beta}$ $(\alpha \neq \beta)$. By virtue of (59) we get

$$da_1=2a_1\bar{\omega}^1_1=4a_1 db$$
$$da_\alpha=2a_\alpha\bar{\omega}^\alpha_\alpha=2a_\alpha db \qquad \text{(not summed for } \alpha).$$

Hence $a_1=A\,e^{4b}$, $a_\alpha=B\,e^{2b}$ $(A, B$ const.). Putting these into (62) we get

$$B\,e^{2b}(c'/c+db/dx^1)=A\,e^{4b}\,c'/c\,.$$

Putting $A/B=D$ we get $e^{2b}=(D-Ec^2)^{-1}$ (E const.)

$$a_1=A(D-Ec^2)^{-2}\,, \qquad a_\alpha=B(D-Ec^2)^{-1}\,.$$

Hence except for a similar mapping we have

$$ds^2=\frac{D}{(D-Ec^2)^2}(dx^1)^2+\frac{c^2}{D-Ec^2}d\sigma^2 \tag{64}$$

Here we will find p_{ij} in (7). By (7), (61) and (56)

$$db_1-b_1db=(db_1/dx^1-b_1^2)\,dx^1\,, \qquad -b_1\omega^1_\alpha=b_1c'\pi^\alpha=(b_1c'/c)\omega^\alpha\,.$$

Hence we get

$$p_{ij}=0\ (i \neq j)\,, \quad p_{11}=db_1/dx^1-b_1^2\,, \quad p_{\alpha\alpha}=b_1c'/c \quad \text{(not summed for } \alpha).$$

Thus the characteristic roots of the geodesic correspondence of $\bar{S}$ with S are all equal except one. In the case $c=\sin x^1$, $D=1$ we have

$$ds^2=\frac{1}{(1-E\sin^2 x^1)^2}(dx^1)^2+\frac{\sin^2 x^1}{(1-E\sin^2 x^1)}d\sigma^2 \tag{65}$$

and if $E=1$ we get (32).

10. Next we seek for the condition that the space (53) can be locally imbedded in the $(n+1)$-dimensional Euclidean space. It is so when and only when there exist auxiliary Pfaffian forms $\omega_i^{n+1}=-\omega_{n+1}^i$ such that

(66) $$\omega^i \wedge \omega_i^{n+1} = 0$$

(67) $$d\omega_i^j = \omega_i^k \wedge \omega_k^j + \omega_i^{n+1} \wedge \omega_{n+1}^j, \qquad d\omega_i^{n+1} = \omega_i^j \wedge \omega_j^{n+1},$$

because, by putting $\omega^{n+1}=0$ and $\omega_{n+1}^{n+1}=0$, these and

$$d\omega^i = \omega^j \wedge \omega_j^i$$

constitute structural equations in the $(n+1)$-dimensional Euclidean space. We assume that an imbedding is realized. We have then by (56)

$$d\omega_1^\alpha = c' d\pi^\alpha + dc' \wedge \pi^\alpha = c'\pi^\beta \wedge \omega_\beta^\alpha + dc' \wedge \pi^\alpha = \omega_1^\beta \wedge \omega_\beta^\alpha + dc' \wedge \pi^\alpha .$$

Hence by (67) $\omega_1^{n+1} \wedge \omega_{n+1}^\alpha = dc' \wedge \pi^\alpha$. We put $dc' = c''dx^1$ and assume $c'' \neq 0$. Thus we see

$$\omega_1^{n+1} = p dx^1 + q_\alpha \pi^\alpha, \qquad \omega_{n+1}^\alpha = r_\alpha dx^1 + s_\alpha \pi^\alpha, \qquad p s_\alpha - q_\alpha r_\alpha = c''$$

(not summed for α).

We assum $n \geqq 3$ and we get

$$\omega_1^{n+1} = p dx^1, \qquad \omega_{n+1}^\alpha = r_\alpha dx^1 + s\pi^\alpha, \qquad ps = c'' .$$

By virtue of the relation $d\omega^{n+1}=0$ we have

$$0 = \omega^1 \wedge \omega_1^{n+1} + \omega^\alpha \wedge \omega_\alpha^{n+1} = -\omega^\alpha \wedge r_\alpha dx^1 = -r_\alpha \omega^\alpha \wedge dx^1$$

and so $r_\alpha = 0$. Hence

(68) $$\omega_1^{n+1} = p dx^1, \qquad \omega_{n+1}^\alpha = s\pi^\alpha,$$

(69) $$ps = c'' .$$

Hence we get by (67)

$$d\omega_1^{n+1} = dp \wedge dx^1, \qquad d\omega_1^{n+1} = \omega_1^\alpha \wedge \omega_\alpha^{n+1} = -c'\pi^\alpha \wedge s\pi^\alpha = 0$$

and so

$$dp \wedge dx^1 = 0 .$$

Thus p is a function of one variable x^1 only and by (69) so is s too. By (68) and by (56)

$$d\omega_{n+1}^{a}=ds\wedge\pi^{a}+sd\pi^{a}=ds\wedge\pi^{a}+s\pi^{\beta}\wedge\omega_{\beta}^{a}=ds\wedge\pi^{a}+\omega_{n+1}^{\beta}\wedge\omega_{\beta}^{a}$$

$$d\omega_{n+1}^{a}=\omega_{n+1}^{1}\wedge\omega_{1}^{a}+\omega_{n+1}^{\beta}\wedge\omega_{\beta}^{a}=-pc'dx^{1}\wedge\pi^{a}+\omega_{n+1}^{\beta}\wedge\omega_{\beta}^{a}$$

and we get

$$ds=-pc'dx^{1}\,. \tag{70}$$

Next we consider (67) and get

$$d\omega_{a}^{\beta}-\omega_{a}^{\gamma}\wedge\omega_{\gamma}^{\beta}=\omega_{a}^{1}\wedge\omega_{1}^{\beta}+\omega_{a}^{n+1}\wedge\omega_{n+1}^{\beta}=-((c')^{2}+s^{2})\pi^{a}\wedge\pi^{\beta}\,.$$

Hence $d\sigma^{2}=\sum_{a}(\pi^{a})^{2}$ is of constant curvature. We put

$$(c')^{2}+s^{2}=K \qquad (K=\text{const.})\,. \tag{71}$$

Conversely if $d\sigma^2$ is of constant positive curvature, we determine s by (71) and p by (69). Then by differentiation of (71) $c'dc'+s\,ds=0$ and by (69) we get (70). Then an imbedding is realized by (68). Thus we get the following theorem.

THEOREM. *If $n\geqq 3$ and c is not a linear function in x^1, the space with the metric* (53) *is of imbedding class* 1, *when and only when the space with the metric $d\sigma^2$ is of positive constant curvature.*

As a corollary we have: if the space (26) is of constant positive curvature, then we can realize geodesic preserving deformation (65) from (26) to (32) in the $(n+1)$-dimensional Euclidean space.

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