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On the Differential Forms of the First Kind on Algebraic Varieties II.

Shoji KOIZUMI.

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We shall give some supplementary remarks to my previous paper on the same subject[1]. As in Weil's definition, we shall call differential forms to be of the first kind on a Variety $\boldsymbol{U}$, when they are finite at every simple Point on every Variety birationally equivalent to $\boldsymbol{U}$. This definition is equivalent to my previous one in [K], if $\boldsymbol{U}$ has a birationally equivalent model which is a complete Variety without singularities.

1. We shall prove the following theorem as an extension of the theorem 2 in [K].

THEOREM 1. Let ω be a differential form of the first kind of degreer on a Product-Variety $\boldsymbol{U}\times\boldsymbol{V}$, then we have the following expression

$$\omega=\sum\sigma_i\tau_i$$

where σ_i, τ_i are, respectively, differential forms of the first kind on $\boldsymbol{U}$, $\boldsymbol{V}$, of degree d_i, $r-d_i$. Moreover, if ω, $\boldsymbol{U}$ and $\boldsymbol{V}$ have a common field k of definition which is perfect, σ_i, τ_i are defined over k.

PROOF. Without loss of generality we may suppose that ω, $\boldsymbol{U}$ and $\boldsymbol{V}$ are defined over a perfect field k. Let $\boldsymbol{P}$ and $\boldsymbol{Q}$ be independent generic Points over k, of $\boldsymbol{U}$ and $\boldsymbol{V}$, respectively. If (t) and (u) are respectively, sets of uniformizing parameters at $\boldsymbol{P}$ and $\boldsymbol{Q}$, on $\boldsymbol{U}$ and $\boldsymbol{V}$, then

$$\begin{aligned}\omega&=\sum_{(i,j)}z_{i_1,\dots,i_s;j_1,\dots,i_{r-s}}dt_{i_1}\cdots dt_{i_s}du_{j_1}\cdots du_{j_{r-s}}\\&=\sum_j\Big(\sum_i z_{(i,j)}dt_{i_1}\cdots dt_{i_s}\Big)du_{j_1}\cdots du_{j_{r-s}}\end{aligned}$$

where $z_{i_1,\dots,i_s;\,j_1,\dots,j_{r-s}}$ are contained in $k(\boldsymbol{P},\boldsymbol{Q})$ and (i,j) means $i_1,\dots,i_s;\,j_1,\dots,j_{r-s}$. If we consider $\sum_i z_{(i,j)}dt_{i_1}\cdots dt_{i_s}$ as defined on $\boldsymbol{U}$ over the field $k(\boldsymbol{Q})$, they are of the first kind.

1) Journal of the Mathematical Society of Japan Vol. 1, No. 3, 1949. This note will be denoted by [K], and we shall use the same terminologies and notations as in [K].

Therefore from the theorem 4 in $[K]$[2], we obtain

$$\sum_{i} z_{(i,j)}\, dt_{i_1}\cdots dt_{i_s} = \sum u_{(j)k}\sigma_k$$

where $u_{(j)k}$ are contained in $k(Q)$, and σ_k are the differential forms of the first kind on U defined over k.
So it follows

$$\omega = \sum \sigma_i \tau_i$$

where σ_i are as above and τ_i are the differential forms on V over k. We shall denote $dt_{i_1}\cdots dt_{i_s}$ by T_α $(\alpha = 1, \cdots\cdots, N)$, and may take the following base $\{\sigma_{i,j}\}$ of the vector space over k spanned by $\{\sigma_i\}$

$$\sigma_0 = u_0$$

$$\begin{cases} \sigma_{1,1} = u^1_{1,1} T_1 + u^2_{1,1} T_2 + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_{1,a_1} = u^1_{1,a_1} T_1 + u^2_{1,a_1} T_2 + \cdots \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{cases} \sigma_{i,1} = u^i_{i,1} T_i + u^{i+1}_{i,1} T_{i+1} + \cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \sigma_{i,a_i} = u^i_{i,a_i} T_i + u^{i+1}_{i,a_i} T_{i+1} + \cdots \end{cases}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

where $(u^i_{i,1}, u^i_{i,2}, \cdots, u^i_{i,a_i})$ are contained in $k(P)$ and are linearly independent over k. Then ω has the following expression

$$\begin{aligned} \omega &= \sigma_0\tau_0 + \sigma_{1,1}\tau_{1,1} + \cdots + \sigma_{1,a_1}\tau_{1,a_1} + \cdots + \sigma_{i,1}\tau_{i,1} + \cdots + \sigma_{i,a_i}\tau_{i,a_i} + \cdots \\ &= u_0\tau_0 \pm (u^1_{1,1}\tau_{1,1} + \cdots + u^1_{1,a_1}\tau_{1,a_1})\, T_1 \pm \cdots \\ &\qquad \pm (u^i_{1,1}\tau_{1,1} + \cdots + u^i_{i,1}\tau_{i,1} + \cdots + u^i_{i,a_i}\tau_{i,a_i})\, T_i + \cdots \end{aligned}$$

Since the differential forms $u^i_{1,1}\tau_{1,1} + \cdots + u^i_{i,1}\tau_{i,1} + \cdots + u^i_{i,a_i}\tau_{i,a_i}$ on V are of the first kind, by the proposition 10 in $[K]$, we may show by the mathematical induction that $\tau_{i,j}$ are of the first kind. q. e. d.

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2) It is not necessary for this theorem for U to have a complete model without singularities, which is birationally equivalent to U over k. From the theory of normal varieties we may easily deduce that the theorem holds true if k in perfect.

2. Next, we consider an extension of the theorem 2 in [K]. We begin with the following definition.

DEFINITION 1. *Let ω be a differential form on a Variety $\boldsymbol{U}$. ω and $\boldsymbol{U}$ are defined over a field k and $\boldsymbol{P}$ is a generic Point of $\boldsymbol{U}$ over k. We say that ω has the property (F) at a point $\boldsymbol{P}'$ on $\boldsymbol{U}$, if ω has the following expression*

$$\omega=\sum u_i dv_i,$$

where u_i and v_i are contained in the specialization ring of $\boldsymbol{P}'$ in $k(\boldsymbol{P})$

It is evident that the property (F) at $\boldsymbol{P}'$ is equivalent to the finiteness at $\boldsymbol{P}'$, when $\boldsymbol{P}'$ is a simple Point on $\boldsymbol{U}$.

PROPOSITION 1. *Let ω be a differential form on a complete Variety $\boldsymbol{U}$. If ω has the property (F) everywhere on $\boldsymbol{U}$, ω is of the first kind.*

PROOF. Let V^n be a birationally equivalent variety over k to $\boldsymbol{U}$, and P be a simple $(n-1)$-dimensional point over k on V. If $\boldsymbol{Q}$ on $\boldsymbol{U}$ is a birationally corresponding Point to P, the specialization ring of P includes that of $\boldsymbol{Q}$. Therefore if ω has the property (F) at $\boldsymbol{Q}$ on $\boldsymbol{U}$, it has the same property at P on V. This proves the proposition.

PROPOSITION 2. *Let $\boldsymbol{V}$ be a simple Subvariety of $\boldsymbol{U}$ and $\boldsymbol{P}$ be a point on $\boldsymbol{V}$. If a differential form ω on $\boldsymbol{U}$ is finite on $\boldsymbol{V}$, it induces a differential form ω' on $\boldsymbol{V}$. Moreover,, if ω has the property (F) at $\boldsymbol{P}$ on $\boldsymbol{U}$, ω' has also the same property at $\boldsymbol{P}$ on $\boldsymbol{V}$.*

PROOF. The first assertion follows from the proposition 6 in [K]. Let k be a common field of definition for $\boldsymbol{U}$, $\boldsymbol{V}$ and ω, and $\bar{\boldsymbol{P}}$, $\bar{\boldsymbol{Q}}$ be, respectively, the generic Points of $\boldsymbol{U}$, $\boldsymbol{V}$ over k. Every element in the specialization ring of $\boldsymbol{P}$ in $k(\bar{\boldsymbol{P}})$ has a uniquely determined specialization over $\bar{\boldsymbol{P}}\rightarrow\bar{\boldsymbol{Q}}$ with respect to k, and that specialization is contained in the specialization ring of $\boldsymbol{P}$ in $k(\bar{\boldsymbol{Q}})$. This proves the proposition.

From these two propositions we can obtain at once the following theorem.

THEOREM 2. *Let $\boldsymbol{U}$ be a complete Variety without singularities, and $\boldsymbol{V}$ be a simple Subvariety of $\boldsymbol{U}$. If a differential form ω on $\boldsymbol{U}$ is of the first kind, it induces a differential form ω' on $\boldsymbol{V}$ of the first kind.*

Mathematical Institute, Nagoya University.