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PARALLEL MEAN CURVATURE TORI IN $\mathbb{C}P^2$ AND $\mathbb{C}H^2$

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Abstract. We explicitly determine tori that have a parallel mean curvature vector, both in the complex projective plane and the complex hyperbolic plane.

1. Introduction. Parallel mean curvature tori in real four-dimensional space forms are tori with constant mean curvature(cmc) that lie on a totally geodesic hypersurface or an umbilical hypersurface of the ambient space, as proved independently by Hoffman [15], Chen [4], Yau [24]. The study of such cmc tori in real three-dimensional space forms is a topic in the theory of submanifolds (see, for example, Wente [23], Abresch [1], Pinkall and Sterling [20], Bobenko [3], and Andrews and Li [2]). In the above-mentioned studies, parallel mean curvature tori in the complex two-plane $\mathbb{C}^2$ are classified by identifying $\mathbb{C}^2$ with real Euclidean four-space. In this paper, we study such tori in non-flat two-dimensional complex space forms. The main result of this paper tells us that the non-flat case is much simpler than the flat one. In fact, Theorem 4.4 of this paper states that a non-zero parallel mean curvature torus in a non-flat two-dimensional complex space form is flat and totally real. Then, in Theorem 4.5, applying a result of Ohnita [19] or Urbano [22], we explicitly determine parallel mean curvature tori in the complex projective plane $\mathbb{C}P^2$ and the complex hyperbolic plane $\mathbb{C}H^2$, endowed with their canonical structures of Kaehler surfaces.

We remark that parallel mean curvature spheres in $\mathbb{C}P^2$ and $\mathbb{C}H^2$ have been classified by Hirakawa [14] and Fetcu [9]. Additional studies regarding parallel mean curvature surfaces in various ambient spaces that are neither real nor complex space forms can be found in Torralbo and Urbano [21], Fetcu and Rosenberg [10, 11, 12], and Ferreira and Tribuzy [8].

In this paper, we first study the local structure of a parallel mean curvature surface in a two-dimensional complex space form using different methods from those in our previous works [16, 17]. The new ingredient in this paper is Lemma 2.4, which proves the existence of special isothermal coordinates fitting to the geometry of a parallel mean curvature vector. Then, we determine explicitly the second fundamental form of the surface for these coordinates. By coupling these results with generalised Hopf differentials found in Ogata [18] and Fetcu [9], we prove our main results, Theorems 4.4 and 4.5.

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2. Preliminaries. Let $\overline{M}[4\rho]$ be a complex two-dimensional complex space form with constant holomorphic sectional curvature 4ρ. Furthermore, let M be an oriented and connected real two-dimensional Riemannian manifold with Gaussian curvature K and $x : M \longrightarrow \overline{M}[4\rho]$ be an isometric immersion, with Kaehler angle α such that the mean curvature vector H is nonzero and parallel for the normal connection on the normal bundle of the immersion. We remark that the case with $H = 0$ has been studied in Chern and Wolfson [5] and Eschenburg, Guadalupe, and Tribuzy [7].

Let $M_0 = \{p \in M \mid x$ is neither holomorphic nor anti-holomorphic at $p\}$. M_0 is an open dense subset of M. Because all of the calculations and formulas on M_0 presented in Ogata [18] are valid until page 400, according to the remark in Hirakawa [14], there exists a local field of unitary coframes $\{w_1, w_2\}$ on M_0 such that, by restricting it to x, the Riemannian metric ds^2 on M_0 can be written as $ds^2 = \phi\bar{\phi}$, where $\phi = \cos\alpha/2 \cdot \omega_1 + \sin\alpha/2 \cdot \bar{\omega}_2$. Let a and c be the complex valued functions on M_0 that determine the second fundamental form of x. Then, the Kaehler angle α and the complex 1-form ϕ satisfy

$$d\alpha = (a + b)\phi + (\bar{a} + b)\bar{\phi}\,, \tag{2.1}$$

$$d\phi = (\bar{a} - b)\cot\alpha \cdot \phi \wedge \bar{\phi}\,, \tag{2.2}$$

where $2b = |H| > 0$. By (2.4), (2.5), and (2.6) of Ogata [18], we have

$$K = -4(|a|^2 - b^2) + 6\rho\cos^2\alpha\,, \tag{2.3}$$

$$da \wedge \phi = -\left(2a(\bar{a} - b)\cot\alpha + \frac{3}{2}\rho\sin\alpha\cos\alpha\right)\phi \wedge \bar{\phi}\,, \tag{2.4}$$

$$dc \wedge \bar{\phi} = 2c(a - b)\cot\alpha \cdot \phi \wedge \bar{\phi}\,, \tag{2.5}$$

$$|c|^2 = |a|^2 + \frac{\rho}{2}(-2 + 3\sin^2\alpha)\,, \tag{2.6}$$

where (2.3) is the Gauss equation, (2.4) and (2.5) are the Codazzi-Mainardi equations, and (2.6) is the Ricci equation of x.

REMARK 2.1. (1) The immersion x is holomorphic (resp. anti-holomorphic) at $p \in M$ if and only if $\alpha = 0$ (resp. $\alpha = \pi$) at p. Hence, it holds that $\sin\alpha \neq 0$ on M_0.

(2) The unitary coframes $\{w_1, w_2\}$ used in (2.1)–(2.5) are uniquely determined up to the orientations of both $\overline{M}[4\rho]$ and M_0 (see [18]), hence the complex one-form ϕ on M_0 is unique up to its sign and conjugacy.

Let us take an isothermal coordinate for the Riemannian metric $\phi\bar{\phi}$ on M_0 that makes M_0 a Riemann surface with a local complex coordinate z, and put

$$\Phi_1 = (8ba - 3\rho\sin^2\alpha)\phi^2\,, \quad \Phi_2 = \bar{c}\phi^2\,.$$

We will use the following facts that were proved in Ogata [18] and Fetcu [9].

LEMMA 2.2. *The quadratic forms Φ_1 and Φ_2 on M_0 are holomorphic.*

For completeness, we will present a proof of Lemma 2.2. The quadratic forms Q and Q' on M in Fetcu [9] are written in our terminology as

$$Q = (8b(\bar{c}+a) - 3\rho \sin^2\alpha)\phi^2\,, \quad Q' = (8b(\bar{c}-a) + 3\rho \sin^2\alpha)\phi^2\,. \tag{2.7}$$

Let $\phi = \lambda dz$ ($\lambda \neq 0$). By using (2.1)–(2.5), we see that

$$d(\lambda^2(8b(\bar{c}+a) - 3\rho\sin^2\alpha)) \wedge \phi = 0\,, \quad d(\lambda^2(8b(\bar{c}-a) + 3\rho\sin^2\alpha)) \wedge \phi = 0\,,$$

showing that both forms of (2.7) are holomorphic on M_0. By taking both the addition and the subtraction of the two forms of (2.7), we proved Lemma 2.2

The local structure of the immersion satisfying $a = \bar{a}$ on M_0 has been determined in Kenmotsu and Zhou [16] and Kenmotsu [17]. We remark that the classification table of those surfaces can also be found in Hirakawa [14]. By applying these results, we obtain the following lemma.

LEMMA 2.3. *Let $\rho \neq 0$ and let $x : M \longrightarrow \overline{M}[4\rho]$ be an isometric immersion with a non-zero parallel mean curvature vector. If $a = \bar{a}$ on M_0, then either $K \equiv 0$ on M and x is totally real, or $K \leq -2b^2$ on M and x is not totally real.*

PROOF. If x is totally real, then $\alpha = \pi/2$, so $K = 0$ on M_0 by (2.1) and (2.3). By the continuity of K, we have also $K = 0$ on $\overline{M_0} = M$. Suppose that x is not totally real. When $\rho \neq 0$, we have that $\rho = -3b^2$ [16, 17], and by Theorem 2.1 (iii) in Hirakawa [14] we have $K \leq -2b^2$. In fact, we see $K =$ constant $= -2b^2$ on M_0 if α is constant, and $K = -2(2(a+b)^2 + b^2) \leq -2b^2$ on M_0 if α is not constant. By the continuity of K, we proved Lemma 2.3. □

From this point on, we will study an immersion satisfying $d\alpha \neq 0$ and $a \neq \bar{a}$ at a point of M_0. An immersion x is called a general type if it satisfies these two conditions on M_0.

We proved in [17] that if x is of a general type, then a is a function of α, say $a = a(\alpha)$, satisfying the first order ordinary complex differential equation

$$\frac{da}{d\alpha} = \frac{\cot\alpha}{\overline{a(\alpha)} + b}\left(-2ba(\alpha) + 2|a(\alpha)|^2 + \frac{3\rho}{2}\sin^2\alpha\right), \quad (a + b \neq 0) \tag{2.8}$$

and the Kaehler angle α satisfies the second order partial differential equation

$$\alpha_{z\bar{z}} - F(\alpha)\alpha_z\alpha_{\bar{z}} = 0\,, \tag{2.9}$$

where

$$F(\alpha) = \frac{((a(\alpha) - b)(\overline{a(\alpha)} - b) + 3\rho/2\sin^2\alpha)}{(a(\alpha) + b)(\overline{a(\alpha)} + b)}\cot\alpha\,. \tag{2.10}$$

The following Lemma plays a fundamental role in this paper.

LEMMA 2.4. *Suppose that x is of a general type. Then, there is a coordinate transformation $w = w(z)$ on the Riemann surface M_0 such that in $\phi = \mu dw$, μ is a complex valued function of the single real variable $(w + \bar{w})/2$.*

PROOF. By (2.9), we have that $\alpha_{z\bar{z}}/\alpha_z - F(\alpha)\alpha_{\bar{z}} = 0$, which implies that the function $\log\alpha_z - \int F(\alpha)d\alpha$ is holomorphic for z. Hence, there exists a holomorphic function $G(z)$ such that $\alpha_z = G(z)\exp(\int F(\alpha)d\alpha)$. We set $w = \int G(z)dz$. Then, it follows from (2.1) that $\phi = \lambda dz = \alpha_z/(a(\alpha)+b)dz = \mu dw$, where we set

$$\mu = \frac{\exp\left(\int F(\alpha)d\alpha\right)}{a(\alpha)+b}. \tag{2.11}$$

We remark that μ is a complex valued function of α. Now, we prove that α is a function of u, say $\alpha = \alpha(u)$, where we set $w = u + iv$, $(u, v \in \mathbb{R})$. In fact, by (2.1) and (2.11) we can see that

$$d\alpha = (a+b)\mu dw + (\bar{a}+b)\bar{\mu}d\bar{w} = 2\exp\left(\int F(\alpha)d\alpha\right)du, \tag{2.12}$$

because $F(\alpha)$ is real valued, which proves Lemma 2.4. □

We note that the Kaehler angle α and the complex valued function μ are now functions of a single real variable u for coordinates (u, v). It follows from (2.8), (2.10), and (2.11) that

$$\frac{d\log\mu}{d\alpha} = -\frac{(\overline{a(\alpha)}-b)}{\overline{a(\alpha)}+b}\cot\alpha. \tag{2.13}$$

We can determine c as follows.

LEMMA 2.5. *Suppose that x is of a general type. Then, there is a real number k_1 such that*

$$c = \left(|a(\alpha)|^2 + \frac{\rho}{2}\left(-2+3\sin^2\alpha\right)\right)^{1/2}\frac{(\overline{a(\alpha)}+b)}{a(\alpha)+b}e^{-ik_1v}. \tag{2.14}$$

PROOF. Set $a(\alpha)+b = |a(\alpha)+b|\exp i\theta(\alpha)$ and $c = |c|\exp i\nu(u, v)$, where $\theta(\alpha)$ and $\nu(u, v)$ are real valued functions of α and u and v respectively. By (2.11), the absolute value and the argument of μ are functions of α and $-\theta(\alpha)$ (mod 2π), respectively. Hence, we have that

$$\mu^2\bar{c} = |\mu|^2(\alpha)|c|(\alpha)\exp i(-2\theta(\alpha) - \nu(u, v)).$$

We note that $\theta = \theta(\alpha(u))$ is a function of u only. Hence, the absolute value of $\mu^2\bar{c}$ is a function of u only. Together with Lemma 2.2, this implies that

$$\frac{d}{d\alpha}\left(|\mu|^2|c|\right)\frac{d\alpha}{du} + |\mu|^2|c|\frac{\partial\nu}{\partial v} = 0, \quad 2\theta'(\alpha)\frac{d\alpha}{du} + \frac{\partial\nu}{\partial u} = 0.$$

By the second equation above, we see that $\nu(u, v) = -2\theta(\alpha(u)) + \xi(v)$ for some real valued function ξ of v. By inserting this to the first equation, we find that there is a real number k_1 satisfying

$$\frac{d}{du}\left(\log\left(|\mu|^2|c|(\alpha(u))\right)\right) = k_1, \quad \xi'(v) = -k_1. \tag{2.15}$$

By the second formula of (2.15), we have that $\nu(u,v) = -2\theta(\alpha(u)) - k_1 v + k_2$ for some $k_2 \in \mathbb{R}$. Because c is defined up to a complex factor of norm 1, we may assume that $k_2 = 0$. Then, by the fact that $(\bar{a}+b)/(a+b) = \exp(-2i\theta)$, Lemma 2.5 is proved. □

The constant k_1 admits the following expression.

LEMMA 2.6. *Suppose that x is of a general type. Then, we have*

$$2k_1 = \rho\mu \frac{\left(8|a|^2 + 9b(a+\bar{a})\sin^2\alpha - 8b^2 + 18b^2\sin^2\alpha\right)}{(\bar{a}+b)\left(|a|^2 + \rho/2(-2+3\sin^2\alpha)\right)}\cot\alpha\,. \tag{2.16}$$

PROOF. By the first equation of (2.15) and (2.6), we have

$$k_1 = \frac{d\alpha}{du}\left(\frac{d\log\mu}{d\alpha} + \frac{d\log\bar{\mu}}{d\alpha} + \frac{1}{2}\frac{1}{|c|^2}\left(\frac{da}{d\alpha}\overline{a(\alpha)} + a(\alpha)\frac{d\bar{a}}{d\alpha} + 3\rho\sin\alpha\cos\alpha\right)\right).$$

It follows from (2.8), (2.11), (2.12), and (2.13) that (2.16) holds, proving Lemma 2.6. □

Lemma 2.2 can be sharpened for the case of a general type.

LEMMA 2.7. *Suppose that x is of a general type. Then, there exist some constants $c_1, c_2 \in \mathbb{C}$ such that*

$$\mu^2(8ba - 3\rho\sin^2\alpha) = c_1\,, \quad \mu^2\bar{c} = c_2 e^{k_1\omega}\,. \tag{2.17}$$

PROOF. By Lemma 2.2, $\mu^2(8ba - 3\rho\sin^2\alpha)$ is a holomorphic function for ω. Since we know $\mu = \mu(\alpha)$, $a = a(\alpha)$ and $\alpha = \alpha(u)$, this is a function of u only, which implies the first formula of (2.17). By (2.11) and (2.14), the argument of $\mu^2\bar{c}$ is $k_1 v$ and by the first formula of (2.15), its absolute value is $c_2 e^{k_1 u}$, which proves Lemma 2.7. □

3. The case with $k_1 = 0$. In this section, we study the case with $k_1 = 0$. The main result of this section is used to prove the main theorems of this paper. Let us recall assumptions to be applied in this section. These are that x is of a general type, $\rho \neq 0$, $a + b \neq 0$, $c \neq 0$, $\sin\alpha \neq 0$ (by Remark 2.1), and $k_1 = 0$.

By (2.16), we have

$$8|a(\alpha)|^2 + 9b(a(\alpha) + \overline{a(\alpha)})\sin^2\alpha - 8b^2 + 18b^2\sin^2\alpha = 0\,. \tag{3.1}$$

Moreover, we prove

LEMMA 3.1. *If x is of a general type with $k_1 = 0$, then it holds that*

$$\cot\alpha\left(\sin^2\alpha - \frac{8}{9}\right)\left(|a|^2 - b^2\right)\left(\rho + 3b^2\right) = 0\,.$$

PROOF. Let us take the derivative of (3.1) for α. Using (3.1) and $d\bar{a}/d\alpha = \overline{da/d\alpha}$, we have

$$8\frac{d}{d\alpha}|a(\alpha)|^2 + 9b\left(\frac{da}{d\alpha} + \overline{\frac{da}{d\alpha}}\right)\sin^2\alpha - 16\cot\alpha\left(|a|^2 - b^2\right) = 0\,. \tag{3.2}$$

By (2.8) and (3.1), we have

$$
\begin{aligned}
(3.3)\quad & \frac{d}{d\alpha}|a(\alpha)|^2 = \frac{\cot\alpha}{|a+b|^2}\left(|a|^2 - b^2\right)\left(4|a|^2 - \frac{4}{3}\rho + 3\rho\sin^2\alpha\right), \\
& \frac{da}{d\alpha} + \overline{\frac{da}{d\alpha}} = 4b\frac{\cot\alpha}{|a+b|^2}\left(|a|^2 - b^2\right) \\
& \qquad + \frac{\cot\alpha}{|a+b|^2}(a+\overline{a}+2b)\left(-2b(a+\bar{a}) + 2b^2 + 2|a|^2 + \frac{3}{2}\rho\sin^2\alpha\right).
\end{aligned}
$$

By (3.1), we have $9b(a+\bar{a}+2b)\sin^2\alpha = -8(|a|^2 - b^2)$. Coupling this with the second formula above, we have

$$
\begin{aligned}
& 9b\left(\frac{da}{d\alpha} + \overline{\frac{da}{d\alpha}}\right)\sin^2\alpha \\
&= \frac{\cot\alpha}{|a+b|^2}\left(|a|^2 - b^2\right)\left(-16|a|^2 + 16b(a+\bar{a}) - 16b^2 + 36b^2\sin^2\alpha - 12\rho\sin^2\alpha\right).
\end{aligned}
$$

Lemma 3.1 is proved by this formula above, (3.2), and (3.3). □

Since x is of a general type, α is not constant, hence we may assume $\cot\alpha \neq 0$, and $\sin^2\alpha - 8/9 \neq 0$. If $|a(\alpha)|^2 = b^2$ on M_0, then we have by (3.1) that $a(\alpha) + \overline{a(\alpha)} = -2b$, and so $(a(\alpha)+b)(\overline{a(\alpha)}+b) = 0$ on M_0, which contradicts the assumptions applied in this section. Hence, Lemma 3.1 implies that

$$
(3.4)\qquad \rho = -3b^2.
$$

By considering (2.14) with $k_1 = 0$, we note that c is a function of u only, because $\alpha = \alpha(u)$. In order to determine $a(\alpha)$ explicitly, we set

$$
(3.5)\qquad \tau = \frac{a(\alpha) - b}{a(\alpha) + b}.
$$

Then, (2.8) and (3.1) are transformed to

$$
\frac{d\tau}{d\alpha} - (1+\tau)(1-\tau)\bar{\tau}\cot\alpha + \frac{9}{8}(1-\tau)^2(1-\bar{\tau})\sin\alpha\cos\alpha = 0,
$$

and $18\sin^2\alpha + (8 - 9\sin^2\alpha)(\tau+\bar{\tau}) = 0$, respectively. Hence, the real part of τ is uniquely determined, and the imaginary part of τ, say $y(\alpha)$, satisfies the first order ordinary differential equation

$$
(3.6)\qquad \frac{dy^2}{d\alpha} + 4\cot\alpha\frac{(4 - 9\sin^2\alpha)}{8 - 9\sin^2\alpha}y^2 + \cot\alpha\frac{(8 - 9\sin^2\alpha)}{4}y^4 = 0.
$$

The solution of (3.6) is given by

$$
(3.7)\qquad y^2(\alpha) = \frac{8c_3}{(8 - 9\sin^2\alpha)(-c_3 + \sin^2\alpha)} \geq 0, \quad (c_3 \in \mathbb{R}),
$$

where the integral constant c_3 satisfies $c_3 \neq 0$ and $8 - 9c_3 \neq 0$. In fact, if $c_3 = 0$, then $y(\alpha)$ identically vanishes, hence $a = \bar{a}$ on M_0, which contradicts the assumption $a \neq \bar{a}$. If

$8-9c_3=0$, then $y^2<0$, giving a contradiction. Since the right hand of (3.7) is non-negative, it holds that

$$\begin{cases} 0<c_3<\sin^2\alpha<\frac{8}{9}, \text{ or } \frac{8}{9}<\sin^2\alpha<c_3\,, & \text{for } c_3>0\,, \\ \frac{8}{9}<\sin^2\alpha\leq 1\,, \quad \text{for } c_3<0\,. \end{cases} \tag{3.8}$$

Later, we will use the explicit formula of a. By (3.5) and (3.7), the real part of $a(\alpha)$, $\Re a(\alpha)$, is given by

$$\Re a(\alpha)=\frac{b}{8-9c_3}\frac{(-16c_3+(8+27c_3)\sin^2\alpha-18\sin^4\alpha)}{\sin^2\alpha}\,, \tag{3.9}$$

and also, the imaginary part of $a(\alpha)$, $\Im a(\alpha)$, is given by, when $c_3>0$,

$$\Im a(\alpha)=\frac{b\sqrt{c_3}}{\sqrt{2}(8-9c_3)}\frac{(8-9\sin^2\alpha)\sqrt{(8-9\sin^2\alpha)(-c_3+\sin^2\alpha)}}{\sin^2\alpha}\,, \tag{3.10}$$

and when $c_3<0$, setting $c_4=-c_3>0$,

$$\Im a(\alpha)=\frac{b\sqrt{c_4}}{\sqrt{2}(8+9c_4)}\frac{(9\sin^2\alpha-8)\sqrt{(9\sin^2\alpha-8)(c_4+\sin^2\alpha)}}{\sin^2\alpha}\,, \tag{3.11}$$

where $\sin\alpha$ satisfies (3.8). We note that by (3.7), (3.10) and (3.11), those intervals in (3.8) are the maximal ones of $\sin^2\alpha$ on which $\Im a(\alpha)$ does not vanish. We compute the Gaussian curvature of M_0 as follows: By (3.1) and (3.5), we have

$$|a(\alpha)|^2=b^2\frac{1+(\tau+\bar\tau)+|\tau|^2}{1-(\tau+\bar\tau)+|\tau|^2}\,,\quad \tau=-\frac{9\sin^2\alpha}{8-9\sin^2\alpha}+iy(\alpha)\,.$$

By these formulas above, (2.3) and (3.7), we have

$$K=\frac{-2b^2}{(8-9c_3)}\left(\left(9\sin^2\alpha-8\right)^2+(8-9c_3)\right)\,, \tag{3.12}$$

which proves the following Lemma 3.2

LEMMA 3.2. *If* $8-9c_3>0$, *then the Gaussian curvature* K *is upper bounded by a negative constant on* M_0, *that is, it holds that* $K\leq-2b^2<0$ *on* M.

Hence, we have proved part of the following theorem

THEOREM 3.3. *Let* $\rho\neq 0$, *and let* $x:M\longrightarrow\overline{M}[4\rho]$ *be an isometric immersion with a non-zero parallel mean curvature vector of a general type. If* $k_1=0$, *then* $\rho=-3b^2$, *and the first and second fundamental forms of* x *are explicitly determined by* (2.11), (2.14), (3.9), (3.10), (3.11), *and a real number* c_3. *In particular, the Gaussian curvature* K *is bounded above by* $-2b^2$ *when* $8-9c_3>0$. *Conversely, for* $b>0$, *there is a one-parameter family of parallel mean curvature immersions of a general type with* $k_1=0$ *from a simply connected domain* D *in* $\mathbb{R}^2$ *into* $\overline{M}[-12b^2]$, *with* $|H|=2b$.

Proof of the converse: For given $b > 0$ and $c_3 \in \mathbb{R}$, we can find a simply connected domain D with coordinates (u, v), a real valued function $\alpha = \alpha(u)$, complex valued functions $a = a(\alpha)$ and $c = c(\alpha)$, and a complex one form ϕ on D, by considering (3.7), (3.9), and (2.14) with $k_1 = 0$. Then, we can prove that these functions satisfy the structure equations (2.1)–(2.6), and that c_3 is the parameter of the family. We omit a detailed computation of the proof, because the converse is not used in this paper.

REMARK 3.4. The case with $k_1 \neq 0$ remains to be studied. This will be addressed in future work, because the study of this case is not necessary for the present paper.

4. Parallel mean curvature tori. In this section, we suppose that M is homeomorphic to a torus, and we denote it by $\mathbb{T}^2$. Let $\mathbb{T}^2_0 = \{p \in \mathbb{T}^2 \mid x$ is neither holomorphic nor anti-holomorphic at $p\}$. First, we study the case of $a = \bar{a}$ on $\mathbb{T}^2_0$.

LEMMA 4.1. *Let $\rho \neq 0$ and let $x : \mathbb{T}^2 \longrightarrow \overline{M}[4\rho]$ be an isometric immersion with a non-zero parallel mean curvature vector. Suppose that x satisfies $a = \bar{a}$ on $\mathbb{T}^2_0$. Then, $K = 0$ on $\mathbb{T}^2$ and x is totally real.*

PROOF. By Lemma 2.3, if x is not totally real, then $K \leq -2b^2$ on $\mathbb{T}^2$. However this presents a contradiction by the Gauss-Bonnet Theorem, proving Lemma 4.1. □

From now on, we suppose that $a \neq \bar{a}$ at some point of $\mathbb{T}^2_0$. In particular, we have $8ba - 3\rho\sin^2\alpha \neq 0$ at the point of $\mathbb{T}^2_0$, hence by Lemma 2.7 $c_1 \neq 0$ and Φ_1 is a non-zero quadratic form on $\mathbb{T}^2_0$, hence $Q - Q'$ is a non-zero holomorphic quadratic form on $\mathbb{T}^2$ by 2.7 and Fetcu [9]. By Riemann-Roch's Theorem, the dimension of the vector space over $\mathbb{C}$ of holomorphic quadratic forms on a torus is one, and $Q - Q'$ is a base of the vector space. Hence, the holomporphic quadratic form $Q + Q'$ on $\mathbb{T}^2$ is a constant multiple of $Q - Q'$. By Lemma 2.7, we have

$$\mu^2\bar{c} = \text{constant}\,, \text{ and } k_1 = 0\,. \tag{4.1}$$

We remark that c in the above formula does not vanish on $\mathbb{T}^2_0$. In fact, if $c = 0$ at a point of $\mathbb{T}^2_0$, then it must identically vanish by (4.1). Such surfaces are classified by Hirakawa [14]. We apply Proposition 3.5 of [14] to our situation that is $a \neq \bar{a}$ at a point of $\mathbb{T}^2_0$, and get $K = -2b^2$ on $\mathbb{T}^2_0$. Hence we have that $K = \text{constant} = -2b^2$ on $\mathbb{T}^2$, which presents a contradiction by the Gauss-Bonnet Theorem because of $b \neq 0$. By taking the ratio of the two formulas of (2.17), we have that for some $\gamma(\neq 0) \in \mathbb{C}$,

$$8ba - 3\rho\sin^2\alpha - b\gamma\bar{c} = 0 \quad \text{on } \mathbb{T}^2_0\,. \tag{4.2}$$

We have

LEMMA 4.2. *If $a \neq \bar{a}$ at a point of $\mathbb{T}^2_0$, then $|\gamma|^2 = 2(8 - 9c_3) > 0$.*

PROOF. Since we have $k_1 = 0$ by (4.1), we can apply the results of Section 3 for $\mathbb{T}^2_0$. Then, by (3.4) and (4.2) we have that $8a(\alpha) + 9b\sin^2\alpha = \gamma\bar{c}$ on $\mathbb{T}^2_0$. We take the absolute value of the formula above and then we use (2.6) and (3.1) to get

$$4b(8-9\sin^2\alpha)^2 = |\gamma|^2(-9\sin^2\alpha\Re a(\alpha) + 16b - 27b\sin^2\alpha)\,,$$

and by (3.9) we get $(8-9\sin^2\alpha)^2(|\gamma|^2 - 2(8-9c_3)) = 0$. If $|\gamma|^2 - 2(8-9c_3) \neq 0$, then $8-9\sin^2\alpha = 0$ on $\mathbb{T}^2$, which implies $\alpha =$ constant on $\mathbb{T}^2$, so $a = \bar{a}$ on $\mathbb{T}_0^2$, giving a contradiction. We proved Lemma 4.2. □

LEMMA 4.3. *It holds that $a = \bar{a}$ on $\mathbb{T}_0^2$.*

PROOF. Let $U = \{p \in \mathbb{T}_0^2 \mid a \neq \bar{a}$ at $p\}$. Suppose that U is non-empty. We note that, by Theorem 3.3 and Lemma 4.2, K is upper bounded by a negative constant on U. If $U = \mathbb{T}_0^2$, then we have a contradiction by the Gauss-Bonnet Theorem. From now on, we consider the case that there is a point $p_0 \in \mathbb{T}_0^2$ with $a = \bar{a}$ at p_0. We remark $c_3 \neq 0$ because of (3.7). For $c_3 > 0$, by (3.10), $\Im a$ tends to zero if and only if $\sin^2\alpha$ tends to c_3 or $8/9$. Therefore, for the case of $c_3 > 0$, $(c_3, 8/9)$ is the maximal interval of $\sin^2\alpha$ satisfying $a \neq \bar{a}$. For the case of $c_3 < 0$, the similar consideration proves that $(8/9, 1)$ is the maximal interval of $\sin^2\alpha$ satisfying $a \neq \bar{a}$. We have $0 < \sin^2\alpha(p_0) \le c_3$ or $8/9 \le \sin^2\alpha(p_0)$. If $\sin^2\alpha(p_0) < c_3$, then there is an open neighborhood W of p_0 such that $\sin^2\alpha < c_3$ on W. Since we have $a = \bar{a}$ on W, it holds that $K \le -2b^2$ at p_0 by Lemma 2.3. For the case of $8/9 < \sin^2\alpha(p_0)$, the similar consideration as before implies also $K \le -2b^2$ on the point of $a = \bar{a}$. Therefore, for those points $p \in \mathbb{T}_0^2$ satisfying $a(p) = \bar{a}(p)$, K is also upper bounded by a negative constant. We proved that K is upper bounded above by a negative constant on both sets U and $\mathbb{T}_0^2 \setminus U$, which contradicts the Gauss-Bonnet Theorem by taking the integral of K over $\mathbb{T}^2$. Hence, we proved Lemma 4.3. □

By Lemmas 4.1, 4.2 and 4.3, we proved the following main result of this paper:

THEOREM 4.4. *Let $\rho \neq 0$, and let $x : \mathbb{T}^2 \longrightarrow \overline{M}[4\rho]$ be an isometric immersion with a non-zero parallel mean curvature vector. Then, the Gaussian curvature of $\mathbb{T}^2$ vanishes identically and x is totally real.*

Let $\mathbb{C}P^n$ and $\mathbb{C}H^n$ be the complex projective space and complex hyperbolic space, respectively, endowed with Kaehler metrics of constant holomorphic sectional curvature. In Ohnita [19], and also independently in Urbano [22], n-dimensional totally real submanifolds with non-negative sectional curvature in $\mathbb{C}P^n$ and $\mathbb{C}H^n$ have been classified in the context of the theory of symmetric spaces. For the flat case, these immersions are explicitly described in Dajczer and Tojeiro [6] when the ambient space is $\mathbb{C}P^n$, and Hirakawa [13] when the ambient space is $\mathbb{C}H^2$. By combining Theorem 4.4 with those results, we determined tori with non-zero parallel mean curvature in two-dimensional non-flat complex space forms. For completeness, we state the following as a theorem

THEOREM 4.5. *Let $\mathbb{T}^2$ be a real two-dimensional compact orientable Riemannian manifold with genus one. Then, an isometric immersion from $\mathbb{T}^2$ into $\mathbb{C}P^2$ or $\mathbb{C}H^2$ has a non-zero parallel mean curvature vector if and only if the image is a totally real flat torus in $\mathbb{C}P^2$ or $\mathbb{C}H^2$, respectively.*

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