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SQUARE MEANS VERSUS DIRICHLET INTEGRALS FOR HARMONIC FUNCTIONS ON RIEMANN SURFACES

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Abstract. We show rather unexpectedly and surprisingly the existence of a hyperbolic Riemann surface W enjoying the following two properties: firstly, the converse of the celebrated Parreau inclusion relation that the harmonic Hardy space $HM_2(W)$ with exponent 2 consisting of square mean bounded harmonic functions on W includes the space $HD(W)$ of Dirichlet finite harmonic functions on W, and a fortiori $HM_2(W) = HD(W)$, is valid; secondly, the linear dimension of $HM_2(W)$, hence also that of $HD(W)$, is infinite.

1. Introduction. It is a traditional use of notation to adopt $H(R)$ as the class of harmonic functions on a Riemann surface R in the harmonic classification theory of Riemann surfaces (see e.g. [13]). In this paper we are mainly concerned with the linear subspace $HM_2(R)$ of $H(R)$ consisting of those harmonic functions u such that u^2 admits a harmonic majorant h on R, i.e., $u^2 \leqq h$ on R. This property is usually referred to as being *square mean bounded* (cf. e.g. [1, p. 216]). The part M_2 in the notation $HM_2(R)$ thus suggests the square mean. Let $\{\Omega\}$ be the directed net of regular subregions Ω of an open (i.e., noncompact) Riemann surface R, which serves as an exhaustion $\Omega \uparrow R$ of R. Take a $u \in H(R)$ and we examine when u^2 admits a harmonic majorant. We denote by $H^{\Omega}_{u^2}$ the harmonic function on Ω with boundary values u^2 on $\partial\Omega$. Since u^2 is subarmonic on R, by a simple observation based upon the maximum principle, we see that

$$H^{\Omega}_{u^2} \leqq H^{\Omega'}_{u^2} \quad (\Omega \subset \Omega') \tag{1.1}$$

on Ω and thus we conclude that

$$\lim_{\Omega\uparrow R} H^{\Omega}_{u^2} =: h \tag{1.2}$$

exists and is either a positive harmonic function on R, i.e., $h \in H(R)^+$, or $h \equiv +\infty$. The former case occurs if and only if $h(o) < +\infty$ for one and hence for every reference point $o \in R$ and the convergence of (1.2) is of locally uniform on R. The former case is thus exactly the case $u \in HM_2(R)$ and $u^2 \leqq h$ on R in (1.2) which is not only a harmonic majorant of u^2 but also the *least* harmonic majorant of u^2. This justifies the term, square mean bounded, since

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$$H^{\Omega}_{u^2}(o) = \int_{\partial\Omega} u^2 d\mu_\Omega \leqq h(o)$$

for every Ω, i.e., the square mean $\int_{\partial\Omega} u^2 d\mu_\Omega$ is bounded by $h(o)$ for every Ω containing o, where μ_Ω is the harmonic measure on $\partial\Omega$ with the reference point o.

On the other hand the subharmonicity of u^2, or the superharmonicity of $-u^2$, can be observed from the following different view point. Since $-(u^2 - H^{\Omega}_{u^2})$ is the Green potential $\int_\Omega g(\cdot, z; \Omega) d\nu(z)$, where $g(\cdot, z; \Omega)$ is the Green function on Ω with its pole at $z \in \Omega$, with the measure ν whose density $d\nu(z)/dxdy$ $(z = x + iy)$ is given by the $1/2\pi$ times

$$\Delta(u^2 - H^{\Omega}_{u^2}) = \Delta u^2 = 2|\nabla u|^2$$

on Ω with the gradient ∇u of u, we have the Riesz decomposition

$$u^2 = H^{\Omega}_{u^2} - \frac{1}{\pi}\int_\Omega g(\cdot, z; \Omega)|\nabla u(z)|^2 dxdy \tag{1.3}$$

on Ω. As a counterpart of (1.1) we also have

$$\frac{1}{\pi}\int_\Omega g(\cdot, z; \Omega)|\nabla u(z)|^2 dxdy \leqq \frac{1}{\pi}\int_{\Omega'} g(\cdot, z; \Omega')|\nabla u(z)|^2 dxdy \quad (\Omega \subset \Omega') \tag{1.4}$$

on Ω and a fortiori, corresponding to (1.2), we deduce

$$\lim_{\Omega\uparrow R} \frac{1}{\pi}\int_\Omega g(\cdot, z; \Omega)|\nabla u(z)|^2 dxdy = \frac{1}{\pi}\int_R g(\cdot, z; R)|\nabla u(z)|^2 dxdy \tag{1.5}$$

exists on R, which is also either finite or infinite corresponding to $h(o) < +\infty$ or $h(o) = +\infty$ in (1.2). Here the limit $g(\cdot, z; R) := \lim_{\Omega\uparrow R} g(\cdot, z; \Omega)$ exists on R, which is either the Green function on R with its pole at z in R or $+\infty$ on R. In the former (resp. latter) case, R is referred to as being *hyperbolic* (resp. *parabolic*). In the hyperbolic case, the convergence of the above limit is of locally uniform on $R \setminus \{z\}$. Therefore we get the following result.

THEOREM A. *A nonconstant harmonic function u on R is square mean bounded, i.e., $u \in HM_2(R)$ if and only if R is hyperbolic and*

$$\int_R g(o, z; R)|\nabla u(z)|^2 dxdy < +\infty \tag{1.6}$$

for one and hence for every reference point o in R.

We are unable to locate the place where the above statement exactly in this identical form is mentioned but essentially it is found, for example, in the celebrated paper of Parreau [12] and also in Doob [3] in the case of higher dimensions. Anyway an important point here is the following. For locally Sobolev functions u on R, i.e., $u \in W^{1,2}_{\mathrm{loc}}(R)$, we can consider the squared seminorm called the *Dirichlet integral*

$$D(u; R) := \int_R du \wedge *du = \int_R |\nabla u(z)|^2 dxdy \leqq +\infty \tag{1.7}$$

of u taken over R. The linear subspace $HD(R)$ of $H(R)$ consisting of functions $u \in H(R)$ with finite Dirichlet integrals $D(u; R) < +\infty$ is one of long standing important classes of functions in the classification theory of Riemann surfaces. Viewing the integral in (1.6) as a weighted Dirichlet integral, we are interested in comparing its finiteness with that of genuine one in (1.7). Since the density in (1.6) is bounded (and even "vanishing" at the ideal boundary) except in a vicinity of the singularity o which is of order $\log(1/|z|)$ in terms of local parameter z at o and thus integrable in the vicinity of o with respect to the measure $dxdy$, we can say that the condition $D(u; R) < +\infty$ assures the inequality (1.6). This gives a notable Parreau inclusion relation:

$$HM_2(R) \supset HD(R) \tag{1.8}$$

for *any* open Riemann surface R. This is the relation we are mainly concerned with in this paper.

It may be a bit digressing from the subject at hand but let us mention the following. The simplest subclass of $H(R)$ except $\{0\}$ is the class $\boldsymbol{R}$ viewed as that of constant functions on R, where basically $\boldsymbol{R}$ is used to indicate the real number field. Trivially

$$HM_2(R) \supset \boldsymbol{R}$$

is the case for every surface R. One might ask when the inverse inclusion relation $HM_2(R) \subset \boldsymbol{R}$ holds, or equivalently, when $HM_2(R) = \boldsymbol{R}$ holds. In the classification theory we denote by $\mathcal{O}_{HM_2}$ the class of Riemann surfaces R with $HM_2(R) = \boldsymbol{R}$. The primary theme of the classification theory is to characterize the class $\mathcal{O}_{HX}$ for various properties X of functions related to harmonic functions and also compare $\mathcal{O}_{HX}$ with $\mathcal{O}_{HY}$ for different properties X and Y. The inverse inclusion $HM_2(R) = \boldsymbol{R}$, or equivalently $R \in \mathcal{O}_{HM_2}$, thus yields quite trivially the finite linear dimensionality of $HM_2(R)$, i.e., $\dim HM_2(R) = 1$ in the present case, which is a kind of degeneracy of R. In analogy to the above mentioned frame, we propose to study the *inverse inclusion problem* to (1.8): what kind of degenracy occurs to R when the inverse incusion relation to (1.8) holds, i.e., $HM_2(R) \subset HD(R)$ or equivalently $HM_2(R) = HD(R)$. More concretely, we ask whether the inverse inclusion relation

$$HM_2(R) = HD(R) \tag{1.9}$$

implies the degeneracy

$$\dim HM_2(R) < +\infty\,. \tag{1.10}$$

Taking an exponent p in $(1, +\infty)$ other than 2, we can consider the p mean bounded class $HM_p(R)$ exactly in the same fashion as in the case $p = 2$, i.e., $u \in HM_p(R)$ if and only if $u \in H(R)$ and there exists an $h \in H(R)$ such that $|u|^p \leqq h$ on R. The classes $HM_p(R)$ $(1 \leqq p \leqq \infty)$ with $HM_1(R) = HP(R)$, the class of essentially positive harmonic functions on R (cf. Section 2), and $HM_\infty(R) = HB(R)$, the class of bounded harmonic functions on R, are also called harmonic *Hardy spaces*. Corresponding to the same question as above asked to $p \in (1, +\infty) \setminus \{2\}$, the first named author obtained the following result (see [5] and the reference therein).

THEOREM B. *If the class identity $HM_p(R) = HD(R)$ for a p with $1 < p \leqq +\infty$ and $p \neq 2$ holds, then the underlying Riemann surface R is so degenerate as to satisfy* $\dim HM_p(R) < +\infty$.

(For the part $HM_\infty(R)(\equiv HB(R)) = HD(R)$, see also [6], [7] and [9]. See also [10] for a different and actually easier and simpler proof of the above theorem for $p \in (1, \infty) \setminus \{2\}$). Here, as is easily seen, $HM_p(R) \supset HM_2(R) \supset HD(R)$ for every R if $1 < p < 2$ and therefore $HM_p(R) = HD(R)$ is certainly an inverse inclusion for $1 < p < 2$ and a fortiori Theorem B is the answer to the inverse inclusion problem. However, there are no inclusion relations between $HM_p(R)$ and $HD(R)$ for $2 < p \leqq +\infty$ for general R. In this sense the relation $HM_p(R) = HD(R)$ has nothing to do with the inverse inclusion and the meaning of Theorem B might not be too clear beyond its mere formality for $2 < p < +\infty$. Anyway, in view of this fact, i.e., Theorem B, it might be quite natural to expect that Theorem B continues to hold even for $p = 2$. Therefore it may be a bit surprising that the answer to the above question concerning (1.9) and (1.10) is negative, and to report on which is the main purpose of this paper. Namely, we will prove the following result based upon the construction of a suitable Riemann surface.

THE MAIN THEOREM. *There is an open Riemann surface W such that although the inverse inclusion relation $HM_2(W) = HD(W)$ holds and yet $HM_2(W)$ contains sufficiently many functions in the sense that* $\dim HM_2(W) = +\infty$.

At the end of this introduction we state the outline of this paper. The paper consists of 7 sections including the present section. Since we essentially make use of the Wiener compactification theory throughout this paper, we explain briefly its core part in Section 2 to an extent we really use in this paper. Particular care is also done in this section on quasibounded harmonic functions from the view point how they are controlled by their behaviors on the Wiener harmonic boundary δR. In addition to two really basic properties of the class $HM_2(R)$, the representation theorem $HM_2(R) = L^2(\delta R, \omega)$ (ω being the harmonic measure on δR) of fundamental importance is given in Section 3. The proof of our main theorem essentially starts from Section 4. We coined the term afforested surface in [11]. Surfaces under this name are now well recognized to be very useful in many instances including the present construction required in the main theorem. We will construct an afforested surface W in this section, which is the surface sought in the main theorem. The results obtained or explained in Sections 2 and 3 are speciallized in Section 5, from the general Riemann surfaces R to the particular surface W constructed in Section 4. In general we cannot say anything about the relation between the value $w(o)$ $(o \in R)$ (harmonic measure) and its Dirichlet integral $D(w; R)$ for harmonic measure function w on general Riemann surfaces R. However, on our W we have a finite positive constant K such that $D(w; W) \leqq K \cdot w(o)$ $(o \in W)$ for every harmonic measure function w on W. A specified form of this will be proved in Section 6. This result is fatally impotant. We will not mention a remarkable fact that the mutual Dirichlet integral $-\infty \leqq D(w_1, w_2; R) \leqq 0$ for harmonic measures w_1 and w_2 on a general surface R with $w_1 \wedge w_2 = 0$ but a special part of this result for the special surface $R = W$ is also added in this section, which is used essentially in the proof of the main theorem. The proof of the main theorem is completed in Section 7.

2. Wiener harmonic boundary. Since we will make an essential use of the theory of Wiener compactifications of Riemann surfaces R, we briefly compile it here to an extent really needed in our proof of the main theorem (cf. e.g. [2], [13]). The results stated in this section are of course all well known for specialists but still we added proofs to certain core facts for convenience of the reader. A continuous function f on R is referred to as a (continuous) *Wiener function* if $|f|$ is dominated by some superharmonic function on R and harmonizable in the following sense: the net $(H_f^\Omega)_\Omega$ converges to an $h_f \in H(R)$ locally uniformly on R, i.e.,

$$h_f = \lim_{\Omega \uparrow R} H_f^\Omega$$

locally uniformly on R, where (Ω) is the directed net by inclusion of regular subregions Ω of R and H_f^Ω is the PWB (i.e., Perron-Wiener-Brelot) solution of the harmonic Dirichlet problem on Ω with the boundary data f on $\partial\Omega$. A function g on R is said to be a *quasipotential* or occasionally *Wiener potential* if $|g|$ is dominated by a potential on R, i.e., a superharmonic function with the vanishing greatest harmonic minorant on R (cf. e.g. [4]). We denote by $\mathcal{W}(R)$ the class of continuous Wiener functions on R and by $\mathcal{W}_0(R)$ the subclass of $\mathcal{W}(R)$ consisting of continuous quasipotentials on R so that we have the following form of the Riesz decomposition theorem:

$$\mathcal{W}(R) = HP(R) \oplus \mathcal{W}_0(R) \quad \text{(direct sum)}, \tag{2.1}$$

i.e., any $f \in \mathcal{W}(R)$ is uniquely decomposed into the sum $f = h_f + g_f$ of the harmonic component $h_f \in HP(R)$ and the quasipotential component $g_f \in \mathcal{W}_0(R)$, where $HP(R)$ is the class of $u \in H(R)$ such that $|u|$ admits a harmonic majorant so that $HP(R)$ can also be characterized by $HP(R) := H(R)^+ \ominus H(R)^+$, the class of differences of two nonnegative harmonic functions on R. For this reason $h \in HP(R)$ is occasionally said to be *essentially positive* on R. The class $HP(R)$ forms a vector lattice with lattice operations $\vee$ as join and $\wedge$ as meet defined as follows: $u \vee v$ (resp. $u \wedge v$) for two harmonic functions u and v in $H(R)$ denotes the least (resp. greatest) harmonic majorant (resp. minorant) of u and v on R. Then $u \vee v$ and $u \wedge v$ exist on R and again belong to $HP(R)$ as far as u and v belong to $HP(R)$. Since it can be easily ascertained that $HP(R)$ really forms a vector lattice, the canonical way of decomposing $HP(R) = H(R)^+ \ominus H(R)^+$ is the Jordan decomposition

$$u = u^+ - u^- \quad (u^+ := u \vee 0, \quad u^- := (-u) \vee 0) \tag{2.2}$$

for any $u \in HP(R)$ with u^+ and u^- in $H(R)^+$. The decomposition (2.1) can also be used to define the class $\mathcal{W}(R)$, i.e., a continuous Wiener function on R is a continuous function on R expressed as a sum of an essentially positive harmonic function on R and a continuous quasipotential on R. It can happen that there are no nonzero pontentials on R and in such a case R is referred to as being *parabolic* and otherwise *hyperbolic*. We use the notation $\mathcal{O}_G$ for the class of parabolic Riemann surfaces R which are also characterized by the equivalent condition that there is no Green function on R. We denote by $\mathcal{O}_{HP}$ the class of Riemann surfaces R such that $HP(R) = \boldsymbol{R}$. Then we know (cf. e.g. [13]) that

$$\mathcal{O}_G \subset \mathcal{O}_{HP}.$$

If $R \in \mathcal{O}_G$, then we understand that $\mathcal{W}(R) = \mathcal{W}_0(R) = CB(R)$, the class of bounded continuous functions on R.

We need to add a few words on an important subclass $\mathcal{R}(R)$ of $\mathcal{W}(R)$ explained below. The Dirichlet space $L^{1,2}(R)$ is the class of *Dirichlet functions* f on R characterized by $f \in W^{1,2}_{\text{loc}}(R)$, the locally Sobolev (1,2) function space on R, and $D(f; R) < +\infty$. A continuous Dirichlet function f on R, i.e., $f \in L^{1,2}(R) \cap C(R)$, is also called as a *Royden function* on R and the totality of Royden functions f on R is denoted by $\mathcal{R}(R)$. We also denote by $\mathcal{R}_0(R)$ the class of $f \in \mathcal{R}(R)$ such that there is a sequence $(f_n)_{n \in \boldsymbol{N}} \subset \mathcal{R}(R)$ with the following properties: each f_n has a compact support in R ($n \in \boldsymbol{N}$); $f_n \to f$ ($n \to \infty$) locally uniformly on R; and, $D(f_n - f; R) \to 0$ ($n \to \infty$). We call $f \in \mathcal{R}_0$ a continuous *Dirichlet potential* or *Royden potential*. We have

$$\mathcal{R}(R) \subset \mathcal{W}(R)\,, \quad \mathcal{R}_0(R) \subset \mathcal{W}_0(R)\,. \tag{2.3}$$

The Riesz decomposition (2.1) applied to $\mathcal{R}(R)$ is in particular referred to as the Royden-Brelot decomposition and takes the form

$$\mathcal{R}(R) = HD(R) \oplus \mathcal{R}_0(R) \quad \text{(orthogonal decomposition)}\,, \tag{2.4}$$

i.e., every $f \in \mathcal{R}(R)$ is uniquely decomposed into the sum $f = h_f + g_f$ of $h_f \in HD(R)$ and $g_f \in \mathcal{R}_0(R)$ satisfying the Dirichlet principle:

$$D(f; R) = D(h_f; R) + D(g_f; R)\,, \tag{2.5}$$

where, as already introduced in the introduction,

$$HD(R) := \{u \in H(R);\, D(u; R) < +\infty\}$$

and it is known (cf. e.g. [13]) that the following Virtanen-Royden property is fullfilled: $HD(R)$ is a vector sublattice of $HP(R)$, i.e., $HD(R) \subset HP(R)$ and $u \vee v$ and $u \wedge v$ belong to $HD(R)$ for every pair of u and v in $HD(R)$, and, for example, $u = \lim_{n\to\infty} u \wedge n$ locally uniformly on R and $\lim_{n\to\infty} D(u - u \wedge n; R) = 0$, where $n \in \boldsymbol{N}$, the set of positive integers.

The *Wiener compactification* R^* of any given Riemann surface R is characterized by the following 4 properties: R^* is a compact Hausdorff space; R^* contains R as its open and dense subspace; every $f \in \mathcal{W}(R)$ can be extended uniquely to R^* as a $[-\infty, +\infty]$-valued continuous function on R^*; and, thus extended class $\mathcal{W}(R)$ separates points in R^*. We know the unique existence of R^* for every R. The set

$$\gamma = \gamma R := R^* \setminus R$$

is the *Wiener boundary* of R. The *Wiener harmonic boundary*, or simply harmonic boundary when Wiener is clear,

$$\delta = \delta R$$

of R is the set of points $\zeta \in \gamma$ such that

$$\lim_{z \in R, z \to \zeta} g(z) = 0 \tag{2.6}$$

for every (not necessarily continuous) quasipotential g on R so that δ is also compact. Consider the harmonic Dirichlet problem on R with a boudary data φ on γ treated by the usual PWB (i.e., Perron-Wiener-Brelot) procedure. In case φ is resolutive we denote by H_φ^R the PWB-solution. It is seen that every bounded continuous function on γ is resolutive. We can say that δ characterized by (2.6) is also defined as the set of regular points $\zeta \in \gamma$ in the sense of the PWB procedure:

$$\lim_{z\in R, z\to\zeta} H_\varphi^R = \varphi(\zeta) \tag{2.7}$$

for every bounded continuous function φ on γ. Observe that three properties $R \in \mathcal{O}_G$, $\mathcal{W}(R) = \mathcal{W}_0(R) = CB(R)$, and $\delta R = \emptyset$ are mutually equivalent. (In this case the Wiener compactification R^* of R is nothing but identical with the Čech compactification of R, which is the maximal ideal space of the normed ring $CB(R)$ of bounded continuous functions on R.) Another important role of δ is the so called comparison principle. We denote by $HB(X)$ the linear space of bounded harmonic functions u on an open subset X of a Riemann surface R and by $\overline{Y}$ (resp. ∂Y) the closure relative to R^* (resp. the relative bounday relative to R) of any subset Y of R^*. For any u and v in $HB(X)$, the *comparison principle* says that

$$\limsup_{z\in X, z\to\zeta} u(z) \leqq \liminf_{z\in X, z\to\zeta} v(z)$$

for every $\zeta \in (\partial X) \cup (\overline{X} \cap \delta R)$ implies that $u \leqq v$ on X. A *harmonic measure* $\omega = \omega_o$ on γ with the reference point $o \in R$ is given by

$$H_\varphi^R(o) = \int_\gamma \varphi d\omega$$

for every $\varphi \in C(\gamma)$. Since $\omega(\gamma \setminus \delta) = 0$, we may replace γ by δ in the above definition. Thus the values of the boundary data φ on $\gamma \setminus \delta$ in H_φ^R does not affect on H_φ^R and therefore we only have to give the boundary data φ on δ when we treat the PWB-solution H_φ^R.

In this final part of Section 2 we mention the Parreau decomposition of essentially positive harmonic functions. A function $u \in HP(R)$ is said to be *quasibounded* on R if

$$u = \lim_{s,t\in \boldsymbol{R}^+, s,t\uparrow+\infty} (u \wedge s) \vee (-t)\,, \tag{2.8}$$

where the convergence is of locally uniform on R. We denote by $HP_q(R)$ or by $HB'(R)$ the class of quasibounded essentially positive harmonic functions on R so that

$$HB(R) \subset HB'(R) \equiv HP_q(R) \subset HP(R)\,.$$

A function $u \in HP(R)$ is said to be *singular* if

$$(u \wedge s) \vee (-t) = 0 \tag{2.9}$$

on R for every s and t in $\boldsymbol{R}^+$. We denote by $HP_s(R)$ the class of singular essentially positive harmonic functions on R. Then the identity

$$HP(R) = HP_q(R) \oplus HP_s(R) \quad \text{(direct sum)} \tag{2.10}$$

is referred to as the *Parreau decomposition* since it was pointed out by Parreau [12]. It is easily checked that $h \in HP_q(R)$ (resp. $HP_s(R)$) if and only if h^+ and h^- simultaneously belong to $HP_q(R)$ (resp. $HP_s(R)$). The following criterion is handy to use.

PROPOSITION 2.11. *An essentially positive harmonic function $u \in HP(R)$ on R is quasibounded on R if and only if*

$$u = H^R_{u|\delta R} \tag{2.12}$$

on R, and u is singular on R if and only if

$$u|\delta R = 0\,. \tag{2.13}$$

In paticular, the quasibounded component of u is $H^R_{u|\delta}$ and the singular component of u is $u - H^R_{u|\delta}$ in the Parreau decomposition of u:

$$u = H^R_{u|\delta} + (u - H^R_{u|\delta})\,. \tag{2.14}$$

Before proceeding to the proof of the above proposition we insert here a remark on some property of PWB-solutions. For two resolutive $[-\infty, +\infty]$-valued continuous functions φ and ψ on $\delta = \delta R$, we define $\varphi \cup \psi$ and $\varphi \cap \psi$ by

$$(\varphi \cup \psi)(\zeta) = \max(\varphi(\zeta), \psi(\zeta))\,, \quad (\varphi \cap \psi)(\zeta) = \min(\varphi(\zeta), \psi(\zeta))$$

for every $\zeta \in \delta$, which are again resolutive $[-\infty, +\infty]$-valued continuous functions on δ. We have the consistency between $(\vee, \wedge)$ and $(\cup, \cap)$ in the following sense:

$$H^R_\varphi \vee H^R_\psi = H^R_{\varphi\cup\psi}\,, \quad H^R_\varphi \wedge H^R_\psi = H^R_{\varphi\cap\psi} \tag{2.15}$$

on R. This will be used conveniently in the following proof of Proposition 2.11.

PROOF OF PROPOSITION 2.11. For simplicity we also write u for $u|\delta$ for $[-\infty, +\infty]$-valued continuous functions u on R^* such as $u \in HP(R)$. First we show the condition (2.12) for $u \in HP(R)$ implies the quasiboundedness of u on R. In fact,

$$u = H^R_u = \lim_{s,t\in\boldsymbol{R}^+, s,t\uparrow+\infty} H^R_{(u\cap s)\cup(-t)}$$

$$= \lim_{s,t\in\boldsymbol{R}^+, s,t\uparrow+\infty} (H^R_u \wedge H^R_s) \vee H^R_{-t} = \lim_{s,t\in\boldsymbol{R}^+, s,t\uparrow+\infty} (u \wedge s) \vee (-t)$$

on R, i.e., $u = \lim_{s,t\in\boldsymbol{R}^+, s,t\uparrow+\infty}(u \wedge s) \vee (-t)$ so that $u \in HP_q(R)$. Conversely, we assume that $u \in HP(R)$ is quasibounded on R and we are to show that u satisfies (2.12), i.e., $u = H^R_u$ on R. We may assume $u \geqq 0$ on R without loss of generality. Let $u_n := u \wedge n$ ($n \in \boldsymbol{N}$). Then $u_n \uparrow u$ locally uniformly on R and $u_n = u \cap n \uparrow u$ pointwise on δ as $n \uparrow +\infty$. Then

$$u = \lim_{n\uparrow+\infty} u_n = \lim_{n\uparrow+\infty} H^R_{u_n} = \lim_{n\uparrow+\infty} H^R_{u\cap n} = H^R_u$$

on R, as required.

Next we show the equivalence of (2.13) $u|\delta = 0$ and $u \in HP_s(R)$. We may again assume that $u > 0$ on R, and in this case, $u \in HP_s(R)$ is characterized by $u \wedge t = 0$ on R for every $t \in \boldsymbol{R}^+$, which is equivalent to that $u \geqq h \in HB(R)^+$ implies $h \equiv 0$ on R. Suppose first $u|\delta = 0$. Then $u \geqq u \wedge t$ on R for every $t \in \boldsymbol{R}^+$ and this implies that $0 = u|\delta \geqq (u \wedge t)|\delta \geqq 0$. Since $u \wedge t \in HB(R)^+$, the maximum principle (as a consequence of the comparison principle) shows that $u \wedge t = 0$ on R so that $u \in HP_s(R)$. Conversely, assume that $u \in HP_s(R)$. Then $u \wedge t = 0$ on R for every $t \in \boldsymbol{R}^+$ and, since $0 = (u \wedge t)|\delta = (u \cap t)|\delta = (u|\delta) \cap t$, we obtain $(u|\delta) \cap t = 0$ pointwise on δ for every $t \in \boldsymbol{R}^+$, which implies $u|\delta = 0$, i.e., (2.13), as required. □

COROLLARY 2.16. *An essentially positive harmonic function $u \in HP(R)$ on R is quasibounded if and only if, firstly, $u \in L^1(\delta, \omega)$, i.e.,*

$$\int_\delta |u| d\omega < +\infty , \tag{2.17}$$

and, secondly, the Poisson representation

$$u(z) = \int_\delta u d\omega_z \quad (z \in R) \tag{2.18}$$

holds, where ω_z is the harmonic measure on δ with the reference point $z \in R$.

PROOF. Observe that $u \in HB'(R)$ is equivalent to $u^+ + u^- \in HB'(R)$ and $u^+ + u^- = |u|$ on δ. In general, we have $H^R_\varphi(z) = \int_\delta \varphi d\omega_z$ for every $[-\infty, +\infty]$-valued continuous function $\varphi \in L^1(\delta, \omega)$. Hence, by Proposition 2.11, we see that $u \in HB'(R)$ if and only if both of (2.17) and (2.18) are valid. □

At this point it might be merely a grandmotherly solicitude to once more confirm that the Lebesgue space $L^1(\delta, \omega)$ is the quotient space consisting of equivalence classes of ω-integrable functions, where two ω-integrable functions are equivalent when they are identical ω-a.e. on δ. Thus, if we say that a function f belongs to $L^1(\delta, \omega)$, then we simply mean that f is ω-integrable over δ, i.e., $\int_\delta |f| d\omega < +\infty$. But if we say $f \in L^1(\delta, \omega)$ has a property $\mathcal{P}$ (such as continuity, positivity, and the like), then it means that there is a representative g of the equivalence class containing f (i.e., g is ω-integrable and $g = f$ ω-a.e. on δ) such that g has the property $\mathcal{P}$. For example, let us recall the Bauer theorem (cf. e.g. [2]) maintaining that a function f on δ is resolutive if and only if $f \in L^1(\delta, \omega)$, and in this case

$$H^R_f(z) = \int_\delta f d\omega_z$$

for every $z \in R$. This theorem actually consists of the following two assertions: firstly, if f is resolutive, then f is ω-integrable; secondly, if $f \in L^1(\delta, \omega)$, then we can find a resolutive g on δ with $g = f$ ω-a.e. on δ. Keeping these, possibly overcautious and unnecessary, remark in mind, however, we rephrase the above corollary 2.16 as stated below. We denote by $C(R; \overline{\boldsymbol{R}})$ with $\overline{\boldsymbol{R}} = [-\infty, \infty]$ the space of $[-\infty, \infty]$-valued continuous functions on δ.

PROPOSITION 2.19. *The following relations hold*:

$$HB'(R)|\delta = L^1(\delta, \omega) \subset C(\delta; \overline{\boldsymbol{R}}) . \tag{2.20}$$

PROOF. Once the first identity is established, then the last inclusion follows at once since $HB'(R) \subset C(R^*; \overline{\boldsymbol{R}})$ as a consequence of the definition of R^*. In view of (2.17), we see that the space $HB'(R)|\delta$ is included in the space $L^1(\delta,\omega)$ and therefore to complete the proof of (2.20) we only have to show that

$$L^1(\delta,\omega)\subset HB'(R)|\delta\,. \tag{2.21}$$

For the purpose choose an arbitrary $f\in L^1(\delta,\omega)$. By the Bauer theorem there is a $g\in L^1(\delta,\omega)$ such that $g=f$ ω-a.e. on δ and g is resolutive on δ. Set $u:=H^R_g$ on R. Since trivially $u=H^R_u$ on R, we deduce that $H^R_{u-g}=0$ on R. Observe that, since $(u-g)^+=(u-g)\cup 0$,

$$H^R_{(u-g)^+}=H^R_{(u-g)\cup 0}=H_{u-g}\vee H^R_0=0\vee 0=0\,.$$

Similarly, since $(u-g)^-=-((u-g)\cap 0)$, we have

$$H^R_{(u-g)^-}=-H^R_{(u-g)\cap 0}=-(H^R_{u-g}\wedge H^R_0)=-(0\wedge 0)=0\,.$$

Finally, since $|u-g|=(u-g)^+ +(u-g)^-$, we see that $H^R_{|u-g|}=0$ on R and of course at o. Hence

$$\int_\delta |u-g|d\omega=H^R_{|u-g|}(o)=0$$

so that $|u-g|=0$ ω-a.e., and with which $f=g$ ω-a.e. on δ implies $f=u$ ω-a.e. on δ, proving (2.21). □

3. Basics for square mean boundedness. We state three well-known fundamental properties of the Hardy space $HM_2(R)$ of square mean bounded harmonic functions on a Riemann surfaces $R\notin\mathcal{O}_G$. By the same reason as in Section 2 we also add proofs for them to stress how the Wiener compactification theory can be conveniently used for these purposes, which is of course again entirely unnecessary for those familiar with classical harmonic Hardy spaces and the ideal boundary theory or automorphic functions in the uniformization theory. We saw in the introduction that $R\in\mathcal{O}_G$ implies $HM_2(R)=\boldsymbol{R}$ and therefore we impose the restriction on R here and hereafter in this paper that R is hyperbolic in order to avoid the trivial situation. Recall that $HP(R)$ is a vector lattice with respect to the harmonic lattice operations $\vee$ and $\wedge$. Concerning this, we have the following fact.

PROPOSITION 3.1. *The space $HM_2(R)$ is a vector sublattice of the space $HP(R)$.*

PROOF. We only have to show that $u^+:=u\vee 0$ exists and belongs to $HM_2(R)$ for any given $u\in HM_2(R)$. If $u\equiv 0$ on R, then nothing is left to prove and hence we may assume that $u\not\equiv 0$ on R. Then there exists an $h\in H(R)^+\setminus\{0\}$ such that $u^2\leqq h$ on R. The key observation in this present proof is to note the following simple fact: $h^{1/2}$ is superharmonic on R as a consequence of $\Delta h^{1/2}=(-1/4h^{3/2})|\nabla h|^2\leqq 0$ on R. From $u\cup 0\leqq |u|$ and $u^2\leqq h$, or $|u|\leqq h^{1/2}$, on R, it follows that

$$u\cup 0\leqq h^{1/2}$$

on R. In view of the fact that u^+ is the least harmonic majorant of the subharmonic function $u\cup 0$ on R and $h^{1/2}$ is one of superharmonic majorant of $u\cup 0$ on R, we deduce that $u\cup 0\leqq u^+\leqq h^{1/2}$, or $(u^+)^2\leqq h$, on R, which amounts to the same that u^+ exists and belongs to $HM_2(R)$ so that $HM_2(R)$ is a vector sublattice of $HP(R)$, as required. □

The following is the most decisively important basic property of the class $HM_2(R)$ and the furthest from the triviality among three basic properties discussing in this section.

PROPOSITION 3.2. *Every square mean bounded harmoic function on R is quasibounded on R, i.e., $HM_2(R) \subset HB'(R) \equiv HP_q(R)$.*

PROOF. Take an arbitrary $u \in HM_2(R)$ and we are to show that $u \in HB'(R) \equiv HP_q(R)$. Let $u = u^+ - u^-$ be the Jordan decomposition of u in the vector lattice $HM_2(R)$ so that u^+ and u^- are in $HM_2(R)$. If we can show that u^+ and u^- belong to $HB'(R)$, then we may conclude that $u \in HB'(R)$. For this reason we can assume that $u > 0$, or $u \in HM_2(R)^+ \setminus \{0\}$, on R to prove $u \in HB'(R)$. Let $u = u_q + u_s$ be the Parreau decomposition of u into $u_q \in HP_q(R) \equiv HB'(R)$ and $u_s \in HP_s(R)$. Trivially $u_q^2 \leqq u^2$ and $u_s^2 \leqq u^2$ since $u_q \geqq 0$ and $u_s \geqq 0$ on R along with $u \geqq 0$ on R. We are thus to prove that if $u \in HM_2(R)^+ \cap HP_s(R)^+$, then $u \equiv 0$ on R. We prove this by contradiction. Assume contrariwise $u > 0$ on R so that $2a := u(o) > 0$, where o is the reference point in R. Let G be the component of the open set $\{z \in R; v(z) > 0\}$ containing o, where $v := u - a$. Since $u \in HM_2(R)^+$, there is an $h \in H(R)^+$ with $u^2 \leqq h$ on R. Hence

$$v^2 = (u-a)^2 \leqq 2u^2 + 2a^2 \leqq 2h + 2a^2 =: k \in H(R)^+ .$$

In particular, $v^2 \leqq k$ on G. By the definition of G, we have $v|\partial G = 0$. Observe that $G_\Omega := G \cap \Omega$ for regular subregions Ω of R containing o constitute an exhaustion $\{G_\Omega\}$ of G. Let $w_\Omega \in HB(G_\Omega)^+$ with boundary data $w_\Omega = 0$ on $(\partial G) \cap \Omega$ and $w_\Omega = 1$ on $(\partial \Omega) \cap G$. Since $(w_\Omega)_\Omega$ forms a decreasing net, the limit

$$w_G := \lim_{\Omega \uparrow R} w_\Omega$$

exists on G and the convergence is the local uniform one on $G \cup \partial G$ and is referred to as the relative harmonic measure (function) of G. Then either $w_G \equiv 0$ or $w_G > 0$ on G and if we have the former case, then we denote the situation by $G \in SO_{HB}$ (see [13]). By $u \in HP_s(R)$, we have $u|\delta = 0$ and a fortiori $v|\delta = -a < 0$. Hence we see that

$$\overline{G} \cap \delta = \emptyset ,$$

which is known to be equivalent to $G \in SO_{HB}$ (see [13]). Therefore we have

$$w_G = \lim_{\Omega \uparrow R} w_\Omega = 0$$

on G. By using the harmonic measure ν_Ω on ∂G_Ω with the reference point $o \in G_\Omega$ and the Schwarz inequality we see that

$$v(o) = H_v^{G_\Omega}(o) = H_{vw_\Omega}^{G_\Omega}(o) = \int_{\partial G_\Omega} v w_\Omega d\nu_\Omega$$

$$= \int_{\partial G_\Omega} v\sqrt{w_\Omega}\sqrt{w_\Omega} d\nu_\Omega \leqq \left(\int_{\partial G_\Omega} v^2 w_\Omega d\nu_\Omega \right)^{1/2} \left(\int_{\partial G_\Omega} w_\Omega d\nu_\Omega \right)^{1/2}$$

$$= H_{v^2 w_\Omega}^{G_\Omega}(o)^{1/2} \cdot H_{w_\Omega}^{G_\Omega}(o)^{1/2} = H_{v^2}^{G_\Omega}(o)^{1/2} \cdot H_{w_\Omega}^{G_\Omega}(o)^{1/2}$$

$$\leqq H_k^{G_\Omega}(o)^{1/2} \cdot H_{w_\Omega}^{G_\Omega}(o)^{1/2} = \sqrt{k(o)w_\Omega(o)}\,,$$

i.e., we have $v(o)/\sqrt{k(o)} \leqq \sqrt{w_\Omega(o)}$ for every admissible Ω. On letting $\Omega \uparrow R$, we derive a contradiction $0 < v(o)/\sqrt{k(o)} \leqq \sqrt{w_\Omega(o)} \downarrow 0$. □

Recall that the harmonic measure $\omega = \omega_o$ on the harmonic boundary $\delta = \delta R$ of R with the reference point $o \in R$ is the Borel measure on δ such that

$$H_\varphi^R(o) = \int_\delta \varphi d\omega$$

for every $\varphi \in C(\delta)$. If we denote by ω_z the harmonic measure on δ with the other reference point $z \in R$ switched from o, then we have

$$H_\varphi^R(z) = \int_\delta \varphi d\omega_z$$

for every $\varphi \in C(\delta)$. The following characterization of $HM_2(R)$ in terms of $HM_2(R)|\delta$ will be heavily used in our proof of the main theorem.

PROPOSITION 3.3. *A function u on R belongs to the class $HM_2(R)$ if and only if firstly u belongs to $HB'(R) \equiv HP_q(R)$ and secondly $u|\delta$ belongs to the Lebesgue space $L^2(\delta, \omega)$, i.e.,*

$$\int_\delta u^2 d\omega < +\infty\,. \tag{3.4}$$

PROOF. Suppose $u \in HM_2(R)$. By Proposition 3.2, u is in $HB'(R)$. For the proof of (3.4) we consider the Jordan decomposition $u = u^+ - u^-$ of u and note that u^+ and u^- belong again to $HM_2(R)$. Since $u^+ = u \cup 0$ and $u^- = -(u \cap 0)$ on δ, we have $|u| = u^+ + u^-$ on δ. Therefore (3.4) is equivalent to $\int_\delta (u^\pm)^2 d\omega < +\infty$. For this reason we may assume that $u > 0$ on R to show (3.4). Let $u^2 \leqq h \in H(R)^+$ on R. Then $H^R_{(u\wedge t)^2}$ is the least harmonic majorant of the subharmonic function $(u \wedge t)^2$ on R for any $t \in \boldsymbol{R}^+$ so that

$$\int_\delta (u \cap t)^2 d\omega = \int_\delta (u \wedge t)^2 d\omega = H_{(u\wedge t)^2}(o) \leqq h(o)$$

for every $t \in \boldsymbol{R}^+$. On letting $t \uparrow \infty$ we deduce $\int_\delta u^2 d\omega \leqq h(o) < +\infty$, i.e., (3.4) is derived. Conversely, when $u \in HB'(R)$ satisfies (3.4), we are to show that $u \in HM_2(R)$. Using the Poisson expression

$$u(z) = \int_\delta u d\omega_z \quad (z \in R)$$

of $u \in HB'(R)$, by the Schwartz inequality we deduce

$$u(z)^2 = \left(\int_d u \cdot 1 d\omega_z\right)^2 \leqq \left(\int_\delta u^2 d\omega_z\right) \cdot \left(\int_\delta d\omega_z\right) = \int_\delta u^2 d\omega_z$$

for $z \in R$. In view of (3.4), $\int_\delta u^2 d\omega_z$ defines a harmonic function $h \geqq 0$ on R and thus $u^2 \leqq h \in H(R)^+$, i.e., $u \in HM_2(R)$, as required. □

COROLLARY 3.5. *The following representation theorem holds for* $HM_2(R)$:

$$HM_2(R)|\delta = L^2(\delta, \omega) \tag{3.6}$$

so that every $f \in L^2(\delta, \omega)$ *has a continuous representative in the extended sense.*

PROOF. By Propositions 3.3 and 2.19, we infer that

$$HM_2(R)|\delta = (HB'(R)|\delta) \cap L^2(\delta, \omega) = L^1(\delta, \omega) \cap L^2(\delta, \omega) = L^2(\delta, \omega)$$

and we are done. □

4. An afforested surface. We have finished two preparatory discussions, one on the Wiener compactifications and the other on the basic properties about square mean bounded harmonic functions, but we still need to add one more preliminary material which we call afforested surfaces introduced in [11]. By a *slit* γ in a Riemann surface X we mean a simple arc in X such that there exists a parametric disc $U := \{|z| < 1\}$ on X in which γ is represented as $\gamma = [-r, r] := \{z \in U; \Im z = 0, |\Re z| \leqq r\}$ $(0 < r < 1)$. By a *common slit* γ in two Riemann surfaces X and Y we understand that there are slits γ_X in X and γ_Y in Y such that γ_X and γ_Y can be identified with a slit γ both in X and Y. We denote by

$$(X \setminus \gamma) \bowtie_\gamma (Y \setminus \gamma)$$

the Riemann surface obtained by pasting $X \setminus \gamma$ to $Y \setminus \gamma$ crosswise along γ. An *afforested surface* $R := \langle P, (T_i)_{i\in \boldsymbol{N}}, (\sigma_i)_{i\in \boldsymbol{N}}\rangle$ consists of three ingredients: an open Riemann surface P, called a *plantation*; a sequence $(T_i)_{i\in \boldsymbol{N}}$ of open Riemann surfaces T_i each of which is called a *tree*; and a sequence $(\sigma_i)_{i\in \boldsymbol{N}}$ of common slits σ_i in P and T_i $(i \in \boldsymbol{N})$ called *roots* of trees T_i or *root holes* in P, where σ_i are assumed to be mutually disjoint and not accumulating in P (cf. the remark stated at the end of this section). Then the afforested surface $R := \langle P, (T_i)_{i\in \boldsymbol{N}}, (\sigma_i)_{i\in \boldsymbol{N}}\rangle$ is given by

$$R := \cdots \left(\left(\left(P \setminus \bigcup_{i\in \boldsymbol{N}} \sigma_i \right) \bowtie_{\sigma_1} (T_1 \setminus \sigma_1) \right) \bowtie_{\sigma_2} (T_2 \setminus \sigma_2) \right) \cdots, \tag{4.1}$$

which is called the *afforested surface* formed by foresting each tree T_i to P at its root σ_i for every $i \in \boldsymbol{N}$. Although after all it amounts to the same, some might feel the following expression of R is easier to understand than the above original form (4.1) of R:

$$R = \bigcup_{n\in \boldsymbol{N}} \left(\left(P \setminus \bigcup_{i\in \boldsymbol{N}} \sigma_i \right) \bowtie_{\sigma_n} (T_n \setminus \sigma_n) \right).$$

We are ready to proceed to the proof of the main theorem of this paper: there exists a Riemann surface W such that

$$HM_2(W) = HD(W) \tag{4.2}$$

and yet W carries sufficiently many square mean bounded harmonic functions in the sense that

$$\dim HM_2(W) = \infty. \tag{4.3}$$

The surface W is given as an afforested surface $W := \langle P, (T_i)_{i\in \boldsymbol{N}}, (\sigma_i)_{i\in \boldsymbol{N}}\rangle$ as described below. First as the plantation P of W we take simply the complex plane $\boldsymbol{C}$:

$$P := \boldsymbol{C}\,. \tag{4.4}$$

One of the major contributions in the history of the classification theory of Riemann surfaces is the establishment of the nonemptiness of the class $\mathcal{O}_{HP} \setminus \mathcal{O}_G$, i.e., the discovery of a hyperbolic Riemann surface T with $HP(T) = \boldsymbol{R}$ independently by Sario and Tôki (cf. e.g. [1], [13]). We fix such a T and let

$$T_i \equiv T \in \mathcal{O}_{HP} \setminus \mathcal{O}_G \quad (i \in \boldsymbol{N})\,. \tag{4.5}$$

Fix a sequence $(U_i)_{i\in\boldsymbol{N}}$ of mutually disjoint discs U_i in $\boldsymbol{C}$ not accumulating in $\boldsymbol{C}$ explicitly given by

$$U_i := \Delta(4i, 1) \quad (i \in \boldsymbol{N})\,, \tag{4.6}$$

where $\Delta(c, r)$ (resp. $\overline{\Delta}(c, r)$) is the open (resp. closed) disc with radius $r > 0$ centered at $c \in \boldsymbol{C}$. Fix a reference point o arbitrarily in $\boldsymbol{C} \setminus \bigcup_{i\in\boldsymbol{N}} U_i$ but for definiteness we let $o = 0$, the origin of $\boldsymbol{C}$, unless otherwise is explicitly mentioned. We denote by M_i the *Harnack constant* of the compact set $\{o\} \cup \partial U_i$ with respect to the class $H(\boldsymbol{C} \setminus \bigcup_{k\in\boldsymbol{N}}(1/2)U_k)^+$, where $(1/2)U_k = \Delta(4k, 1/2)$, so that M_i is the infimum of $\lambda \in [1, \infty]$ such that

$$\lambda^{-1}u(z_1) \leqq u(z_2) \leqq \lambda u(z_1)$$

for every $u \in H(\boldsymbol{C} \setminus \bigcup_{k\in\boldsymbol{N}}(1/2)U_k)^+$ and for every pair (z_1, z_2) of points z_1 and z_2 in $\{o\} \cup \partial U_i$. Hence $M_i \in [1, \infty)$ and, in particular,

$$\sup_{\partial U_i} u \leqq M_i \cdot u(o)$$

for every $u \in H(\boldsymbol{C} \setminus \bigcup_{k\in\boldsymbol{N}}(1/2)U_k)^+$. In the copy T_i of the fixed $T \in \mathcal{O}_{HP} \setminus \mathcal{O}_G$ we take a disc $\hat{U}_i$ viewed as a copy of U_i and take a slit σ_i in U_i given by

$$\sigma_i := [-s_i, s_i] + 4i \quad (s_i \in (0, 1/2)) \tag{4.7}$$

which is denoted also by $\hat{\sigma}_i$ if it is considered in $\hat{U}_i$ $(i \in \boldsymbol{N})$. We require for $(s_i)_{i\in\boldsymbol{N}}$ to satisfy the two conditions

$$0 < s_{i+1} \leqq s_i < 1/2 \quad (i \in \boldsymbol{N})\,, \tag{4.8}$$

i.e., $(s_i)_{i\in\boldsymbol{N}}$ is a nonincreasing sequence, and, more seriously,

$$K := 2\pi \sum_{n\in\boldsymbol{N}} \frac{M_n}{\log(1/s_n)} < +\infty\,, \tag{4.9}$$

which is achieved if $(s_n)_{n\in\boldsymbol{N}} \downarrow 0$ enough rapidly. For example the choice

$$s_n := \exp\Big(- 2^n \sum_{1\leqq j\leqq n} M_j\Big) \quad (n \in \boldsymbol{N})$$

suffices. Then the afforested surface $W := \langle \boldsymbol{C}, (T_i)_{i\in \boldsymbol{N}}, (\sigma_i)_{i\in \boldsymbol{N}}\rangle$ determined by (4.4), (4.5), and (4.7) with the specification (4.8) and (4.9) is the one sought, i.e.,

$$W := \cdots (((\boldsymbol{C} \setminus \Sigma) \bowtie_{\sigma_1} (T_1 \setminus \sigma_1)) \bowtie_{\sigma_2} (T_2 \setminus \sigma_2)) \cdots , \tag{4.10}$$

where we have set

$$\Sigma := \bigcup_{n\in \boldsymbol{N}} \sigma_n . \tag{4.11}$$

Corresponding to the remark right after (4.1), the surface W in (4.10) may also be given by the following equivalent form:

$$W = \bigcup_{n\in \boldsymbol{N}} ((\boldsymbol{C} \setminus \Sigma) \bowtie_{\sigma_n} (T_n \setminus \sigma_n)) .$$

We conclude this section by stating a remark on the set σ_n, which is expected to be useful for better understanding for what follows in the sequel. Originally σ_n is the common slit $[-s_n, s_n]$ in the parametric disc U_n in $\boldsymbol{C}$ and that $\hat{U}_n$ in T_n, where U_n and $\hat{U}_n$ are identified. The set σ_n is also the boundary of $\boldsymbol{C}\setminus\sigma_n$ or of $T_n\setminus\sigma_n$ over σ_n. In this case σ_n can be understood as a Jordan curve $\sigma_n^+ \cup \sigma_n^-$ or rather $\sigma_n^+ - \sigma_n^-$, where σ_n^+ (resp. σ_n^-) is the upper (resp. lower) edge of σ_n, i.e., the so called Carathéodory boundary $\sigma_n^+ - \sigma_n^-$ of $\boldsymbol{C} \setminus \sigma_n$ or $T_n \setminus \sigma_n$ lying over σ_n. When we consider σ_n in W it becomes an analytic Jordan curve homeomorphic to $\sigma_n^+ - \sigma_n^-$. Thus σ_n stands for either simply a slit $[-s_n, s_n]$, a Jordan curve $\sigma_n^+ - \sigma_n^-$ as a Carathéodory boundary of $\boldsymbol{C} \setminus \sigma_n$ or $T_n \setminus \sigma_n$ over σ_n, or an analytic Jordan curve in W. In case we talk about σ_n we will often not mention explicitly which one of the above three we mean but mostly it can be easily understood which one of these three is the case from the context or the situation.

5. Potential theory on W. Our goal is to show that the W given in (4.10) and (4.11) satifies (4.2) and (4.3). Keeping this in mind we start by establishing some basic potential theoretic properties of the above W constructed in (4.10) and (4.11). Both conditions (4.8) and (4.9) are essential in establishing the validity of (4.2) and (4.3) shown in the sequel but in the present section only (4.8) is needed and (4.9) is redundant for the time being. We use essentially the Wiener compactification W^*, the Wiener boundary $\gamma = \gamma W$, and the most important Wiener harmonic boundary $\delta = \delta W$ of W. We denote by δT_n the Wiener harmonic boundary of T_n. We can view that $\delta T_n \subset \delta W$. Since $T_n = T \in \mathcal{O}_{HP} \setminus \mathcal{O}_G$, δT_n consists of only one point, say d_n. Thus

$$\delta T_n = \{d_n\} \subset \delta W \quad (n \in \boldsymbol{N}) .$$

We will see that the subset

$$\hat{\delta} W := \bigcup_{n\in \boldsymbol{N}} \delta T_n = \{d_1, d_2, \ldots, d_n, \ldots\} \subset \delta W$$

plays an important role and is the core part of δW. The following fact shows this situation clearly. Note that functions in $HB'(W) \equiv HP_q(W)$ are $[-\infty, +\infty]$-valued continuous on W^* and are determined uniquely by their values on δW (cf. Section 2).

PROPOSITION 5.1 (The unicity principle). *If* $u \in HB'(W) \equiv HP_q(W)$ *satisfies* $u|\hat{\delta}W = 0$, *then* $u \equiv 0$ *on* W. *In particular*, $\delta W \setminus \hat{\delta}W$ *is of harmonic measure zero and hence* $\hat{\delta}W$ *is dense in* δW.

PROOF. We may suppose that $u \in HB(W)^+$. Contrary to the assertion, assume that $\alpha := \sup_W u > 0$. Since $u|\delta T_n = 0$ and $u \leqq \alpha$ on $\hat{\sigma}_n$, which is as before σ_n considered in $\hat{U}_n$, the usual maximum principle yields that $u|\partial\hat{U}_n \leqq \beta < \alpha$, where β is independent of $n \in \boldsymbol{N}$ by virtue of (4.8). Since $u \leqq \alpha$ on ∂U_n and $u \leqq \beta$ on $\partial\hat{U}_n$, by considering u on $(U_n \setminus \sigma_n) \uplus_{\sigma_n = \hat{\sigma}_n} (\hat{U}_n \setminus \hat{\sigma}_n)$ in the sense that $(U_n \setminus \sigma_n) \uplus_{\sigma_n} (\hat{U}_n \setminus \sigma_n)$ since $\sigma_n = \hat{\sigma_n}$, we see that $u \leqq (\alpha + \beta)/2$ on $\sigma_n = \hat{\sigma}_n$ so that $u \leqq (\alpha + \beta)/2$ on T_n for every $n \in \boldsymbol{N}$. Let $E(z)$ be the harmonic function on $\boldsymbol{C} \setminus \sigma_1$ such that E has vanishing boundary values on σ_1 and $\lim_{z\to\infty} E(z) = +\infty$. Comparing boundary values of $(1/m)E + (\alpha + \beta)/2$ and u on $\partial(\boldsymbol{C} \setminus \Sigma)$, we see that

$$0 \leqq u \leqq (1/m)E + (\alpha + \beta)/2$$

on $\boldsymbol{C} \setminus \Sigma$ for every $m \in \boldsymbol{N}$. On letting $m \uparrow \infty$ we have that $u \leqq (\alpha + \beta)/2$ on $\boldsymbol{C} \setminus \Sigma$. Hence $u \leqq (\alpha + \beta)/2$ on W, which implies $\alpha \leqq (\alpha + \beta)/2$, or $\alpha \leqq \beta$, a contradiction. □

PROPOSITION 5.2 (The comparison principle). *If* $u \in HB'(W) \equiv HP_q(W)$ *satisfies* $u|\hat{\delta}W \geqq 0$, *then* $u \geqq 0$ *on* W.

PROOF. By the $[-\infty, +\infty]$-valued continuity of u and by the unicity principle, we see that $u|\delta W \geqq 0$. Then by the comparison principle in Section 2, we must conclude that $u \geqq 0$ on W. □

Fix an arbitrary $i \in \boldsymbol{N}$ and note that d_i is an isolated point in δW. We can find $\varphi \in C(\delta W)$ such that $\varphi(d_i) = 1$ and $\varphi|(\delta W \setminus \{d_i\}) = \varphi|(\delta W \setminus \delta T_i) = 0$. Let $e_i = H_\varphi^R$. By using the unicity principle we can conclude that e_i is the unique function in $HB(W)$ such that $e_i|\delta T_i = 1$ and $e_i|(\delta W \setminus \delta T_i) = 0$. In particular

$$e_i|\delta T_j = \delta_{ij} \quad \text{(the Kronecker delta)} \tag{5.3}$$

for every pair $(i, j) \in \boldsymbol{N} \times \boldsymbol{N}$. Let ω be the harmonic measure (cf. Section 2) on δW with respect to the reference point $o = 0 \in W$. Then

$$\omega(\{d_i\}) = \int_{\delta W} e_i d\omega = e_i(o) \tag{5.4}$$

and, again by the unicity principle, $\sum_{i \in \boldsymbol{N}} e_i \equiv 1$ on W so that

$$\sum_{i \in \boldsymbol{N}} e_i(o) = \sum_{i \in \boldsymbol{N}} \omega(\{d_i\}) = \omega(\hat{\delta}W) = 1\,.$$

Hence we see that

$$\omega = \sum_{i \in \boldsymbol{N}} e_i(0)\delta_i\,, \tag{5.5}$$

where δ_i is the Dirac measure on δW supported at δT_i $(i \in \boldsymbol{N})$.

The next proposition is nothing but the rephrasing of Corollary 2.16, but we repeat its proof to get better understanding fitting to the present specified situation.

PROPOSITION 5.6 (The representation theorem). *An essentially positive harmonic function u on R is quasibounded on R if and only if there exists a unique sequence $(a_i)_{i\in\boldsymbol{N}}$ in $\boldsymbol{R}$ with*

$$\sum_{i\in\boldsymbol{N}} |a_i| e_i(o) < \infty \tag{5.7}$$

such that u is represented by

$$u = \sum_{i\in\boldsymbol{N}} a_i e_i \tag{5.8}$$

on R. The convergence in (5.8) *is the local uniform one on R.*

PROOF. Suppose first that $u \in HB'(W) \equiv HP_q(W)$ and put $a_i = u(d_i)$ $(i \in \boldsymbol{N})$, which are easily seen to be finite. We are to show that $(a_i)_{i\in\boldsymbol{N}}$ satisfies two conditions (5.7) and (5.8). We start from the special case of $u \geqq 0$ on R. Consider functions

$$u_i := \sum_{1\leqq j\leqq i} a_j e_j \quad (i \in \boldsymbol{N}).$$

By the comparison principle, $(u_i)_{i\in\boldsymbol{N}}$ is seen to be an increasing sequence in $HB(W)$ dominated by u on R. Hence there exists $v \in HP_q(W)$ such that $u_i \leqq \lim_{j\uparrow\infty} u_j = v \leqq u$ for every $i \in \boldsymbol{N}$. Clearly $v = u$ on $\hat{\delta} W$ and a fortiori the unicity principle implies that $v = u$ on W, or (5.8) is valid locally uniformly on W. Clearly $\sum_{i\in\boldsymbol{N}} |a_i| e_i(0) = \sum_{i\in\boldsymbol{N}} a_i e_i(o) = u(o) < \infty$, i.e., (5.7) is valid. Then next we turn to the general case and let $u = u^+ - u^-$ be the Jordan decomposition of u in the vector lattice $HP_q(W)$. Like we have set $a_i = u(d_i)$ we let $a_i^{\pm} = u^{\pm}(d_i)$ for every $i \in \boldsymbol{N}$. Then, since $u^+ = u \cup 0$ and $u^- = -(u \cap 0)$ on δW, we see that

$$a_i = a_i^+ - a_i^-$$

and

$$|a_i| = a_i^+ + a_i^- = \text{ either } a_i \text{ or } -a_i$$

according again to $a_i \geqq 0$ or $a_i < 0$ for every i in $\boldsymbol{N}$. Therefore, from $\sum_{i\in\boldsymbol{N}} |a_i^{\pm}| e_i(o) = \sum_{i\in\boldsymbol{N}} a_i^{\pm} e_i(o) < \infty$, it follows that

$$\sum_{i\in\boldsymbol{N}} |a_i| e_i(o) = \sum_{i\in\boldsymbol{N}} a_i^+ e_i(o) + \sum_{i\in\boldsymbol{N}} a_i^- e_i(o) < \infty,$$

i.e., (5.7) is valid, and similarly, from the fact that $u^{\pm} = \sum_{i\in\boldsymbol{N}} a_i^{\pm} e_i$ locally uniformly on R, it follows that

$$u = u^+ - u^- = \sum_{i\in\boldsymbol{N}} a_i^+ e_i - \sum_{i\in\boldsymbol{N}} a_i^- e_i = \sum_{i\in\boldsymbol{N}} (a_i^+ - a_i^-) e_i = \sum_{i\in\boldsymbol{N}} a_i e_i$$

locally uniformly on W, i.e., the condition (5.8) is also deduced.

Conversely suppose the u is given by (5.8) with the sequence $(a_i)_{i\in\boldsymbol{N}}$ satisfying the condition (5.7). Then, since $u(d_i) = a_i$ for every $i \in \boldsymbol{N}$ and $u^+ = u \cup 0$ and $u^- = -(u \cap 0)$ on δW, we see that

$$u^+ = \sum_{a_i \geqq 0} a_i e_i \quad \text{and} \quad u^- = -\sum_{a_i<0} a_i e_i ,$$

both of which are convergent at o and hence locally uniformly on W. Thus both of u^+ and u^- belong to $HP_q(W)$ so that we can conclude that $u = u^+ - u^- \in HP_q(W)$, as required. □

At the end of this section we state a characterization of $HM_2(W)$ in terms of boundary values of functions in $HM_2(W)$ on $\hat{\delta}W$, which is again simply a rephrasing of Proposition 3.3 or rather a specification to the present concrete surface W.

PROPOSITION 5.9 (Criterion of square mean boundedness). *An essentially positive harmonic function u on W belongs to $HM_2(W)$ if and only if there exists a unique sequence $(a_i)_{i\in\boldsymbol{N}}$ in $\boldsymbol{R}$ with*

$$\sum_{i\in\boldsymbol{N}} a_i^2 e_i(o) < \infty \tag{5.10}$$

such that the representation

$$u = \sum_{i\in\boldsymbol{N}} a_i e_i \tag{5.11}$$

holds on W, where the above series converges locally uniformly on W.

PROOF. Suppose first that $u \in HM_2(W)$. Since $HM_2(W) \subset HB'(W)$ by Proposition 3.2, we see by Proposition 5.6 the existence of the unique sequence $(a_i)_{i\in\boldsymbol{N}}$ with (5.7) and (5.8). Thus in particular (5.11) is valid. Applying Proposition 3.3 to our present u, we obtain

$$\sum_{i\in\boldsymbol{N}} a_i^2 e_i(o) = \int_{\delta W} u^2 d\omega < \infty ,$$

i.e., (5.10) is valid. Conversely, assume that we have a sequence $(a_i)_{i\in\boldsymbol{N}}$ with (5.10) and (5.11) for u. Then

$$\left(\int_{\delta W} |u| d\omega\right)^2 \leqq \int_{\delta W} u^2 d\omega = \sum_{i\in\boldsymbol{N}} a_i^2 e_i(o) < +\infty$$

with (5.10) and (5.11) show that $u \in HB'(W)$ by Corollary 2.16 and $u|\delta W$ belongs to the Lebesgue space $L^2(\delta W, \omega)$. Thus Proposition 3.3 shows that u belongs to the class $HM_2(W)$. □

6. Dirichlet integrals of harmonic measures. Mostly there are no predominating relations between harmonic measures $\mu_z(K) = \int_K d\mu_z$ of compact subsets $K \subset \delta R$, where μ_z is the harmonic measure on the harmonic boundary δR of some general Riemann surface R with respect to any reference point $z \in R$, and their Dirichlet integrals $D(\mu_\cdot(K); R)$; there always exists a sequence $(K_n)_{n\in\boldsymbol{N}}$ of compact subsets K_n of δR such that $D(\mu_\cdot(K_n); R) \downarrow 0$ and $\mu_\cdot(K_n) \uparrow 1$ as $n \uparrow \infty$ and on the other hand it is not too rare that there exists a sequence

$(K_n)_{n\in\boldsymbol{N}}$ such that $D(\mu.(K_n); R) \uparrow \infty$ and $\mu.(K_n) \downarrow 0$ as $n \uparrow \infty$. Hence it is remarkable if there is an R such that $D(\mu.(K); R)$ is uniformly estimated by $\mu.(K)$ for every K. One of the central purpose of this section is to show that the Riemann surface W in (4.10) is of this sort.

The harmonic measure ω on the harmonic boundary $\delta = \delta W$ of the surface W is given by (5.5) and hence ω is determined by its values of each one point subset $\{d_n\} = \delta T_n$ of $\hat{\delta} = \hat{\delta} W$: $\omega(\{d_n\}) = \omega(\delta T_n) = e_n(o)$ $(n \in \boldsymbol{N})$. Hence the comparison of the harmonic measure $e_n(o)$ and the Dirichlet integral $D(e_n; W)$ is the essential task in this situation.

PROPOSITION 6.1 (The main estimates). *The set of inequalities*

$$D(e_n; W) \leqq K \cdot e_n(o) \quad (n \in \boldsymbol{N}) \tag{6.2}$$

holds with the constant K given in (4.9).

PROOF. For each fixed $n \in \boldsymbol{N}$ we consider the subsurface

$$W_n := \left(\boldsymbol{C} \setminus \bigcup_{m\in\boldsymbol{N}} \sigma_m\right) \bowtie_{\sigma_n} (T_n \setminus \sigma_n) \tag{6.3}$$

of W and the function f_n defined on W with the following conditions: $f_n \in C(W^*)$; $f_n \in H(W_n)$; $f_n = 1$ on $\delta T_n = \{d_n\}$; $f_n = 0$ on $W \setminus W_n$. Instead of showing (6.2) directly, we first complete the painstaking task of establishing

$$D(f_n; W) \leqq K \cdot f_n(0) \quad (n \in \boldsymbol{N}) \tag{6.4}$$

and then derive (6.2) from the above (6.4) with the aid of simple relations

$$D(e_n; W) \leqq D(f_n; W) \quad \text{and} \quad e_n(o) \geqq f_n(o) \quad (n \in \boldsymbol{N})\,. \tag{6.5}$$

This is the plan of our proof here and we have to go long way to achieve this object. Before proceeding to the main stream of our proof, we wish to finish here the proof of almost trivial assertion (6.5) above. Since $f_n \in \mathcal{R}(W)$ as is easily seen (cf. (2.3)) and clearly e_n is seen to be the harmonic part of f_n in the Royden-Brelot decomposition of f_n (cf. (2.4) and (2.5)), we see that the first inequality of (6.5) is valid. Using the comparison principle in Section 2 and Proposition 5.1 on the subregion W_n of W we see that $e_n \geqq f_n$ on W_n and thus on W. Hence in particular $e_n(o) \geqq f_n(o)$, the second inequality of (6.5), is deduced. Thus the validity of (6.4) implies that, by using (6.5) just established,

$$D(e_n; W) \leqq D(f_n; W) \leqq K \cdot f_n(o) \leqq K \cdot e_n(o)\,,$$

i.e., (6.2), so that the proof of Proposition 6.1 is complete. Thus from now on we concentrate ourselves to the proof of (6.4).

With the aim of studying (W_n, f_n) for every $n \in \boldsymbol{N}$ we introduce two more kinds of auxiliary subregions and functions. For every $n \in \boldsymbol{N}$ we fix arbitrarily a regular exhaustion $(T_{ni})_{i\in\boldsymbol{N}}$ of T_n with $T_{ni} \supset \overline{\hat{U}}_n$. For each fixed $n \in \boldsymbol{N}$ we consider the subregion

$$W_{ni} := \left(\boldsymbol{C} \setminus \bigcup_{m\in\boldsymbol{N}} \sigma_m\right) \bowtie_{\sigma_n} (T_{ni} \setminus \sigma_n) \tag{6.6}$$

of W for every $i \in \boldsymbol{N}$ and the function f_{ni} on W characterized by the conditions $f_{ni} \in C(W^*)$, $f_{ni} \in H(W_{ni})$, $f_{ni} = 1$ on $T_n \setminus T_{ni}$, and $f_{ni} = 0$ on $\bigcup_{m \in \boldsymbol{N} \setminus \{n\}} T_m$. Finally consider the subregion

$$W_{nij} := \left(\Delta(0, 4j+2) \setminus \bigcup_{1 \leqq m \leqq j} \sigma_m \right) \uplus_{\sigma_n} (T_{ni} \setminus \sigma_n) \tag{6.7}$$

for every $n \in \boldsymbol{N}$, $i \in \boldsymbol{N}$, and $j \in \boldsymbol{N}$ with $j \geqq n$ and the function f_{nij} determined by the properties $f_{nij} \in C(W^*)$, $f_{nij} \in H(W_{nij})$, $f_{nij} = 1$ on $T_n \setminus T_{ni}$, and $f_{nij} = 0$ on $W \setminus (W_{nij} \cup (T_n \setminus T_{ni}))$. For the existence of the above functions f_n, f_{ni}, and f_{nij}, see [13, Chapter III], for example.

It is clear that the sequence $(f_{nij})_{j \geqq n}$ is increasing and $f_{nij} \leqq f_{ni}$ on W and therefore, by the unicity principle, we have

$$\lim_{j \uparrow \infty} f_{nij} = f_{ni} \tag{6.8}$$

locally uniformly on W. Observe that $f_{nij} - f_{ni(j+k)} = 0$ on $W \setminus W_{ni(j+k)}$ for every $j \geqq n$ and $k \in \boldsymbol{N}$. By the Stokes formula we see that

$$\begin{aligned} D(f_{nij} - f_{ni(j+k)}, f_{ni(j+k)}; W) &= D(f_{nij} - f_{ni(j+k)}, f_{ni(j+k)}; W_{ni(j+k)}) \\ &= \int_{\partial W_{ni(j+k)}} (f_{nij} - f_{ni(j+k)}) * df_{ni(j+k)} = 0\,, \end{aligned}$$

which implies that $D(f_{nij}, f_{ni(j+k)}; W) = D(f_{ni(j+k)}; W)$. Here $D(u, v; W) := \int_W du \wedge *dv$ is, by definition, the *mutual Dirichlet integral* of u and v over W so that $D(u, u; W) = D(u; W)$. Thus we can now conclude that

$$D(f_{nij} - f_{ni(j+k)}; W) = D(f_{nij}; W) - D(f_{ni(j+k)}; W) \tag{6.9}$$

for every $j \geqq n$ and $k \in \boldsymbol{N}$. By a standard simple argument, we can conclude by keeping (6.8) in mind that (6.9) implies the validity of

$$\lim_{j \uparrow \infty} D(f_{ni} - f_{nij}; W) = 0 \tag{6.10}$$

for every fixed n and i in $\boldsymbol{N}$.

To derive the counterpart of (6.8) and (6.9) for the sequence $(f_{ni})_{i \in \boldsymbol{N}}$ we once more use the sequence $(f_{nij})_{j \geqq n}$ for each $i \in \boldsymbol{N}$. As in the case of (6.8) it is again clear that the sequence $(f_{nij})_{i \in \boldsymbol{N}}$ is decreasing for any pair of $n \in \boldsymbol{N}$ and $j \geqq n$ and $f_{n(i+k)j} \leqq f_{nij} \leqq 1$ on W. Letting $j \uparrow \infty$ in the last sequence of inequalities, we deduce that $f_n \leqq f_{n(i+k)} \leqq f_{ni} \leqq 1$ on W, i.e., the sequence $(f_{ni})_{i \in \boldsymbol{N}}$ is decreasing and $f_n \leqq f_{ni} \leqq 1$ on W for every $i \in \boldsymbol{N}$. Once more, in view of the unicity principle we can conclude that

$$\lim_{i \uparrow \infty} f_{ni} = f_n \tag{6.11}$$

locally uniformly on W. Similarly as above we still use $(f_{nij})_{j \geqq n}$ for any fixed $n \in \boldsymbol{N}$ and for every $i \in \boldsymbol{N}$. Observe that $f_{nij} - f_{n(i+k)j} = 0$ on $\partial W_{n(i+k)j}$ for every $i \in \boldsymbol{N}$ and $k \in \boldsymbol{N}$ with fixed $n \in \boldsymbol{N}$ and every $j > n$. By exactly the same procedure as we did in the derivation of (6.9), we obtain

$$D(f_{nij} - f_{n(i+k)j}; W) = D(f_{nij}; W) - D(f_{n(i+k)j}; W)$$

for any fixed $n \in \boldsymbol{N}$ and any i and k in $\boldsymbol{N}$ and further for every $j \geqq n$. Making $j \uparrow \infty$ in the above displayed identity, we obtain

$$D(f_{ni} - f_{n(i+k)}; W) = D(f_{ni}; W) - D(f_{n(i+k)}; W) \tag{6.12}$$

for every i and k in $\boldsymbol{N}$ with any fixed $n \in \boldsymbol{N}$. As we derived (6.10) from (6.9), since we also have (6.11), we can deduce that

$$\lim_{i\uparrow\infty} D(f_n - f_{ni}; W) = 0 \tag{6.13}$$

for every fixed $n \in \boldsymbol{N}$.

By the Stokes formula we deduce

$$D(f_{nij}; W) = D(f_{nij}; W_{nij}) = \int_{W_{nij}} df_{nij} \wedge *df_{nij}$$

$$= \int_{\partial W_{nij}} f_{nij} * df_{nij} = \int_{\partial T_{ni}} *df_{nij}\ .$$

The total flux of f_{nij} across ∂W_{nij} is zero as a consequence of again the Stokes formula we have

$$D(f_{nij}; W) = -\sum_{1\leqq m\leqq j; m\neq n} \int_{\sigma_m} *df_{nij} - \int_{\partial\Delta(0,4j+2)} *df_{nij} \tag{6.14}$$

for every $j \geqq n$. Observe that $(f_{nij})_{j\geqq n}$ increasingly converges to f_{ni} locally uniformly on W, $0 \leqq - * df_{nij} \uparrow - * df_{ni}$ $(j \uparrow \infty)$ uniformly on each σ_m $(m \neq n)$, and $- * df_{nij} \geqq 0$ on $\partial\Delta(0, 4j + 2)$, for any fixed $j \geqq n$. Thus the identity (6.14) yields

$$D(f_{nij}; W) \geqq -\sum_{1\leqq m\leqq j; m\neq n} \int_{\sigma_m} *df_{nij} \geqq -\sum_{1\leqq m\leqq j'; m\neq n} \int_{\sigma_m} *df_{nij}$$

for every $j \geqq j'$. Letting $j \uparrow \infty$ in the above displayed inequality, (6.10) assures that

$$D(f_{ni}; W) \geqq -\sum_{1\leqq m\leqq j'; m\neq n} \int_{\sigma_m} *df_{ni}\ .$$

On letting $j' \uparrow \infty$, we conclude that

$$D(f_{ni}; W) \geqq -\sum_{m\in\boldsymbol{N}\setminus\{n\}} \int_{\sigma_m} *df_{ni} \tag{6.15}$$

for any fixed $n \in \boldsymbol{N}$ and $i \in \boldsymbol{N}$. On the other hand we also derive from (6.14) that

$$D(f_{nij}; W) \leqq -\sum_{m\in\boldsymbol{N}\setminus\{n\}} \int_{\sigma_m} *df_{nij} - \int_{\partial\Delta(0,4j+2)} *df_{nij} \tag{6.16}$$

for any $j \geqq n$. We consider the annulus $A_{nj} := \Delta(0, 4j+2) \setminus \overline{U}_n$ $(j \geqq n)$ and the modulus function $w \in H(A_{nj}) \cap C(\overline{A}_{nj})$ with $w|\partial U_n = 1$ and $w|\partial\Delta(0, 4j+2) = 0$. Clearly $f_{nij} \leqq w$ on A_{nj} and therefore $0 \leqq -*df_{nij} \leqq -*dw$ on $\partial\Delta(0, 4j+2)$. Hence we deduce that

$$0 \leqq -\int_{\partial\Delta(0,4j+2)} *df_{nij} \leqq -\int_{\partial\Delta(0,4j+2)} *dw$$

$$= \int_{\partial A_{nj}} w * dw = D(w; A_{nj}) = \frac{2\pi}{\mathrm{mod} A_{nj}} .$$

Here $\mathrm{mod} A_{nj}$ is the modulus of the annulus A_{nj}, which is easily seen by a geometric observation to tends to ∞. Hence

$$\lim_{j\uparrow\infty}\left(-\int_{\partial\Delta(0,4j+2)} *df_{nij}\right) = 0 . \tag{6.17}$$

Applying this when we make $j \uparrow \infty$ in (6.16), we obtain

$$D(f_{ni}; W) \leqq -\sum_{m\in \boldsymbol{N}\setminus\{n\}} \int_{\sigma_m} *df_{ni} . \tag{6.18}$$

Putting (6.15) and the above (6.18) together, we can conclude that

$$D(f_{ni}; W) = -\sum_{m\in \boldsymbol{N}\setminus\{n\}} \int_{\sigma_m} *df_{ni} \tag{6.19}$$

for every $i \in \boldsymbol{N}$ and for any fixed $n \in \boldsymbol{N}$.

Finally recall that the sequence $(f_{ni})_{i\in \boldsymbol{N}}$ is decreasing and converges to f_n locally uniformly on W and therefore

$$-*df_{ni} \geqq -*df_{n(i+k)} \geqq -*df_n \geqq 0$$

on each σ_m $(m \neq n)$ and $-*df_{ni} \downarrow -*df_n$ $(i \uparrow \infty)$ uniformly there. A fortiori we can show, with a bit of consideration on the double limit business, that

$$\lim_{i\uparrow\infty}\left(-\sum_{m\in \boldsymbol{N}\setminus\{n\}} \int_{\sigma_m} *df_{ni}\right) = -\sum_{m\in \boldsymbol{N}\setminus\{n\}} \int_{\sigma_m} *df_n ,$$

and by making $i \uparrow \infty$ in (6.19) we can deduce the identity

$$D(f_n; W) = -\sum_{m\in \boldsymbol{N}\setminus\{n\}} \int_{\sigma_m} *df_n \tag{6.20}$$

for any fixed $n \in \boldsymbol{N}$, which will play an important role in the derivation of (6.4).

We now come to the final stage of showing (6.4) based upon (6.20), i.e., we evaluate $\int_{\sigma_m} *df_n$ for every $m \in \boldsymbol{N} \setminus \{n\}$ for each fixed $n \in \boldsymbol{N}$. As the conformal coordinate z around σ_m to calculate the above integral $\int_{\sigma_m} *df_n$ we employ the Joukowski coordinate z determined by the relation

$$w = \frac{s_m}{2}\left(\frac{z}{s_m} + \frac{s_m}{z}\right) + 4m , \tag{6.21}$$

where w is the natural coordinate on $\boldsymbol{C}$. Let V_m be the Jordan region in the z-plane bounded by the image of ∂U_m in the w-plane under the correspondence (6.21). Then the Joukowski mapping (6.21) maps $V_m \setminus \overline{\Delta}(0, s_m)$ onto $U_m \setminus \sigma_m$ in the w-plane conformally sending the circle $\partial U_m = \partial\Delta(0, 1) + 4m$ onto the Jordan curve ∂V_m and the slit σ_m, viewed as the Jordan curve $\sigma_m^+ - \sigma_m^-$ with σ_m^+ (resp. σ_m^-) the upper (resp. lower) edge of σ_m, onto the circle $\partial\Delta(o, s_m)$. Then we are to evaluate the integral

$$-\int_{\sigma_m} *df_n = \int_0^{2\pi} \left[\frac{\partial}{\partial r} f_n(re^{i\theta})\right]_{r=s_m} s_m d\theta \quad (z = re^{i\theta}). \tag{6.22}$$

We let $C_m := \partial\Delta(0, \mu_m)$ be the maximal circle contained in $\overline{V}_m$, which is seen to touch ∂V_m at the point μ_m that correspond to $1 + 4m$ in ∂U_m by (6.21). Then it is easily seen that

$$\mu_m = 1 + \sqrt{1 - s_m^2} > 1 .$$

Since $\max_{\partial U_m} f_n \leqq M_m f_n(o)$, it holds that $0 \leqq f_n \leqq M_m f_n(o)$ on $U_m \setminus \sigma_m$, or on $V_m \setminus \overline{\Delta}(0, s_m)$, so that $\max_{C_m} f_n \leqq M_m f_n(o)$. Since $f_n|\partial\Delta(0, s_m) = 0$, we see that

$$0 \leqq f_n(z) \leqq M_m f_n(o) \frac{\log(|z|/s_m)}{\log(\mu_m/s_m)} \quad (s_m \leqq |z| \leqq \mu_m) .$$

Hence $\mu_m > 1$ implies that

$$0 \leqq \left[\frac{\partial}{\partial r} f_n(re^{i\theta})\right]_{r=s_m} \leqq \frac{M_m f_n(o)}{\log(1/s_m)} \cdot \frac{1}{s_m} \quad (0 \leqq \theta \leqq 2\pi)$$

so that by (6.22) we conclude that

$$0 \leqq -\int_{\sigma_m} *df_n \leqq \frac{2\pi M_m}{\log(1/s_m)} f_n(o) \quad (m \in \boldsymbol{N} \setminus \{n\}) .$$

Therefore, by (6.20) and (4.9), we see that

$$\begin{aligned} D(f_n; W) &= \sum_{m \in \boldsymbol{N}\setminus\{n\}} \left[-\int_{\sigma_m} *df_n\right] \leqq \sum_{m \in \boldsymbol{N}\setminus\{n\}} \frac{2\pi M_m}{\log(1/s_m)} f_n(o) \\ &= \left[2\pi \sum_{m \in \boldsymbol{N}\setminus\{n\}} \frac{M_m}{\log(1/s_m)}\right] f_n(o) = K \cdot f_n(o) \end{aligned}$$

so that (6.4) is deduced. □

To show that the surface W given in (4.10) satisfies the required two properties (4.2) and (4.3) we need one more preparation in addition to the essentially important fact (6.2) stated in Proposition 6.1. It again concerns about the symmetric matrix $(D(e_i, e_j; W); i, j \in \boldsymbol{N})$ as follows:

PROPOSITION 6.23. *Every element not contained in the diagonal of the matrix* $(D(e_i, e_j; W); i, j \in \boldsymbol{N})$ *is not positive, i.e.,*

$$D(e_i, e_j; W) \leqq 0 \tag{6.24}$$

for every pair (i, j) *of* i *and* j *in* $\boldsymbol{N}$ *with* $i \neq j$.

PROOF. We fix an arbitrarily chosen and then fixed regular exhaustion $(T_{jm})_{m\in \boldsymbol{N}}$ of each T_j $(j \in \boldsymbol{N})$ such that T_{j1} contains the closure of $\hat{U}_j$. Then we consider the regular subregion W_m of W given by

$$W_m := ((\cdots(((S_m \uplus_{\sigma_1} (T_{1m} \setminus \sigma_1)) \uplus_{\sigma_2} (T_{2m} \setminus \sigma_2)))\cdots) \uplus_{\sigma_m} (T_{mm} \setminus \sigma_m)) \tag{6.25}$$

for every $m \in \boldsymbol{N}$, where

$$S_m := \Delta(0, 4m+2) \setminus \bigcup_{1\leqq j\leqq m} \sigma_j$$

(cf. (4.1) and the comment right after it). Thus $W \setminus \overline{W}_m$ consists of m components $T_j \setminus \overline{T}_{jm}$ $(1 \leqq j \leqq m)$ and the afforested surface

$$\langle \boldsymbol{C} \setminus \overline{\Delta}(0, 4m+2), (T_j)_{m<j<\infty}, (\sigma_j)_{m<j<\infty}\rangle \subset W .$$

Using these subregions W_m $(m \in \boldsymbol{N})$ we form a regular exhaustion $(W_m)_{m\in\boldsymbol{N}}$ of W. Let $\gamma_{0m} := \partial\Delta(0, 4m+2)$ and $\gamma_{jm} := \partial T_{jm}$ $(1 \leqq j \leqq m)$. Then

$$\partial W_m = \gamma_{0,m} + \gamma_{1m} + \cdots + \gamma_{mm} .$$

For each fixed $n \in \boldsymbol{N}$ we consider an approximating sequence $(e_{nm})_{m\in\boldsymbol{N}}$ of e_n as follows. For each $m \in \boldsymbol{N}$ we consider the function e_{nm} on W given by the conditions $e_{nm} \in C(W^*)$, $e_{nm} \in H(W_m)$, $e_{nm}|(T_n \setminus T_{nm}) = 1$, and $e_{nm} = 0$ on each component of $W \setminus W_m$ except on $T_n \setminus T_{nm}$. In case $n > m$, $e_{nm} \equiv 0$ on W. Since $e_{nm} - e_{n(m+k)} = 0$ on ∂W_{m+k}, the Stokes formula assures that

$$\begin{aligned} D(e_{nm} - e_{n(m+k)}, e_{n(m+k)}; W) &= D(e_{nm} - e_{n(m+k)}, e_{n(m+k)}; W_{m+k}) \\ &= \int_{\partial W_{m+k}} (e_{nm} - e_{n(m+k)}) * de_{n(m+k)} = 0 . \end{aligned}$$

From this it follows that

$$D(e_{nm} - e_{n(m+k)}; W) = D(e_{nm}; W) - D(e_{n(m+k)}; W) \tag{6.26}$$

for every $m \in \boldsymbol{N}$ and every $k \in \boldsymbol{N}$. This shows that $(D(e_{nm}; W))_{m\in\boldsymbol{N}}$ is a decreasing sequence and $(e_{nm})_{m\in\boldsymbol{N}}$ is a Cauchy sequence in the $D(\cdot; W)^{1/2}$-seminorm. On the other hand, since $0 \leqq e_{nm} \leqq 1$ on W and harmonic on W_m, the class $\{e_{nm}; m \in \boldsymbol{N}\}$ forms a normal family. Choose any two subsequences $(u_m)_{m\in\boldsymbol{N}}$ and $(v_m)_{m\in\boldsymbol{N}}$ of $(e_{nm})_{m\in\boldsymbol{N}}$ and let u (resp. v) be the local uniform limit of $(u_m)_{m\in\boldsymbol{N}}$ (resp. $(v_m)_{m\in\boldsymbol{N}}$) on W. By applying the Fatou lemma to (6.26), we see that

$$\lim_{m\to\infty} (D(u_m - u; W) + D(v_m - v; W)) = 0$$

so that

$$\lim_{m\to\infty} D((u_m - v_m) - (u - v); W) = 0 .$$

At the same time $u-v$ is the local uniform limit of uniformly bounded sequence $(u_m-v_m)_{m\in\boldsymbol{N}}$ on W. Since $u_m - v_m$ has a compact support in W, we can conclude that $u - v$ is a Dirichlet potential (cf. [13]) and hence a Wiener potential. Thus $u - v = 0$ on $\hat{\delta} W$ and a fortiori $u - v \equiv 0$ on W. We have thus seen that $(e_{nm})_{m\in\boldsymbol{N}}$ converges to a $w \in H(W)$ both locally uniformly on W and in $D(\cdot; W)^{1/2}$-seminorm. Therefore $(e_{nm} - e_n)_{m\in\boldsymbol{N}}$ converges to $w - e_n \in H(W)$

both locally uniformly on W and in $D(\cdot; W)^{1/2}$-seminorm. Again we see that $e_{nm} - e_n = 0$ on $\hat{\delta}W$ and thus on δW, which implies that $w - e_n = 0$ on δW (cf. [13]). This proves that

$$e_n = \lim_{m\to\infty} e_{nm} \tag{6.27}$$

locally uniformly on W and simultaneously

$$\lim_{m\to\infty} D(e_n - e_{nm}; W) = 0\,. \tag{6.28}$$

Take arbitrarily two different numbers i and j in $\boldsymbol{N}$ (i.e., $i \neq j$) fixed for a while and choose $m \in \boldsymbol{N}$ arbitrarily with $m > \max\{i, j\}$. Then

$$\begin{aligned} D(e_{im}, e_{jm}; W) = D(e_{im}, e_{jm}; W_m) &= \int_{\partial W_m} e_{im} * de_{jm} \\ &= \sum_{k=0}^{m} \int_{\gamma_{km}} e_{im} * de_{jm} = \int_{\gamma_{im}} *de_{jm}\,. \end{aligned}$$

Observe that $e_{jm} = 0$ on γ_{im} $(i \neq j)$ and $e_{jm} > 0$ on W_m. Then the outer normal derivative of e_{jm} at every point in γ_{im} is nonpositive and thus $*de_{jm} \leqq 0$ on γ_{im}. Therefore

$$D(e_{im}, e_{jm}; W) = \int_{\gamma_{im}} *de_{jm} \leqq 0\,.$$

In view of (6.28), we deduce, by letting $m \uparrow \infty$ in the above displayed relation, that $D(e_i, e_j; W) \leqq 0$, as required. □

7. Proof of the main theorem. Thus far we have finished all the necessary preparations and hence we can now complete the proof of the main theorem stated in Section 1 quite easily in a few lines. Take the Riemann surface W constructed in (4.10) and we are to show the validity of (4.2) $HM_2(W) = HD(W)$ and (4.3) $\dim HM_2(W) = \infty$. The latter is now trivial. Let $E(W) := \{e_i; i \in \boldsymbol{N}\}$ be the family of functions e_i determined by (5.3) $e_i(d_j) = \delta_{ij}$ (the Kronecker delta), where $\{d_j\} = \delta T_j$. Since $E(W) \subset HB(W)$, we see that $E(W) \subset HM_2(W)$. Choose any finite subset $\{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\} \subset E(W)$ and let

$$\lambda_{i_1} e_{i_1} + \lambda_{i_2} e_{i_2} + \cdots + \lambda_{i_k} e_{i_k} \equiv 0 \tag{7.1}$$

on W for constants $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_k} \in \boldsymbol{R}$. Considering the above (7.1) at each point d_{i_μ} $(1 \leqq \mu \leqq k)$ by using (5.3) $e_{i_\nu}(d_{i_\mu}) = \delta_{i_\nu i_\mu}$, we see that $\lambda_{i_1} = \lambda_{i_2} = \cdots = \lambda_{i_k} = 0$. This proves that $E(W)$ is a linearly independent infinite subset of $HM_2(W)$ so that (4.3) $\dim HM_2(W) = \infty$ is valid.

We turn to the proof of (4.2) $HM_2(W) = HD(W)$. As we have seen in (1.8), it holds that $HM_2(R) \supset HD(R)$ for any Riemann surface R and in particular $HM_2(W) \supset HD(W)$. Thus we only have to show that $HM_2(W) \subset HD(W)$ in order to conclude (4.2). Since $HM_2(W)$ is a vector sublattice of $HP(W)$ (cf. Proposition 3.1), what really we have to show is that $HM_2(W)^+ \subset HD(W)$. For the purpose we take an arbitrary $u \in HM_2(W)^+$ and we will show that $u \in HD(W)$. By Proposition 5.9, there exists a unique sequence $(a_i)_{i\in \boldsymbol{N}} \subset \boldsymbol{R}^+$ such that

$$u = \sum_{i\in \boldsymbol{N}} a_i e_i \tag{7.2}$$

on W, where the series is convergent locally uniformly on W, and

$$\sum_{i\in \boldsymbol{N}} a_i^2 e_i(o) < \infty \,. \tag{7.3}$$

For each $n \in \boldsymbol{N}$ we consider the function

$$u_n := \sum_{1\leqq i\leqq n} a_i e_i \,,$$

which belongs to the class $HB(W)$. Clearly $(u_n)_{n\in \boldsymbol{N}}$ is increasing and

$$u = \lim_{n\to\infty} u_n \tag{7.4}$$

locally uniformly on W. Hence, by the Fatou lemma in the integration theory,

$$D(u;\, W) \leqq \liminf_{n\to\infty} D(u_n;\, W)\,. \tag{7.5}$$

Observe that

$$\begin{aligned} D(u_n;\, W) = D\Bigg(\sum_{1\leqq i\leqq n} a_i e_i, \sum_{1\leqq j\leqq n} a_j e_j;\, W\Bigg) &= \sum_{1\leqq i,j\leqq n} a_i a_j D(e_i, e_j;\, W) \\ &= \sum_{1\leqq i\leqq n} a_i^2 D(e_i;\, W) + \sum_{1\leqq i,j\leqq n, i\neq j} a_i a_j D(e_i, e_j;\, W)\,. \end{aligned}$$

Since $a_j \geqq 0$ ($i \in \boldsymbol{N}$), we deduce by (6.24) that

$$D(u_n;\, W) \leqq \sum_{1\leqq i\leqq n} a_i^2 D(e_i;\, W) \quad (n \in \boldsymbol{N})\,.$$

By (6.2) with (7.3), we finally conclude that

$$D(u_n;\, W) \leqq \sum_{1\leqq i\leqq n} a_i^2 \cdot K e_i(o) \leqq K \sum_{i\in \boldsymbol{N}} a_i^2 e_i(o) =: C < +\infty\,.$$

This with (7.5) implies that $D(u;\, W) \leqq C < +\infty$ so that $u \in HD(W)$, as required. The proof of the main theorem is herewith complete. □

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