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Theorems in the geometry of numbers for Fuchsian groups.

By Masatsugu TSUJI

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1. We introduce a non-euclidean metric in $|z|<1$ by

$$ds=\frac{2|dz|}{1-|z|^2},$$

so that the non-euclidean radius r of a circle $|z|=\rho<1$ is

$$r=\log\frac{1+\rho}{1-\rho} \tag{1}$$

and the non-euclidean measure $\sigma(E)$ of a measurable set E in $|z|<1$ is

$$\sigma(E)=\iint_E \frac{4\,rdrd\theta}{(1-r^2)^2} \qquad (z=re^{i\theta}),$$

hence the non-euclidean area of a disc $\varDelta: |z|\leqq r<1$ is

$$\sigma(\varDelta)=\frac{4\pi r^2}{1-r^2}. \tag{2}$$

Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain. Let E be a measurable set in $|z|<1$ and E_ν $(\nu=0,1,2,\cdots)$ be its equivalents by G and $A(r, E_\nu)$ be the non-euclidean measure of the part of E_ν contained in $|z|\leqq r$ and put

$$A(r,E)=\sum_{\nu=0}^{\infty} A(r,E_\nu). \tag{3}$$

If $\sigma(D_0)<\infty$, then I have proved in another paper[1]) that

$$\int_0^r \frac{A(r,E)}{r}\,dr=\frac{2\pi\,\sigma(E)}{\sigma(D_0)}\log\frac{1}{1-r}+O(1) \qquad (r\to 1). \tag{4}$$

1) M. Tsuji: Theory of Fuchsian groups. Jap. Journ. Math. 21 (1951).

By means of (4), we shall prove

THEOREM 1. *Let G be a Fuchsian group and D_0 be its fundamental domain, such that $\sigma(D_0) < \infty$. Let E be a measurable set in $|z| < 1$. If $\sigma(E) > \sigma(D_0)$, then the equivalents E_ν of E overlap.*

PROOF. Suppose that E_ν do not overlap, then

$$A(r, E) \leqq \int_0^r \int_0^{2\pi} \frac{4\, t dt d\theta}{(1-t^2)^2} = \frac{4\pi r^2}{1-r^2},$$

so that

$$\int_0^r \frac{A(r, E)}{r} dr \leqq 2\pi \log \frac{1}{1-r} + O(1) \qquad (r \to 1),$$

hence by (4),

$$\frac{2\pi\sigma(E)}{\sigma(D_0)} \log \frac{1}{1-r} + O(1) \leqq 2\pi \log \frac{1}{1-r} + O(1),$$

so that, making $r \to 1$, we have $\sigma(E) \leqq \sigma(D_0)$. Hence if $\sigma(E) > \sigma(D_0)$, then E_ν overlap.

THEOREM 2. *Let G be a Fuchsian group and D_0 be its fundamental domain, such that $\sigma(D_0) < \infty$, and z_ν $(\nu = 0, 1, 2, \cdots)$ be equivalents of $z = 0$. Let $\Delta : |z| \leqq \rho < 1$ be a disc. If*

$$\sigma(\Delta) \geqq 4\,\sigma(D_0) + \frac{\sigma^2(D_0)}{\pi}, \quad or \quad \rho \geqq \sqrt{1 - \frac{4\pi^2}{(\sigma(D_0) + 2\pi)^2}},$$

then Δ contains one z_ν $(\neq 0)$.

This is an analogue of Minkowski's theorem.

PROOF. Let $\Delta_t : |z| \leqq t$ $(0 \leqq t < 1)$. We increase t from $t = 0$ to $t = 1$ and let ρ be the smallest value of t, such that Δ_ρ contains z_ν $(\neq 0)$. We choose ρ_0, such that the non-euclidean radius of Δ_{ρ_0} is one-half of that of Δ_ρ, so that $\log \frac{1-\rho}{1-\rho} = \log \left(\frac{1+\rho_0}{1-\rho_0}\right)^2$, or

$$\rho = \frac{2\rho_0}{1+\rho_0^2}. \tag{5}$$

Then the equivalents of Δ_{ρ_0} do not overlap, so that by Theorem 1,

$$\sigma(\Delta_{\rho_0}) = \frac{4\pi\rho_0^2}{1-\rho_0^2} \leqq \sigma(D_0),$$

hence from (5),

$$\sigma(\varDelta_\rho)=\frac{4\pi\rho^2}{1-\rho^2}=\frac{16\pi\rho_0^2}{(1-\rho_0)^2}=\frac{16\pi\rho_0^2}{1-\rho_0^2}\left(1+\frac{\rho_0^2}{1-\rho_0^2}\right)$$

$$\leqq 4\sigma(D_0)\bigl(1+\sigma(D_0)/4\pi\bigr)=4\sigma(D_0)+\sigma^2(D_0)/\pi, \text{ or}$$

$$\rho\leqq\sqrt{1-\frac{4\pi}{(\sigma(D_0)+2\pi)^2}}\,, \tag{6}$$

so that, if $\sigma(\varDelta_\rho)>4\sigma(D_0)+\sigma^2(D_0)/\pi$, then $\varDelta$ contains one $z_\nu\ (\neq 0)$. If $\sigma(\varDelta_\rho)=4\sigma(D_0)+\sigma^2(D_0)/\pi$, then considering a slightly larger disc, we see that $\varDelta$ contains one $z_\nu\ (\neq 0)$. Hence our theorem is proved.

2. We consider special cases of Theorem 2. Let F be a closed Riemann surface of genus $p\geqq 2$ spread over the w-plane and we map the universal covering surface $F^{(\infty)}$ of F on $|z|<1$. Then we have a Fuchsian group G in $|z|<1$, whose fundamental domain D_0 is bounded by $4p$ orthogonal circles to $|z|=1$, such that $\sigma(D_0)=4\pi(p-1)$. Then (6) becomes $\rho\leqq\sqrt{1-\frac{1}{(2p-1)^2}}$. Hence

THEOREM 3. *Let G be a Fuchsian group, which corresponds to a closed Riemann surface of genus $p\geqq 2$. If*

$$\rho\geqq\sqrt{1-\frac{1}{(2p-1)^2}}\,,$$

then a disc $\varDelta:|z|\leqq\rho<1$ contains one $z_\nu\ (\neq 0)$.

Let D_0 be a domain in $|z|<1$ bounded by $p\,(\geqq 3)$ orthogonal circles $C_i\ (i=1,2,\cdots,p)$ to $|z|=1$, where C_i, C_{i+1} touch each other at a point on $|z|=1$. We invert D_0 on one of C_i and performing inversions indefinitely, we obtain a modular figure and let G be the group of all inversions and $z_\nu\ (\nu=0,1,2,\cdots)$ be equivalents of $z=0$ by G, then D_0 is its fundamental domain, such that $\sigma(D_0)=\pi(p-2)$, hence by Theorem 3, we have

THEOREM 4. *If $\rho\geqq\sqrt{1-\frac{4}{p^2}}$, then a disc $\varDelta:|z|\leqq\rho<1$ contains one $z_\nu\ (\neq 0)$.*

3. We divide the $z=x+iy$-plane by parallel lines $x=n$ and $y=m$ ($n, m=0, \pm 1, \pm 2,\cdots$) into squares of equal sides, which we call cells. In each cell, let k points be given, which are congruent mod. 1 to those in other cells. We call the totality of these points lattice points. Let E be a measurable set of measure $m\,E$. Then we can translate

E into E', such that the number of lattice points contained in E' is $\geqq k\, m\, E$ and we can translate E into E'', such that the number of lattice points contained in E'' is $\leqq k\ m\, E$[2]. We shall prove an analogous theorem for Fuchsian groups.

Let G be a Fuchsian group and D_0 be its fundamental domain, and k points $z_0^1, \cdots, z_0^k$ be given in D_0 and z_ν^i $(\nu=0, 1, 2, \cdots)$ be equivalents of z_0^i. We call the totality of these points lattice points. We call a linear transformation of the form:

$$z' = \frac{z+a}{1+\bar{a}z} \qquad (|a|<1)$$

a (non-euclidean) translation. Then we shall prove

THEOREM 5. *Let G be a Fuchsian group and D_0 be its fundamental domain, such that $\sigma(D_0)<\infty$. Let $\Delta : |z|<\rho_0<1$ be a disc, then we can translate Δ into Δ', such that the number of lattice points contained in Δ' is $\geqq k\dfrac{\sigma(\Delta)}{\sigma(D_0)}$ and we can translate Δ into Δ'', such that the number of lattice points contained in Δ'' is $\leqq k\dfrac{\sigma(\Delta)}{\sigma(D_0)}$.*

PROOF. For a fixed i $(1\leqq i\leqq k)$, we put a mass 1 at each z_ν^i $(\nu=0, 1, 2, \cdots)$, then we have a mass distribution μ^i in $|z|<1$, so that the number of z_ν^i contained in a set E is

$$\mu^i(E) = \int_E d\,\mu^i(a)\,.$$

Let T_a: $z' = \dfrac{z+a}{1+\bar{a}z}$ $(|a|<1)$ be a translation and put $\Delta(a)=T_a(\Delta)$. If $\Delta(a)$ contains z_ν^i, then

$$\left|\frac{a-z_\nu^i}{1-\bar{z}_\nu^i\, a}\right| < \rho_0\,, \tag{7}$$

so that a is contained in an equivalent Δ_ν^i of the disc

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2) Blichfeld: A new principle in the geometry of numbers, with some applications. Trans. Amer. Math. Soc. 15 (1914).

M. Tsuji: On Blichfeld's theorem in the geometry of numbers. Jap. Journ. Math. 19 (1948).

$$\Delta_0^i: \left| \frac{z-z_0^i}{1-\bar{z}_0^i z} \right| < \rho_0, \tag{8}$$

so that

$$\int_{|a|<r} \mu^i\big(\Delta(a)\big)\, d\sigma(a) = \sum_{\nu=0}^{\infty} A(r, \Delta_\nu^i) = A(r, \Delta_0^i).$$

Hence if we put $\mu = \mu^1 + \cdots + \mu^k$, then

$$\int_{|a|<r} \mu\big(\Delta(a)\big)\, d\sigma(a) = A(r, \Delta_0^1) + \cdots + A(r, \Delta_0^k),$$

where $\mu\big(\Delta(a)\big)$ is the number of lattice points contained in $\Delta(a)$.

If we put $M = \underset{|a|<r}{\mathrm{Max}}\, \mu\big(\Delta(a)\big)$, then

$$M \int_{|a|<r} d\sigma(a) = \frac{4\pi M r^2}{1-r^2} \geqq A(r, \Delta_0^1) + \cdots + A(r, \Delta_0^k).$$

We multiply dr/r and integrate on $(0, r)$, then by (4), since $\sigma(\Delta_0^1) = \cdots = \sigma(\Delta_0^k) = \sigma(\Delta)$, we have

$$2\pi M \log \frac{1}{1-r} + O(1) \geqq \sum_{i=1}^{k} \frac{2\pi \sigma(\Delta_0^i)}{\sigma(D_0)} \log \frac{1}{1-r} + O(1)$$

$$= \frac{2\pi k\, \sigma(\Delta)}{\sigma(D_0)} \log \frac{1}{1-r} + O(1),$$

hence, making $r \to 1$, we have $M \geqq k\sigma(\Delta)/\sigma(D_0)$. Since M is an integer, there exists a_0 ($|a_0| < 1$), such that $M = \mu\big(\Delta(a_0)\big)$, so that

$$\mu\big(\Delta(a_0)\big) \geqq k\, \sigma(\Delta)/\sigma(D_0). \tag{9}$$

Similarly putting $m = \mathrm{Min}\, \mu\big(\Delta(a)\big)$, we have for a suitable a_1 ($|a_1| < 1$),

$$\mu\big(\Delta(a_1)\big) \leqq k\, \sigma(\Delta)/\sigma(D_0). \tag{10}$$

Hence our theorem is proved.

If $k=1$ and $\sigma(\Delta) < \sigma(D_0)$, then $\mu\big(\Delta(a_1)\big) = 0$, so that

THEOREM 6. *If $\sigma(\Delta) < \sigma(D_0)$, then we can translate Δ into Δ', such that Δ' does not contain equivalents of $z=0$.*

Mathematical Institute,
Tokyo University.