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On the exceptional set of a certain harmonic function in a unit sphere.

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1. Main theorem.

In the former paper[1], I have proved, by generalizing Beurling's theorem[2], the following theorem.

THEOREM 1. *Let $f(z)$ be regular in $|z|<1$ and*

$$\iint_{|z|<1} |f'(z)|^2 dxdy < \infty, \quad z = x+iy.$$

Then there exists a set E on $|z|=1$, which is of logarithmic capacity zero, such that if $e^{i\theta}$ does not belong to E, then

$$\lim_{z\to e^{i\theta}} f(z) = f(e^{i\theta}) \ (\neq \infty) \text{ exists and uniformly,}$$

when z tends to $e^{i\theta}$ from the inside of any Stolz domain, whose vertex is at $e^{i\theta}$ and for any rectilinear segment l, which connects $e^{i\theta}$ to a point of $|z|<1$,

$$\int_l |f'(z)||dz| < \infty.$$

From this, we have

THEOREM 2. *Let $u(z)$ be harmonic in $|z|<1$ and*

$$\iint_{|z|<1} |\text{grad}\, u(z)|^2 dxdy < \infty.$$

Then there exists a set E on $|z|=1$, which is of logarithmic capacity zero, such that if $e^{i\theta}$ does not belong to E, then

$$\lim_{z\to e^{i\theta}} u(z) = u(e^{i\theta}) \ (\neq \infty) \text{ exists and uniformly,}$$

1) M. Tsuji: Beurling's theorem on exceptional sets. Tohoku Math. Journ. 2 (1950).
2) A. Beurling: Ensembles exceptionels. Acta Math. 72 (1940).

when z *tends to* $e^{i\theta}$ *from the inside of any Stolz domain, whose vertex is at* $e^{i\theta}$ *and for any rectilinear segment* l, *which connects* $e^{i\theta}$ *to a point of* $|z|<1$,

$$\int_l |\text{grad}\, u(z)||dz| < \infty .$$

In this paper, I shall prove the following similar theorem for a harmonic function in a unit sphere.

THEOREM 3. *Let* Δ *be the inside of a unit sphere* S *about the origin* O *and* $u(x,y,z)=u(P)$ $(P=(x,y,z))$ *be harmonic in* Δ *and*

$$\iiint_\Delta |\text{grad}\, u(P)|^2 \frac{dv_P}{\sqrt{1-r^2}} < \infty , \qquad r=\overline{OP},$$

where dv_P *is the volume element. Then there exists a set* E *on* S, *which is of Newtonian capacity zero, such that if* $Q\in S$ *does not belong to* E, *then*

$$\lim_{P\to Q} u(P)=u(Q) \ (\neq \infty) \text{ exists and uniformly,}$$

when P *tends to* Q *from the inside of any Stolz domain*[3], *whose vertex is at* Q *and for any rectilinear segment* l, *which connects* Q *to a point of* Δ,

$$\int_l |\text{grad}\, u(P)|\, ds < \infty ,$$

where ds *is the arc element on* l.

Since $|du|=\left|\dfrac{du}{ds}\right| ds \leqq |\text{grad}\, u(P)|\, ds$,

$$\int_l |du| < \infty ,$$

where the left hand side is the total variation of $u(P)$ on l. First we shall prove some lemmas.

2. Lemmas.

LEMMA 1. $I=\displaystyle\int_0^\infty \frac{t\, dt}{(t^2-2at+1)^{\frac{3}{2}}} = \frac{1}{1-a}, \quad 0<a<1.$

3) A stolz domain is a domain, which is bounned by a cone, whose vertex is at Q and whose generator makes an angle θ_0 $\left(<\frac{\pi}{2}\right)$ with the radius OQ.

PROOF. From

$$I(a, R)=\int_0^R \frac{dt}{\sqrt{t^2-2at+1}}=\log(R-a+\sqrt{R^2-2aR+1})-\log(1-a),$$

we have

$$I=\lim_{R\to\infty}\frac{\partial I(a, R)}{\partial a}=\frac{1}{1-a}.$$

LEMMA 2. *If $u(x,y,z)$ is harmonic, then*

$$|\operatorname{grad} u(P)|=\sqrt{\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2}$$

is subharmonic.

PROOF. We put

$$v=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2,$$

then

$$\frac{\partial v}{\partial x}=2\left(\frac{\partial u}{\partial x}\cdot\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial y}\cdot\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial^2 u}{\partial x\,\partial z}\right),$$

$$\Delta v=2\left(\sum\left(\frac{\partial^2 u}{\partial x^2}\right)^2+2\sum\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right).$$

By Schwarz's inequality,

$$\left(\frac{\partial v}{\partial x}\right)^2\leqq 4v\left(\left(\frac{\partial^2 u}{\partial x^2}\right)^2+\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2+\left(\frac{\partial^2 u}{\partial x\partial z}\right)^2\right),$$

so that

$$\sum\left(\frac{\partial v}{\partial x}\right)^2\leqq 4v\left(\sum\left(\frac{\partial^2 u}{\partial x^2}\right)^2+2\sum\left(\frac{\partial^2 u}{\partial x\partial y}\right)^2\right)=2v\,\Delta v.$$

If we put $w=\sqrt{v}$, then

$$4v^{\frac{3}{2}}\,\Delta w=2v\,\Delta v-\sum\left(\frac{\partial v}{\partial x}\right)^2\geqq 0,\qquad \Delta w\geqq 0,$$

hence $w=|\operatorname{grad} u|$ is subharmonic.

LEMMA 3. *Let O be the origin and* $(\rho, \theta, \varphi)\left(\rho\geqq 0, 0\leqq\theta\leqq\frac{\pi}{2},\right.$

$0\leq\varphi\leq 2\pi\big)$ *be the polar coordinates of a point* $P=(x, y, z)$ *and* D *be the conical domain, such that*

$$D:\ 0<\rho<R,\qquad 0<\theta<\theta_0\left(<\frac{\pi}{2}\right),\qquad 0\leq\varphi\leq 2\pi,$$

and F *be its boundary.*

Let $u(P)=u(\rho, \theta, \varphi)$ *be subharmonic in* D *and continuous in* $\overline{D}$, *except at* O, *such that*

$$\iiint_D |u(P)|^2\, dv_P<\infty,\qquad \iint_F |u(Q)|\, d\sigma_Q<\infty,$$

where dv_P *is the volume element and* $d\sigma_Q$ *is the surface element. Then for* $P\in D$,

(i) $$u(P)\leq\frac{1}{4\pi}\iint_F u(Q)\frac{\partial G(P;Q)}{\partial\nu}d\sigma_Q,$$

where $G(P; Q)$ *is the Green's function of* D *with* P *as its pole and* ν *is the inner normal of* F *at* Q.

(ii) *If* $u(P)\geq 0$ *in* D, *then*

$$\int_0^R u(t, 0, 0)\, dt\leq MR+\frac{1}{\pi}\int_0^{2\pi}d\varphi\int_0^R u(\rho, \theta_0, \varphi)\, d\rho,$$

where $M=\underset{P\in S_R}{\mathrm{Max}}\, u(P)$, S_R *being the part of* F, *which lies on a sphere* $\rho=R$.

Proof. Let $0<\rho<R$ and D_ρ be the part of D, which lies in a half-space $z>\rho$ and $G_\rho(P; Q)$ be its Green's function with P as its pole. Then the boundary F_ρ of D_ρ consists of three parts:

$$F_\rho=S_R+\Sigma_\rho+\sigma_\rho,$$

where Σ_ρ is the part of F_ρ, for which $x^2+y^2+z^2<R^2$, $z>\rho$ and σ_ρ is that, which lies on a plane $z=\rho$.

Since D_ρ is convex, D_ρ is contained in a half-space H_Q, which lies in one side of a tangent plane π_Q of F_ρ at $Q\in F_\rho$.

Since $G_\rho(P; Q)$ is majorated by the Green's function of H_Q, we have, when P is fixed,

$$\frac{\partial G_\rho(P;Q)}{\partial \nu} \leq M_0 \quad (=\text{const.}), \tag{1}$$

for any $Q \in F_\rho$ and for small values of ρ.

Since $\iint_F |u(Q)|\, d\sigma_Q < \infty$, we can find $\rho_0 = \rho_0(\varepsilon)$ for any small $\varepsilon > 0$, such that

$$\int_0^{2\pi}\int_0^{\rho_0} |u(\rho, \theta_0, \varphi)|\, \rho \sin\theta_0 \, d\rho \, d\varphi < \varepsilon . \tag{2}$$

Let $0 < \rho < \rho_0$. Since $u(P)$ is continuous in $\overline{D}_\rho$, we have for $P \in D_{\rho_0}$,

$$u(P) \leq \frac{1}{4\pi}\iint_{F_\rho} u(Q) \frac{\partial G_\rho(P;Q)}{\partial \nu} d\sigma_Q = \frac{1}{4\pi}\iint_{S_R} u(Q) \frac{\partial G_\rho(P;Q)}{\partial \nu} d\sigma_Q$$

$$+ \frac{1}{4\pi}\iint_{\Sigma_{\rho_0}} u(Q) \frac{\partial G_\rho(P;Q)}{\partial \nu} d\sigma_Q + \frac{1}{4\pi}\iint_{\Sigma_\rho - \Sigma_{\rho_0}} u(Q) \frac{\partial G_\rho(P;Q)}{\partial \nu} d\sigma_Q$$

$$+ \frac{1}{4\pi}\iint_{\sigma_\rho} u(Q) \frac{\partial G_\rho(P;Q)}{\partial \nu} d\sigma_Q = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}, \tag{3}$$

where

$$\lim_{\rho\to 0} \mathrm{I} = \frac{1}{4\pi}\iint_{S_R} u(Q) \frac{\partial G(P;Q)}{\partial \nu} d\sigma_Q ,$$

$$\lim_{\rho\to 0} \mathrm{II} = \frac{1}{4\pi}\iint_{\Sigma_{\rho_0}} u(Q) \frac{\partial G(P;Q)}{\partial \nu} d\sigma_Q .$$

By (1), (2),

$$|\mathrm{III}| \leq \frac{M_0}{4\pi}\iint_{\Sigma_\rho - \Sigma_{\rho_0}} |u(Q)|\, d\sigma_Q < \frac{M_0 \varepsilon}{4\pi},$$

so that for $P \in D_{\rho_0}$,

$$u(P) \leq \frac{1}{4\pi}\iint_{S_R} u(Q) \frac{\partial G(P;Q)}{\partial \nu} d\sigma_Q$$

$$+ \frac{1}{4\pi}\iint_{\Sigma_{\rho_0}} u(Q) \frac{\partial G(P;Q)}{\partial \nu} d\sigma_Q + \frac{M_0 \varepsilon}{4\pi} + \overline{\lim_{\rho\to 0}}\, \mathrm{IV} . \tag{4}$$

Since

$$|\mathrm{IV}| \leqq \frac{M_0}{4\pi} \iint_{\sigma_\rho} |u(Q)| \, d\sigma_Q ,$$

$$|\mathrm{IV}|^2 \leqq \left(\frac{M_0}{4\pi}\right)^2 \iint_{\sigma_\rho} d\sigma_Q \iint_{\sigma_\rho} |u(Q)|^2 \, d\sigma_Q = O(\rho^2) \iint_{\sigma_\rho} |u(Q)|^2 d\sigma_Q .$$

Since

$$\int_0^{\rho_0} \frac{|\mathrm{IV}|^2}{\rho^2} \, d\rho \leqq O(1) \int_0^{\rho_0} d\rho \iint_{\sigma_\rho} |u(Q)|^2 \, d\sigma_Q \leqq O(1) \iiint_D |u(P)|^2 dv_P < \infty ,$$

there exists $\rho_\nu \to 0$, such that $\mathrm{IV} \to 0$, hence from (4),

$$u(P) \leqq \frac{1}{4\pi} \iint_{S_R} u(Q) \frac{\partial G(P; Q)}{\partial \nu} \, d\sigma_Q$$

$$+ \frac{1}{4\pi} \iint_{\Sigma_{\rho_0}} u(Q) \frac{\partial G(P; Q)}{\partial \nu} \, d\sigma_Q + \frac{M_0 \varepsilon}{4\pi} ,$$

so that if we make $\rho_0 \to 0$, we have

$$u(P) \leqq \frac{1}{4\pi} \iint_F u(Q) \frac{\partial G(P; Q)}{\partial \nu} , \quad P \in D . \tag{5}$$

Hence (i) is proved.

To prove (ii), let $P = (t, 0, 0)$ $(0 < t < R)$ and $Q = (\rho, \theta_0, \varphi)$ $(0 < \rho < R)$, then we shall prove that

$$\frac{\partial G(P; Q)}{\partial \nu} \leqq \frac{2t \sin \theta_0}{(t^2 - 2t\rho \cos \theta_0 + \rho^2)^{\frac{3}{2}}} , \tag{6}$$

where ν is the inner normal of F at Q.

Let π_Q be the tangent plane of F at Q. We choose the coordinate axes (ξ, η, ζ), such that Q is the origin and π_Q is the $\xi\eta$-plane, the line $\overrightarrow{OQ}$ coincides with the positive ξ-axis and ν coincides with the positive ζ-axis.

Then D lies in a half-space $\zeta > 0$. Let $G_0(P; M)$ be the Green's function of the half-space $\zeta > 0$, with P as its pole, then $G(P; M)$ is majorated by $G_0(P; M)$, such that

$$G(P;M)\leqq G_0(P;M)=\frac{1}{r}-\frac{1}{r_1},\quad r=\overline{PM},\quad r_1=\overline{P_1M},$$

where P_1 is the image of P with respect to π_Q.

Since $G(P;M)$ and $G_0(P;M)$ vanish at $M=Q$, we have

$$\frac{\partial G(P;Q)}{\partial\nu}\leqq\frac{\partial G_0(P;Q)}{\partial\nu}=\frac{2\cos\varphi}{PQ^2},$$

where φ is the angle between $\overrightarrow{QP}$ and ν.

Since $\cos\varphi=\dfrac{t\sin\theta_0}{PQ}$,

$$\frac{\partial G(P;Q)}{\partial\nu}\leqq\frac{2t\sin\theta_0}{PQ^3}=\frac{2t\sin\theta_0}{(t^2-2t\rho\cos\theta_0+\rho^2)^{\frac{3}{2}}}. \tag{6}$$

Hence putting $t=\rho\tau$, we have by Lemma 1,

$$\int_0^R\frac{\partial G(P;Q)}{\partial\nu}dt\leqq\int_0^R\frac{2t\sin\theta_0\,dt}{(t^2-2t\rho\cos\theta_0+\rho^2)^{\frac{3}{2}}}$$

$$<\frac{2\sin\theta_0}{\rho}\int_0^\infty\frac{\tau\,d\tau}{(\tau^2-2\tau\cos\theta_0+1)^{\frac{3}{2}}}=\frac{2\sin\theta_0}{\rho(1-\cos\theta_0)}$$

$$=\frac{2(1+\cos\theta_0)}{\rho\sin\theta_0}<\frac{4}{\rho\sin\theta_0}. \tag{7}$$

From (5), we have by putting $P=(t,0,0)$

$$u(t,0,0)\leqq\frac{1}{4\pi}\iint_{S_R}u(Q)\frac{\partial G(P;Q)}{\partial\nu}d\sigma_Q$$

$$+\frac{1}{4\pi}\int_0^{2\pi}\int_0^R u(Q)\frac{\partial G(P;Q)}{\partial\nu}\rho\sin\theta_0\,d\rho\,d\varphi,$$

where $Q=(\rho,\theta_0,\varphi)$ in the second integral.

If $u(P)\geqq0$ in D and $M=\underset{P\in S_R}{\mathrm{Max}}\,u(P)$, then since $\displaystyle\iint_{S_R}\frac{\partial G(P;Q)}{\partial\nu}d\sigma_Q\leqq4\pi$, we have by (7)

$$\int_0^R u(t,0,0)\,dt\leqq MR+\frac{1}{4\pi}\int_0^{2\pi}d\varphi\int_0^R u(Q)\rho\sin\theta_0\,d\rho\int_0^R\frac{\partial G(P;Q)}{\partial\nu}dt$$

$$< MR + \frac{1}{\pi}\int_0^{2\pi} d\varphi \int_0^R u(\rho, \theta_0, \varphi)\, d\rho\,.$$

Hence (ii) is proved.

LEMMA 4. *Let D be the same domain as in Lemma 3, such that*

$$D:\ 0<\rho<R\,,\quad 0<\theta<\theta_0\left(<\frac{\pi}{2}\right),\quad 0<\varphi<2\pi\,.$$

Let $u(P)$ be harmonic in $\overline{D}$, except at the origin O, such that

$$\iiint_D |\operatorname{grad} u(P)|^2\, dv_P < \infty\,,\quad \iiint_D |\operatorname{grad} u(P)| \frac{dv_P}{\rho^2} < \infty\,,\quad \rho = \overline{OP}\,.$$

Then for $0<\theta<\theta_1\,(<\theta_0)$, $0<\varphi<2\pi$,

(i) $\lim_{\rho\to 0} u(\rho, \theta, \varphi) = u_0\ (\neq\infty)$ *exists and uniformly,*

(ii) $\int_0^R |\operatorname{grad} u(\rho, \theta, \varphi)|\, d\rho < K\ (=\text{const.})$.

PROOF. By Lemma 2, $|\operatorname{grad} u|$ is subharmonic. Since

$$\iiint_D |\operatorname{grad} u(P)| \frac{dv_P}{\rho^2} = \int_0^{\theta_0} d\theta \int_0^{2\pi}\int_0^R |\operatorname{grad} u(\rho, \theta, \varphi)| \sin\theta\, d\rho\, d\varphi < \infty\,,$$

we have for almost all θ of $[0, \theta_0]$,

$$\int_0^{2\pi}\int_0^R |\operatorname{grad} u(\rho, \theta, \varphi)| \sin\theta\, d\rho\, d\varphi < \infty\,,$$

a fortiori,

$$\int_0^{2\pi}\int_0^R |\operatorname{grad} u(\rho, \theta, \varphi)|\, \rho \sin\theta\, d\rho\, d\varphi < \infty\,.$$

Hence for a non-exceptional θ, $|\operatorname{grad} u(P)|$ satisfies the condition of Lemma 3, so that for a non-exceptional θ,

$$\int_0^R |\operatorname{grad} u(t, 0, 0)|\, dt \leq MR + \frac{1}{\pi}\int_0^{2\pi} d\varphi \int_0^R |\operatorname{grad} u(\rho, \theta, \varphi)|\, d\rho\,,$$

where $$M = \operatorname*{Max}_{P \in S_R} |\operatorname{grad} u(P)|\,.$$

Since $\int_0^{\theta_0} \sin\theta \, d\theta = 1-\cos\theta_0$, by multiplying $\sin\theta$ and integrating over $[0, \theta_0]$, we have

$$\int_0^R |\operatorname{grad} u(t, 0, 0)| \, dt \leq MR$$

$$+\frac{1}{\pi(1-\cos\theta_0)}\int_0^{\theta_0}\int_0^{2\pi}\int_0^R |\operatorname{grad} u(\rho, \theta, \varphi)| \sin\theta \, d\rho \, d\varphi \, d\theta$$

$$= MR + \frac{1}{\pi(1-\cos\theta_0)}\iiint_D |\operatorname{grad} u(P)| \frac{dv_P}{\rho^2}. \tag{1}$$

From this, we see that for $0 \leq \theta \leq \theta_1 (< \theta_0)$, $0 \leq \varphi \leq 2\pi$,

$$\int_0^R |\operatorname{grad} u(\rho, \theta, \varphi)| \, d\rho \leq$$

$$MR + \frac{1}{\pi(1-\cos\delta)}\iiint_D |\operatorname{grad} u(P)| \frac{dv_P}{\rho^2} = K < \infty, \quad (\delta = \theta_0 - \theta_1). \tag{2}$$

Hence (ii) is proved.

From (2), we see that for $0 \leq \theta < \theta_0$, $0 \leq \varphi \leq 2\pi$,

$$\lim_{\rho \to 0} u(\rho, \theta, \varphi) = \lambda(\theta, \varphi) \quad (\neq \infty) \tag{3}$$

exists.

If we put $M_1 = \operatorname{Max}_{P \in S_R} |u(P)|$, then we see from (2) that $u(P)$ is bounded, such that for $0 \leq \theta \leq \theta_1 (< \theta_0)$, $0 \leq \varphi \leq 2\pi$, $0 < \rho \leq R$,

$$|u(\rho, \theta, \varphi)| \leq M_1 + K. \tag{4}$$

Let

$$L(\rho, \theta_1) = \int_0^{2\pi} |\operatorname{grad} u(\rho, \theta_1, \varphi)| \rho \sin\theta_1 \, d\varphi, \tag{5}$$

then by (2),

$$\int_0^R \frac{L(\rho, \theta_1)}{\rho} \, d\rho \leq \int_0^{2\pi} d\varphi \int_0^R |\operatorname{grad} u(\rho, \theta_1, \varphi)| \, d\rho \leq 2\pi K,$$

so that there exists $\rho_\nu \to 0$, such that $L(\rho_\nu, \theta_1) \to 0$. Since

$$|u(\rho_\nu, \theta_1, \varphi) - u(\rho_\nu, \theta_1, \varphi')| \leq \int_{\varphi'}^{\varphi} |\operatorname{grad} u(\rho_\nu, \theta_1, \varphi)| \rho_\nu \sin\theta_1 \, d\varphi \leq L(\rho_\nu, \theta_1) \to 0,$$

we have in (3),

$$\lim_{\rho\to 0} u(\rho, \theta_1, \varphi) = \lim_{\rho\to 0} u(\rho, \theta_1, \varphi'),$$

so that $\lambda(\theta_1, \varphi)$ is independent of φ, such that

$$\lim_{\rho\to 0} u(\rho, \theta_1, \varphi) = \lambda(\theta_1) \quad (0 \leqq \varphi \leqq 2\pi). \tag{6}$$

Since by (4), $u(P)$ is bounded and the origin O is a regular point for Dirichlet problem, we see from (6), that

$$\lim_{\rho\to 0} u(\rho, \theta, \varphi) = u_0 \quad (\neq \infty) \tag{7}$$

uniformly for $0 \leqq \theta \leqq \theta_1$, $0 \leqq \varphi \leqq 2\pi$.

Hence (i) is proved.

LEMMA 5. *Let C be a unit circle on the xy-plane about the origin O and C_1 be a circle of radius 1/2, which touches C at $Q=(1,0)$ internally and Δ, Δ_1 be the inside of C and C_1 respectively. Let P be any point of Δ and $r=\overline{OP}$, $\rho=\overline{PQ}$ and ψ be the angle between $\overrightarrow{OP}$ and $\overrightarrow{PQ}$. Then for $P \in \Delta - \Delta_1$,*

$$\frac{|\cos\psi|}{\rho^2} \leqq \frac{2}{\sqrt{1-r^2}}.$$

PROOF. We remark that $\cos\psi \leqq 0$ for $P \in \Delta - \Delta_1$.
Let $P=(x, y) \in \Delta - \Delta_1$, then $x^2+y^2 \geqq x$,

$$r^2 = x^2+y^2, \qquad \rho^2 = 1+x^2+y^2-2x,$$

so that

$$\rho^2 \geqq 1+x^2+y^2-2(x^2+y^2) = 1-(x^2+y^2) = 1-r^2. \tag{1}$$

Let θ be the angle between $\overrightarrow{QO}$ and $\overrightarrow{QP}$, then

$$r^2 = 1+\rho^2-2\rho\cos\theta, \qquad 1 = r^2+\rho^2-2r\rho|\cos\psi|.$$

If we eliminate r^2 from these equations, we have

$$r|\cos\psi| = \rho-\cos\theta \leqq \rho.$$

Hence if $r \geqq 1/2$, then

$$|\cos\psi| \leqq 2\rho\,. \tag{2}$$

If $0 \leqq r \leqq 1/2$, then $\rho \geqq 1/2$, so that $2\rho \geqq 1$, hence (2) holds in general. From (1), (2), we have

$$\frac{|\cos\psi|}{\rho^2} \leqq \frac{2}{\sqrt{1-r^2}}\,.$$

LEMMA 6. *Let S be a unit sphere about the origin O and E be a closed set on S, which is of Newtonian capacity $\gamma(E)>0$ and D be the complement of E with respect to the whole space.*

Then there exists a positive mass distribution $d\mu(Q)$ on E of total mass 1, such that if we put

$$w(P) = \int_E \frac{d\mu(Q)}{r_{PQ}}\,, \qquad \int_E d\mu(Q) = 1\,,$$

then

$$\iiint_D |\operatorname{grad} w(P)|^2\, dv_P \leqq \frac{4\pi}{\gamma(E)} < \infty\,.$$

PROOF. Let Δ_ρ be an open set, which contains E in its inside and whose boundary F_ρ consists of a finite number of analytic Jordan surfaces, each point of which is of distance $<\rho$ from E and D_ρ be the complement of $\bar{\Delta}_\rho$ with respect to the whole space. Then there exists a positive mass distribution $d\mu_\rho(Q)$ on F_ρ of total mass 1, such that if we put

$$w_\rho(P) = \int_{F_\rho} \frac{d\mu_\rho(Q)}{r_{PQ}}\,, \qquad \int_{F_\rho} d\mu_\rho(Q) = 1\,, \tag{1}$$

then $w_\rho(P)$ is of constant value $\dfrac{1}{\gamma(F_\rho)}$ on F_ρ.

Let $F_\rho^{(\varepsilon)}$ be the niveau surface $w_\rho(P) = \dfrac{1}{\gamma(F_\rho)} - \varepsilon$ ($\varepsilon > 0$) and $D_\rho^{(\varepsilon)}$ be the complement of the inside of $F_\rho^{(\varepsilon)}$. Then by Green's formula,

$$\iiint_{D_\rho^{(\varepsilon)}} |\operatorname{grad} w_\rho(P)|^2\, dv_P = \iint_{F_\rho^{(\varepsilon)}} w_\rho \frac{\partial w_\rho}{\partial \nu}\, d\sigma = \left(\frac{1}{\gamma(F_\rho)} - \varepsilon\right) \iint_{F_\rho^{(\varepsilon)}} \frac{\partial w_\rho}{\partial \nu}\, d\sigma$$

$$=4\pi\left(\frac{1}{\gamma(F_\rho)}-\varepsilon\right)\int_{F_\rho} d\mu_\rho(Q)=4\pi\left(\frac{1}{\gamma(F_\rho)}-\varepsilon\right)<\frac{4\pi}{\gamma(F_\rho)} .$$

Hence for $\varepsilon\to 0$,

$$\iiint_{D_\rho} |\text{grad}\, w_\rho(P)|^2\, dv_P \leq \frac{4\pi}{\gamma(F_\rho)} . \tag{2}$$

Since the total mass of $d\mu_\rho(Q)$ is 1, we can find $\rho_\nu\to 0$, such that $d\mu_{\rho_\nu}(Q)\to d\mu(Q)$, where $d\mu(Q)$ is a positive mass distribution on E of total mass 1. Hence $w_{\rho_\nu}(P)$ tends to

$$w(P)=\int_E \frac{d\mu(Q)}{r_{PQ}}, \quad \int_E d\mu(Q)=1 . \tag{3}$$

Since $\gamma(F_{\rho_\nu})\to\gamma(E)$, we have from (2),

$$\iiint_{D_\rho} |\text{grad}\, w(P)|^2\, dv_P \leq \frac{4\pi}{\gamma(E)} ,$$

so that for $\rho\to 0$,

$$\iiint_{D} |\text{grad}\, w(P)|^2\, dv_P \leq \frac{4\pi}{\gamma(E)} < \infty .$$

3. Proof of Theorem 3.

Let Δ be the inside of a unit sphere S about the origin O and Q be a point of S and $\Delta(Q)$ be the inside of a sphere of radius 1/2, which touches S at Q internally. Let E be a set of $Q\in S$, such that

$$\chi(Q)=\iiint_{\Delta(Q)} |\text{grad}\, u(P)| \frac{\cos\psi}{\rho^2}\, dv_P=\infty , \quad \rho=PQ , \tag{1}$$

where ψ is the angle between $\overrightarrow{OP}$ and $\overrightarrow{PQ}$.

Then we shall prove that $\gamma(E)=0$, where $\gamma(E)$ is the Newtonian capacity of E.

Suppose that $\gamma(E)>0$, then we may assume that E is closed. Let $w(P)$ be the potential function defined by Lemma 6, such that

$$w(P)=\int_E \frac{d\mu(Q)}{r_{PQ}}, \quad \int_E d\mu(Q)=1, \quad \iiint_\Delta |\text{grad}\, w(P)|^2 dv_P<\infty. \tag{2}$$

We put

$$I=\iiint_\Delta |\text{grad}\, u(P)| \frac{\partial w}{\partial r} dv_P, \qquad r=\overline{OP}, \tag{3}$$

then

$$I^2\leq\iiint_\Delta |\text{grad}\, u(P)|^2 dv_P \iiint_\Delta \left(\frac{\partial w}{\partial r}\right)^2 dv_P$$

$$\leq\iiint_\Delta |\text{grad}\, u(P)|^2 dv_P \iiint_\Delta |\text{grad}\, w(P)|^2 dv_P<\infty. \tag{4}$$

Since

$$\frac{\partial w(P)}{\partial r}=\int_E \frac{\cos\psi}{\rho^2} d\mu(Q), \qquad \rho=PQ, \tag{5}$$

we have

$$I=\int_E d\mu(Q) \iiint_\Delta |\text{grad}\, u(P)| \frac{\cos\psi}{\rho^2} dv_P. \tag{6}$$

Since $\cos\psi\geq0$ for $P\in\Delta(Q)$ and $\cos\psi\leq0$ for $P\in\Delta-\Delta(Q)$,

$$I=\int_E d\mu(Q) \iiint_{\Delta(Q)} |\text{grad}\, u(P)| \frac{\cos\psi}{\rho^2} dv_P$$

$$-\int_E d\mu(Q) \iiint_{\Delta-\Delta(Q)} |\text{grad}\, u(P)| \frac{|\cos\psi|}{\rho^2} dv_P. \tag{7}$$

Since by Lemma 5, for $P\in\Delta-\Delta(Q)$, $\dfrac{|\cos\psi|}{\rho^2}\leq\dfrac{2}{\sqrt{1-r^2}}$,

we have

$$\iiint_{\Delta-\Delta(Q)} |\text{grad}\, u(P)| \frac{|\cos\psi|}{\rho^2} dv_P\leq 2\iiint_{\Delta-\Delta(Q)} \frac{|\text{grad}\, u(P)|\, dv_P}{\sqrt{1-r^2}}$$

$$\leq 2\left[\iiint_\Delta \frac{dv_P}{\sqrt{1-r^2}} \iiint_\Delta \frac{|\text{grad}\, u(P)|^2 dv_P}{\sqrt{1-r^2}}\right]^{\frac{1}{2}}$$

$$=2\pi\left[\iiint_\Delta \frac{|\text{grad}\, u(P)|^2 dv_P}{\sqrt{1-r^2}}\right]^{\frac{1}{2}}=K,$$

so that by (1),

$$I \geqq \int_E \chi(Q)\, d\mu(Q) - K = \infty\ ,$$

which contradicts (4). Hence $\gamma(E)=0$.

Hence if $Q \in S$ does not belong to E, then

$$\chi(Q) = \iiint_{\Delta(Q)} |\operatorname{grad} u(P)| \frac{\cos\psi}{\rho^2}\, dv_P < \infty\ . \tag{8}$$

Let $\Delta_{\theta_0}(Q)$ $\left(0<\theta_0<\frac{\pi}{2}\right)$ be the part of $\Delta(Q)$, which lies in a cone, whose vertex is at Q and whose generator makes an angle θ_0 with $\overrightarrow{QO}$, then for $P \in \Delta_{\theta_0}(Q)$, $\cos\psi \geqq a > 0$, where a is a constant, so that

$$\iiint_{\Delta_{\theta_0}(Q)} |\operatorname{grad} u(P)| \frac{dv_P}{\rho^2} < \infty\ . \tag{9}$$

Since

$$\iiint_{\Delta_{\theta_0}(Q)} |\operatorname{grad} u(P)|^2\, dv_P \leqq \iiint_{\Delta} |\operatorname{grad} u(P)|^2 \frac{dv_P}{\sqrt{1-r^2}} < \infty\ , \tag{10}$$

Theorem 3 follows from Lemma 4.

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