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# Symmetrization and univalent functions in an annulus.

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## 1. Introduction and notations.

Recently the theory of symmetrization due to Pólya and Szegö has successfully been utilized in the theory of functions and potential theory by Hayman [3] and Jenkins [4]. In the present paper we will, by the method of symmetrization, obtain several results on omitted values of univalent functions in an annulus, which may be considered as extensions of theorems established by Goodman [1] and Jenkins [4].

For the purpose we take an annulus in the $z$-plane

$$D: \quad Q<|z|<1 \qquad (Q>0)$$

as a doubly-connected basic domain and consider a class $\mathfrak{F}$ of univalent functions $w=f(z)$ which are regular in $D$ and map $D$ onto subdomains of the domain $|w|>Q$ in such a way that the circle $|w|=Q$ corresponds to the circle $|z|=Q$.

In the sequel the Grötzsch's extremal function [2]

$$(1) \qquad w_q=f_0(z, q), \qquad f_0(Q, q)=q \qquad (0<q<Q),$$

which maps $D$ onto an annulus $q<|w_q|<1$ slit from $w_q=1$ to $w_q=\omega_q$ $(q<\omega_q<1)$ along the positive real axis, plays an important rôle and it is explicitly represented in terms of the elliptic function $\wp(u)$ in the form [6]

$$(2) \qquad k'(q)^2 \frac{\wp_q\left(\frac{1}{i} \lg w_q\right)-e_3(q)}{\wp_q\left(\frac{1}{i} \lg w_q\right)-e_2(q)}=k'(Q)^2 \frac{\wp_Q\left(\frac{1}{i} \lg z\right)-e_3(Q)}{\wp_Q\left(\frac{1}{i} \lg z\right)-e_2(Q)},$$

the primitive periods of $\wp_q(u)$ being $2\pi$ and $2i \lg(1/q)$, and $k'(q)$ being a complementary modulus of the elliptic function sn. For the brevity we denote the right-hand side by $S(z, Q)$. Then we have, instead of (2),

$$S(w_q, q)=S(z, Q)$$

or

$$w_q=S^{-1}\big(S(z, Q), q\big) \qquad (0<q<Q), \tag{3}$$

and

$$\omega_q=S^{-1}\big(k'(Q)^2, q\big) \qquad (0<q<Q). \tag{4}$$

## 2. Preliminary lemmas.

For later use we will establish several lemmas in the following.

LEMMA 1. *Let $w=f(z)$ be any function belonging to the class $\mathfrak{F}$ and let $d$ be the shortest distance from the origin $w=0$ to the outer boundary component of the image domain of the annulus $D$ by $w=f(z)$. Then it holds that*

$$PQ\leqq d\leqq 1, \tag{5}$$

*where $P\,(>1)$ is a value satisfying a relation $1/Q=\Phi(P)$, $\lg \Phi(P)$ being the modulus of the so-called Grötzsch's extremal domain $G_P$: $|w|>1$ with a slit along the ray $<P, +\infty>$. The result is best possible.*

PROOF. By means of the invariance of modulus of any ring domain under any conformal mapping and the well-known Grötzsch's theorem [2], the lemma is easily proved. The equality occurs in the left-hand side of (5) if and only if the image domain is the domain transformed by $w'=e^{i\theta}Qw$ ($\theta$ real) from $G_P$, and in the right-hand side of (5) if and only if the image domain is identical with the annulus $D$ itself.

It is known that $\omega_q$ of (4) is a strictly monotone increasing function of $q$ in the interval $(0, Q)$. Further we shall obtain another property of this function in the following

LEMMA 2. *$\omega_q/q$ is a strictly monotone increasing function of $q$ in the interval $(0, Q)$. More precisely stated, there holds*

$$\frac{d\lg\omega_q}{d\lg q}>1 \qquad in \qquad (0, Q), \tag{6}$$

*and further*

$$\lim_{q\to 0}\frac{\omega_q}{q}=P \quad and \quad \lim_{q\to Q}\frac{\omega_q}{q}=\frac{1}{Q} \qquad \Big(\frac{1}{Q}=\Phi(P)\Big). \tag{7}$$

PROOF. Owing to Komatu [6], we can conclude that the function $\omega_q$ of $q$ satisfies the following differential equation

$$\frac{d \lg \omega_q}{d \lg q} = 1 + 2 \sum_{n=1}^{\infty} \frac{\omega_q^n - q^{2n}/\omega_q^n}{1-q^{2n}} \qquad (0<q<Q).$$

Since $\omega_q > q$ in $(0, Q)$, there holds the inequality (6) and therefore the monotonicity of the function $\omega_q/q$ is an immediate consequence of (6). Rewriting (2) in terms of the elliptic $\vartheta$-functions, we obtain

$$\text{(8)} \qquad \prod_{n=1}^{\infty} \left( \frac{1-q^{2n-1}}{1+q^{2n-1}} \right)^4 \frac{\vartheta_3(0)\vartheta_4(v)}{\vartheta_4(0)\vartheta_3(v)} = k'(Q) \qquad \left( v = \frac{\lg \omega_q}{2\pi i} \right).$$

Putting $\lim\limits_{q \to 0} \omega_q/q = A$ and letting $q$ tend to 0 in (8), we have a relation

$$A = \frac{1+k'(Q)}{1-k'(Q)}.$$

Therefore it holds that $A = P\ \big(1/Q = \Phi(P)\big)$ [6]. Thus the first relation of (7) holds and the second of (7) is immediately obtained, since $\omega_q \to 1$ as $q \to Q$.

## 3. Circular symmetrization.

Let $D(f)$ be an image domain by any function $w=f(z) \in \mathfrak{F}$. With $D(f)$ we associate a domain $D^*(f)$ by circular symmetrization in the following manner: for all $s$, $Q<s<\infty$, if $D(f) \cap \{|w|=s\}$ consists of the whole circumference $|w|=s$, $D^*(f) \cap \{|w|=s\}$ shall do the same; otherwise $D^*(f) \cap \{|w|=s\}$ shall consist of a single arc on $|w|=s$ of length equal to that of $D(f) \cap \{|w|=s\}$ and centered at the point $w=-s$. Both ring domains $D(f)$ and $D^*(f)$ have the circle $|w|=Q$ as the inner boundary component.

It was proved by Pólya [9] that if $\Omega$ (or $\Omega^*$) denotes the harmonic measure of the outer boundary component of the respective ring domain $D(f)$ (or $D^*(f)$) there holds the following inequality relating to the Dirichlet integrals of both functions;

$$\text{(9)} \qquad \iint_{D(f)} (\operatorname{grad} \Omega)^2\, dudv \geqq \iint_{D^*(f)} (\operatorname{grad} \Omega^*)^2\, dudv, \quad (w=u+iv).$$

On the other hand it is well known that the Dirichlet integral of

$\Omega$ (or $\Omega^*$) equals $2\pi$/(modulus of the respective ring domain). Therefore we obtain the following

LEMMA 3. *Let $D(f)$ be an image ring domain by $w=f(z)\in\mathfrak{F}$, and denote by $D^*(f)$ the ring domain obtained by circular symmetrization of $D(f)$ in the above manner. Then it holds that*

$$\text{(10)} \qquad \operatorname{Mod} D(f) \leqq \operatorname{Mod} D^*(f),$$

*the notation* Mod *denoting the modulus of the respective domain.*

## 4. Generalization of Jenkins' theorem.

Recently J. A. Jenkins [4] has proved the following theorem: *Let $S$ denote the class of functions $f(z)$ regular and univalent for $|z|<1$ with the expansion $f(z)=z+a_2z^2+\cdots$ about $z=0$. Let $L(f, r)$ denote the length of the set of values on the circle $|w|=r\,(1/4<r<1)$ not covered by values of $f(z)\in S$ for $|z|<1$. Then there holds*

$$\text{(11)} \qquad L(f, r) \leqq 2r\cos^{-1}(8r^{\frac{1}{2}}-8r-1) \qquad (1/4<r<1).$$

*The result is best possible.*

In this section we will generalize the above theorem to the case where an annulus $D: Q<|z|<1$ is a doubly-connected basic domain. For the purpose we use the following canonical slit mapping function $w=H_q(z, p)$. This function maps an annulus $q<|z|<1$ onto the unit circle $|w|<1$ with a concentric circular slit, which is bisected by the positive real axis, as follows: $|w|=1$ corresponds to $|z|=1$, the slit to the circle $|z|=q$ and the origin $w=0$ to a point $z=-p$ $(q<p<1)$. Such mapping function can be determined uniquely and its explicit representation is given in the form [11]

$$\text{(12)} \qquad w=H_q(z, p)=z\,\frac{\theta(qz/p)}{\theta(pz/q)} \qquad (0<q<p<1),$$

$\theta(z)=\sum_{n=-\infty}^{+\infty} q^{n^2}z^n$. Since

$$\theta(z)=\vartheta_3(v/2)=\prod_{n=1}^{\infty}(1-q^{2n})\prod_{n=1}^{\infty}(1+q^{2n-1}z)(1+q^{2n-1}z^{-1}),$$

$$\bigl(z=\exp(v\pi i)\bigr),$$

we can find the values of the function $H_q(z, p)$. An alternative representation of this function was given by Komatu [7]. It is known that the circular slit is situated on the circle $|w|=p$ and centered at the point $w=p$. We denote the slit by $\{|w|=p, |\arg w|\leqq\delta(p, q)\}$. $\delta(p, q)$ depends upon $p$ and $q$.

Now we successively map the annulus $D$ onto new domains by the following functions:

(i) $\zeta=f_0(z, q)$, (ii) $\zeta_1=-\dfrac{q}{\zeta}$,

(iii) $\zeta_2=H_q(\zeta_1, p)$ $(p=q/\omega_q)$, (iv) $w=\dfrac{Q}{\zeta_2}$.

Thus the annulus $D$ is mapped onto a domain: $|w|>Q$ slit along a ray $\langle Q\,\omega_q/q, +\infty\rangle$ and along a concentric circular arc $\{|w|=Q\,\omega_q/q$, $|\arg w|\leqq\delta(q/\omega_q, q)\}$, in such a way that the circle $|w|=Q$ corresponds to the circle $|z|=Q$ and the slit to the circle $|z|=1$.

Using Lemma 2, for any value of $r$ $(PQ<r<1)$ we can uniquely determine $q$ $(0<q<Q)$ such that

$$\frac{Q\,\omega_p}{q}=r. \tag{13}$$

In other words, $q$ is uniquely determined as a function of $r$. Here we denote by $B(r)$ the above slit domain, i. e.

$$B(r):\begin{cases}|w|>Q \text{ slit along the ray } \langle r, +\infty\rangle \text{ and along the}\\ \text{concentric circular arc } \{|w|=r,\ |\arg w|\leqq\delta(Q/r, q)\}.\end{cases}$$

Obviously, $\delta(Q/r, q)$ is a function of $r$ only. For the brevity, we denote this function of $r$ by $\delta_1(r)$.

After above preparatory considerations, we obtain the following

THEOREM 1. *Let $L(f, r)$ denote the length of the set of values on the circle $|w|=r$ $(PQ<r<1)$ not covered by values of $f(z)\in\mathfrak{F}$ for the annulus $D: Q<|z|<1$. Then there holds*

$$L(f, r)\leqq 2r\delta_1(r). \tag{14}$$

*The inequality* (14) *is best possible.*

PROOF. Now suppose that we had $L(f, r)>2r\delta_1(r)$ for some $f\in\mathfrak{F}$ and some $r$ $(PQ<r<1)$. Let $D^*(f)$ be the domain obtained from $D(f)$

by circular symmetrization in the manner explained in § 3. Then $D^*(f)$ would be a proper subdomain of the domain $B(r)$. Hence we obtain an inequality

$$\text{Mod}\, D^*(f) < \text{Mod}\, B(r) \tag{15}$$

and further, by lemma 3, another inequality

$$\text{Mod}\, D(f) \leqq \text{Mod}\, D^*(f)\,. \tag{16}$$

Therefore it holds that

$$\text{Mod}\, D(f) < \text{Mod}\, B(r)\,. \tag{17}$$

Since the latter has the value $\lg(1/Q)$, we would be led to a contradiction. This proves the theorem. The exactness of the inequality (14) is easily recognized.

REMARK. In the above theorem, we determined the upper bound of $L(f, r)$ for any function $f \in \mathfrak{F}$ and for any value of $r$ in the interval $(PQ, 1)$. For any value of $r$, $Q < r \leqq PQ$ or $r \geqq 1$, $L(f, r) \equiv 0$ or $L(f, r) \equiv 2\pi r$, respectively, because of Lemma 1. Hence it is trivial to deal with the problem in such cases.

## 5. Generalization of Goodman's theorem.

Few years ago R. E. Goodman [1] proved the following theorem on omitted values: *Let $S$ denote the class of functions $f(z)$ regular and univalent for $|z|<1$ with the expansion $f(z)=z+a_2z^2+\cdots$ about $z=0$. Let $c$ be fixed, and suppose that for $|z|<1$, $f(z)$ omits all $\xi$ for which*

$$|\xi - c| \leqq R\,.$$

*Then there holds*

$$R \leqq |c| \frac{4|c|-1}{4|c|+1}\,. \tag{18}$$

*The inequality* (18) *is best possible.*

In order to generalize the theorem to the case of doubly-connected basic domain $D$, we now introduce the following linear transformation [5]:

$$\zeta = t\,\frac{R_1}{r_1}\,e^{i\Theta}\,\frac{d(w-w_1)-s(w_2-w_1)}{d(w-w_1)-t(w_2-w_1)}\,;\ \Theta \text{ real},\ d=|w_2-w_1|\,(>0)\,, \tag{19}$$

$s$ and $t\,(s>t)$ being the roots of the equations

$$(20) \qquad \begin{cases} s\cdot t = r_1^2, \\ (d-s)(d-t) = r_2^2; \end{cases}$$

$w_1, w_2, r_1\,(>0)$, $r_2\,(>0)$; $w_1 \neq w_2$ and $R_1\,(>0)$, $R_2\,(>0)$ being given but satisfying a relation

$$(21) \qquad \frac{R_2}{R_1} = \frac{r_2}{r_1}\,\frac{t}{d-t}.$$

By this transformation the whole $w$-plane with two circular holes $|w-w_1| \leqq r_1$ and $|w-w_2| \leqq r_2$ is mapped onto an annulus in the $\zeta$-plane: $R_2 < |\zeta| < R_1$, in such a way that the circle $|\zeta| = R_j (j=1, 2)$ corresponds to the circle $|w-w_j| = r_j\ (j=1, 2)$ and the circle $|\zeta| = R_1 t/r_1$ to the radical axis of two circles $|w-w_j| = r_j\ (j=1, 2)$.

Putting $w_1 = c\,(>0)$, $w_2 = 0$, $r_1 = R_0$, $r_2 = Q\ (<1)$, $R_1 = 1$, $R_2 = q\ (<1)$, $d = c$ and $\Theta = 0$ in (19) (20) and (21), we obtain a transformation

$$(22) \qquad \zeta = \frac{t}{R_0}\,\frac{w-c+s}{w-c+t},$$

$$(23) \qquad \begin{cases} st = R_0^2 \\ (c-s)(c-t) = Q^2, \end{cases} \qquad (t<s)$$

$$(24) \qquad q = \frac{Q}{R_0}\,\frac{t}{c-t}.$$

From (23), $t$ is the smaller root of the quadratic equation

$$(25) \qquad ct^2 - (c^2 + R_0^2 - Q^2)t + cR_0^2 = 0.$$

The inverse function of (22) maps the annulus $q < |\zeta| < 1$ onto the domain $B$: whole plane with two circular holes, $|w| \leqq Q$, $|w-c| \leqq R_0$. It is easily verified that the point $w=\infty$ corresponds to the point $\zeta = t/R_0$ and the point $w = c+R_0$ to the point $\zeta = 1$, and hence the ray $c+R_0 < w < +\infty$ to the segment $1 > \zeta > t/R_0$.

We shall now verify that for a proper choice of $q\ (0<q<Q)$ there holds a relation

$$(26) \qquad \omega_q = \frac{t}{R_0} \qquad (<1),$$

$\omega_q$ being the function of $q$ given by (4). For the purpose, eliminating $t$ from (25) and (26), we obtain

$$(27)\qquad \omega_q R_0^2 - c(\omega_q^2+1)R_0 + (c^2-Q^2)\,\omega_q = 0\,,$$

and analogously from (24) and (26)

$$(28)\qquad R_0 = \frac{c}{\omega_q} - \frac{Q}{q}\,.$$

Inserting (28) into (27), we obtain

$$(29)\qquad c = \frac{Q\,\omega_q(1-q^2)}{q(1-\omega_q^2)}\,.$$

Consider $c$ as a function of $q$ $(0<q<Q)$. Since $(q/\omega_q)(d\omega_q/dq)>1$ by (6) of lemma 2 and $1>q^2(1-\omega_q^2)/\omega_q^2(1-q^2)$, an inequality

$$\frac{d}{dq}\,\frac{1-q^2}{1-\omega_q^2}>0 \qquad (0<q<Q)$$

is easily deduced. Because of the monotonicities of both functions $(1-q^2)/(1-\omega_q^2)$ and $\omega_q/q$ we conclude that (29) considered as a function of $q$ is strictly monotone increasing in the interval $(0, Q)$. Further $c\to PQ$ as $q\to 0$ by (7) and $c\to+\infty$ as $q\to Q$. Hence, for any value of $c$ $(>PQ)$ we can uniquely determine the value of $q$ $(0<q<Q)$ satisfying the relation (29). By the value of $q$ thus obtained and (26) and (28), the value of $t$ and of $R_0$ are determined. It is easily shown that conversely these values satisfy the conditions (23) and (24). Thus the annulus $D: Q<|z|<1$ in the $z$-plane is successively mapped by the function $\zeta=f_0(z, q)$ and the inverse function of (22) onto the domain $B_c$: whole $w$-plane with two circular holes $|w|\leqq Q$, $|w-c|\leqq R_0$ and with the slit along the ray $<c+R_0, +\infty>$. Obviously such mapping function belongs to the class $\mathfrak{F}$.

After above preparatory considerations, we will prove the following

THEOREM 2. *Let $c$ be any fixed value $(|c|>PQ)$ and suppose that in the annulus $D$, $w=f(z)\in\mathfrak{F}$ omits all $\xi$ for which*

$$|\xi-c|\leqq R\,.$$

*Then there holds*

$$(30)\qquad R\leqq R_0 = \frac{|c|}{\omega_q} - \frac{Q}{q}\,,$$

*$q$ being determined by* (29) *and $\omega_q$ by* (4). *The inequality* (30) *is best possible.*

PROOF. Without loss of generality take $c>0$. Now suppose that we had $R>R_0$ for some $f(z)\in\mathfrak{F}$ and some $c$ $(PQ<c)$. Then we can, by the same method with that of proof of Theorem 1, deduce the inequality

$$\mathrm{Mod}\, D(f)<\mathrm{Mod}\, B_c\,.$$

Since the latter has the value $\lg(1/Q)$, we should be led to a contradiction. This proves the theorem. The exactness of the inequality (30) is easily recognized.

REMARK. By Lemma 1 it is easily recognized that, if $Q<|c|\leqq PQ$, $R_0=0$. Hence it is trivial to deal with the problem in such a case.

## 6. A subclass of $\mathfrak{F}$.

In this section we deal with a certain subclass $\mathfrak{F}_c$ of $\mathfrak{F}$, namely a class of univalent functions $\in\mathfrak{F}$ such that in $D: Q<|z|<1$

$$\mathrm{Re}\, f(z)>-c \qquad (c>0)\,.$$

In order to obtain a theorem on omitted values by any function $\in\mathfrak{F}_c$, we start with a preparatory consideration. Consider again the linear transformation (19). Putting $r_1=r_2=Q$, $w_1=-2c$, $w_2=0$, $d=2c$, $\Theta=\pi$, $R_1=1/q$ and $R_2=q$, we obtain

$$\zeta=-\frac{t}{Qq}\,\frac{w+2c-s}{w+2c-t}\,, \tag{31}$$

$$\begin{cases} st=Q^2\,, \\ s+t=2c \end{cases} \qquad (s>t)\,, \tag{32}$$

$$q^2=\frac{t}{2c-t}\,. \tag{33}$$

It is easily verified that by this transformation the half-plane $\mathrm{Re}\, w>-c$ with a circular hole $|w|\leqq Q$ is mapped onto an annulus $q<|\zeta|<1$. Here the circle $|\zeta|=q$ corresponds to the circle $|w|=Q$, the circle $|\zeta|=1$ to the straight line $\mathrm{Re} w=-c$ and the point $\zeta=1$ to the point $w=-c$. Since the straight line $\mathrm{Re}\, w=-c$ is the radical axis of the circles $|w|=Q$ and $|w+2c|=Q$, there holds

$$t = qQ. \tag{34}$$

Eliminating $t$ from (33) and (34), we have

$$c = \frac{Q}{2}\left(q + \frac{1}{q}\right). \tag{35}$$

Considering $c$ as a function of $q$ $(0 < q \leqq Q)$, $c$ is strictly monotone decreasing. Conversely, for any value of $c\left(\geqq (1+Q^2)/2\right)$, $q$ is uniquely determined so as to satisfy the relation (35). By the Grötzsch's function $\zeta = f_0(z, q)$ for the value of so determined $q$ and the inverse function of (31) $(t = qQ)$, the annulus $D$ can be mapped onto a domain $G_c$ in the $w$-plane:

$$G_c: \begin{cases} \text{the half-plane } \operatorname{Re} w > -c \text{ with a circular hole } |w| \leqq Q \\ \text{and with a slit along the segment } \langle -c, \gamma_0 \rangle, \text{ where} \end{cases}$$

$$\gamma_0 = -\frac{(2c-s)+(2c-t)\omega_q}{1+\omega_q} \quad (<0). \tag{36}$$

Here the circle $|w| = Q$ corresponds to the circle $|z| = Q$. Therefore such a mapping function belongs to the class $\mathfrak{F}_c$.

After above consideration, we can prove the following

THEOREM 3. *Let* $f(z) \in \mathfrak{F}_c$ *omit* $\gamma$ *in the annulus* $D: Q < |z| < 1$. *Then there holds*

$$|\gamma| \geqq |\gamma_0| = \frac{(2c-s)+(2c-t)\omega_q}{1+\omega_q}, \qquad \left(c \geqq (1+Q^2)/2\right), \tag{37}$$

*$q$ being determined by* (35) *and* $s$, $t$ *by* (32). *The inequality* (37) *is best possible.*

The proof of this theorem is similar to that of Theorem 1 or 2, and will be omitted. We only notice that in this case for the symmetrized domain $D^*(f)$, $D^*(f) \cap \{|w| = s\}$ $(Q < s)$ should be centered at the point $w = s$.

REMARK. It is easily verified that for any function $f(z) \in \mathfrak{F}_c$

$$c \geqq (1+Q^2)/2.$$

## 7. Bounded univalent functions.

In this section we deal with a subclass $\mathfrak{F}_M$ of $\mathfrak{F}$, namely a class of bounded univalent functions: $|f(z)| < M$ $(M > 1)$. Using Grötzsch's extremal function, consider a function

$$w=Mf_0(z, q) \qquad (q=Q/M), \tag{38}$$

which maps the annulus $D: Q<|z|<1$ onto an annulus $Q<|w|<M$ slit along a segment $<M\omega_{Q/M}, M>$. This function obviously belongs to the class $\mathfrak{F}_M$.

By the method of symmetrization we obtain the following

THEOREM 4. *Let $f(z)\in\mathfrak{F}_M$ omit $\gamma$ in the annulus $D: Q<|z|<1$. Then there holds*

$$|\gamma|\geqq M\omega_{Q/M}, \tag{39}$$

*$\omega_q(q=Q/M)$ being determined by* (4). *The inequality* (39) *is best possible.*

The proof of this theorem is similar with that of Theorem 1, and will be omitted. We only notice that the theorem can also be deduced from Grötzsch's distortion theorem [2] or Komatu's theorem [6] without using the method of symmetrization.

## 8. Steiner symmetrization.

In this section we deal with a certain class of univalent functions different from the class $\mathfrak{F}$, namely the class $\mathfrak{C}$ of all univalent functions $w=f(z)$ in the annulus $D: Q<|z|<1$ which map $D$ onto subdomains of the strip domain $S: 0<\mathrm{Re}\, w<1$ in such a way that the whole boundary of $S$ ($\mathrm{Re}\, w=0$ and $\mathrm{Re}\, w=1$) corresponds to the circle $|z|=1$. For simplicity we denote the inner boundary component of the image ring domain by $\Gamma$ and put $h(f)=\underset{w\in\Gamma}{\mathrm{Osc}}\,(\mathrm{Im}\, w)$. Then we have the following

THEOREM 5. *Let $f(z)$ be any function $\in\mathfrak{C}$. Then there holds the following inequality*

$$h(f)\leqq\frac{1}{\pi}\lg\frac{1}{k'(Q)}, \tag{40}$$

*$k'(Q)$ being the complementary modulus of the* sn*-function with the primitive periods $2\pi$ and $2i\lg(1/Q)$. The inequality* (40) *is best possible.*

PROOF. Denote by $D(f)$ the image ring domain by $w=f(z)$. Let $D^*(f)$ be the domain obtained from $D(f)$ by Steiner symmetrization

[10] with respect to the line $\operatorname{Re} w=\frac{1}{2}$. Speaking more precisely, let $T=S-D(f)$, and let $T^*$ be the set obtained from the set $T$ by Steiner symmetrization with respect to $\operatorname{Re} w=\frac{1}{2}$, i. e. the set intersected by any line $v=$const. $(w=u+iv)$ in a single segment of length equal to the total intersection of $T$ with this line and centered at the point $w=\frac{1}{2}+iv$. Thus $D^*(f)$ is defined as follows: $D^*(f)=S-T^*$. Then, in the case of Steiner symmetrization, Lemma 3 also holds for $f(z)\in\mathfrak{C}$. Therefore we obtain

$$\operatorname{Mod} D(f)\leqq\operatorname{Mod} D^*(f). \tag{41}$$

Now suppose that we had

$$h(f)>\frac{1}{\pi}\lg\frac{1}{k'(Q)}$$

for some $f(z)\in\mathfrak{C}$. Then $D^*(f)$ would be a proper subdomain of the strip domain $S$ with a slit along the line $\operatorname{Re} w=\frac{1}{2}$ and of length $(1/\pi)\lg\left(1/k'(Q)\right)$. Now we denote by $S_0$ such a slit-domain. Hence we obtain another inequality

$$\operatorname{Mod} D^*(f)<\operatorname{Mod} S_0. \tag{42}$$

From (41) and (42) we have

$$\operatorname{Mod} D(f)<\operatorname{Mod} S_0. \tag{43}$$

Since the latter has the value $\lg(1/Q)$, as was shown by the author [8], we should be led to a contradiction. This proves the theorem. The exactness of (40) is obvious.

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