

## MOMENTUM CONSTRUCTION ON RICCI-FLAT KÄHLER CONES

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**Abstract.** We extend Calabi ansatz over Kähler-Einstein manifolds to Sasaki-Einstein manifolds. As an application we prove the existence of a complete scalar-flat Kähler metric on Kähler cone manifolds over Sasaki-Einstein manifolds. In particular there exists a complete scalar-flat Kähler metric on the toric Kähler cone manifold constructed from a toric diagram with a constant height.

**1. Introduction.** A method to construct complete Kähler metrics with good curvature property is known as Calabi ansatz. This method was employed first by Calabi [6] to construct complete Ricci-flat Kähler metrics on the canonical line bundle over a Kähler-Einstein manifolds of positive scalar curvature. Calabi ansatz was extended by various authors (e.g., [12], [15], [16], [11]). In this paper we extend Calabi ansatz to the cone over Sasaki-Einstein manifolds. The method of this paper owes much to the work by Hwang and Singer ([11]) who abstracted the essence of the earlier constructions on the total space of an Hermitian line bundle  $p : L \rightarrow M$  with “ $\sigma$ -constant” curvature over a Kähler manifold  $(M, g_M)$ . This method, also called the *momentum construction*, searches for Kähler forms of the form

$$(1) \quad \omega = p^* \omega_M + dd^c f(t)$$

where  $t$  is the logarithm of the norm function and  $f$  is a function of one variable. In this paper we extend the momentum construction to Sasaki-Einstein manifolds or more generally Sasakian  $\eta$ -Einstein manifolds, i.e., Sasaki manifolds which are transversely Kähler-Einstein. Recall that a Sasaki manifold  $(S, g)$  is an odd dimensional Riemannian manifold such that its cone manifold

$$(C(S), \bar{g}) = (\mathbf{R}^+ \times S, dr^2 + r^2 g)$$

is a Kähler manifold where  $r$  is the standard coordinate of  $\mathbf{R}^+ = \{c \in \mathbf{R}; c \geq 0\}$ . Then  $S$ , which is identified with the submanifold  $\{r = 1\}$ , becomes a contact manifold with the contact form  $\eta := (i(\bar{\partial} - \partial) \log r)|_{r=1}$ . The vector field  $\xi = J \frac{\partial}{\partial r}$  on  $S$  is the Reeb vector field of the contact form, that is,

$$i(\xi)\eta = 1 \quad \text{and} \quad i(\xi)d\eta = 0.$$

We call the flow generated by  $\xi$  the Reeb flow or the characteristic foliation. Since  $\xi - iJ\xi$  is a holomorphic vector field and since  $d\eta$  is non-degenerate on the kernel of  $\eta$ , the local orbit spaces of the Reeb flow (or equivalently the local leaf spaces of the characteristic foliation) inherit Kähler structures. It is a standard fact that  $S$  has a Sasaki-Einstein metric if and only if the local orbit spaces of the Reeb flow have Kähler-Einstein metrics of positive scalar curvature (positive Kähler-Einstein structure for short). If the Kähler form and Ricci form on the local orbit spaces are denoted by  $\omega^T$  and  $\rho^T$ , respectively, then the condition is expressed by

$$(2) \quad \rho^T = \kappa \omega^T$$

for some positive constant  $\kappa$ . Allowing  $\kappa$  to be any real constant, a Sasaki manifold with the condition (2) is called an  $\eta$ -Einstein manifold (see Section 2 for more detail). Typical examples of Sasaki-Einstein manifolds are obtained as follows. Suppose that  $M$  is a positive Kähler-Einstein manifold and take the Sasaki manifold  $S$  to be the total space of the  $U(1)$ -bundle associated with the canonical line bundle  $K_M$ . Then  $S$  is a Sasaki-Einstein manifold.

We perform the momentum construction on the cone  $C(S)$  of a compact  $\eta$ -Einstein Sasaki manifold  $S$  by replacing  $\omega_M$  by the transverse Kähler form  $\omega^T$  and putting  $t = \log r$ . Thus the momentum construction over an  $\eta$ -Einstein Sasaki manifold is of the form

$$(3) \quad \omega = p^* \omega^T + dd^c f(t)$$

where  $p : C(S) \rightarrow S$  is the projection along the radial flow generated by  $r \frac{\partial}{\partial r}$ .

Note that it is proved in [10] (see also [7]) that, given a compact toric Sasaki manifold  $S$  corresponding to a toric diagram with a constant height, one can deform the Sasaki structure by varying the Reeb vector field and then deforming the transverse Kähler form to obtain a Sasaki-Einstein metric. Given a toric Fano manifold  $M$ , we can apply this result to the total space  $S$  of the  $U(1)$ -bundle associated with the canonical line bundle  $K_M$ , and thus we obtain a Sasaki-Einstein metric on  $S$  and a possibly non-standard vector Reeb field such that the local leaf spaces have positive Kähler-Einstein metrics.

Typical results we obtain are as follows.

**THEOREM 1.1.** *Let  $S$  be a compact Sasaki-Einstein manifold and  $C(S)$  its Kähler cone manifold with the cone metric  $dr^2 + r^2g$ . Then we have the following.*

(a) *There exists a complete scalar-flat Kähler metric on  $C(S)$ .*

(b) *For any negative constant  $c$ , there exists a constant  $\gamma > 0$  such that the open submanifold  $\{0 < r < \gamma\}$  in  $C(S)$  admits a complete Kähler metric of negative constant scalar curvature  $c$ .*

*In particular, if  $C(S)$  is a toric Kähler cone manifold corresponding to a toric diagram with a constant height, then there exists a complete scalar-flat Kähler metric on  $C(S)$ .*

In this theorem no metric is Einstein. As a special case of Theorem 1.1 we have the following corollary.

**COROLLARY 1.2.** *Let  $M$  be a toric Fano manifold and  $L$  a holomorphic line bundle such that  $K_M = L^{\otimes p}$ . Then, for any positive integer  $k$ , the total space of  $L^{\otimes k}$  minus the zero section admits a complete scalar-flat Kähler metric. There also exists a complete Kähler metric of negative constant scalar curvature on an open disk bundle of  $L^{\otimes k}$ .*

**THEOREM 1.3.** *Let  $S$  be a compact  $\eta$ -Einstein Sasaki manifold with  $\rho^T = \kappa\omega^T$  for some non-positive constant  $\kappa$  and  $C(S)$  its cone manifold with cone metric  $dr^2 + r^2g$ .*

(a) *If  $\kappa = 0$ , then for any negative constant  $c$ , there exists a constant  $\gamma > 0$  such that the submanifold  $\{0 < r < \gamma\}$  in  $C(S)$  admits a complete Kähler metric of negative constant scalar curvature  $c$ .*

(b) *If  $\kappa < 0$ , we have a negative constant  $c_0$  such that there exists a complete Kähler metric on  $C(S)$  with scalar curvature  $c_0$  and that, for any negative constant  $c < c_0$ , there exists a constant  $\gamma > 0$  such that the open submanifold  $\{0 < r < \gamma\}$  in  $C(S)$  admits a complete Kähler metric of negative constant scalar curvature  $c$ . This metric is Einstein if  $c = (m + 1)\kappa$ .*

Note that in the case of (a), there is no borderline case and no metric is Einstein, as the proof given in Section 4 shows. There are many examples of compact Sasaki manifolds with  $\rho^T = \kappa\omega^T$  for some non-positive constant  $\kappa$  (see Remark 4.1 in Section 4 or Boyer, Galicki and Matzeu [5] for more detail). Theorem 1.3 therefore produces many new complete scalar-flat Kähler manifolds and negative Kähler-Einstein manifolds.

As in the case of Calabi [6], one may try to prove the existence of a complete Ricci-flat Kähler metric on the total space of the canonical line bundle  $K_M$  of a toric Fano manifold  $M$ . In fact one can prove the following result.

**PROPOSITION 1.4.** *Let  $M$  be a toric Fano manifold and  $L$  a holomorphic line bundle over  $M$  such that  $K_M = L^{\otimes p}$  for some positive integer  $p$ . Then for each positive integer  $k$ , there exists a scalar-flat Kähler metric on the total space of  $L^{\otimes k}$  – (zero section) . This metric is Ricci-flat when  $k = p$ , that is when  $L^{\otimes k} = K_M$ .*

It is not clear if the metric extends smoothly to the zero section if the Sasaki-Einstein structure on the  $U(1)$ -bundle of  $K_M$  is irregular. We will give some consideration on this point in Section 4. In physics literature Oota and Yasui [14] have obtained a complete Ricci-flat Kähler metric on the canonical bundle of one-point-blow-up of  $\mathbf{C}P^2$  by a different derivation. More recently, van Coevering [17] has proved the existence of a complete Ricci-flat Kähler metric on the crepant resolutions of a toric Gorenstein cone. This latter result is obtained by solving a complex Monge-Ampère equation.

The organization of this paper is as follows. In Section 2 we review Sasakian geometry and give precise statements of the results obtained in [10]. In Section 3 we apply the Calabi ansatz to Sasakian  $\eta$ -Einstein manifolds and derive basic formulae. In Section 4 we give proofs of the theorems stated in this introduction. In Section 5 we give a proof of Proposition 1.4 and other related results, and also discuss about the possibility of extending the metric to the zero section.

We refer the reader to the review of Boyer and Galicki [4] in which the problem of resolving the cone metrics are taken up from wider view points.

This work was inspired by conversations with Shi-Shyr Roan and Naichung Conan Leung at Sugadaira conference in 2006. The author is especially grateful to Professor Leung for suggesting to consider the canonical bundles of toric Fano manifolds.

**2. Sasaki manifolds.** A Sasaki manifold is a Riemannian manifold  $(S, g)$  whose cone manifold  $(C(S), \bar{g})$  with  $C(S) \cong S \times \mathbf{R}^+$  and  $\bar{g} = dr^2 + r^2g$  is a Kähler manifold where  $r$  is the standard coordinate on  $\mathbf{R}^+$ . From this definition,  $S$  is odd-dimensional and we put  $\dim_{\mathbf{R}} S = 2m + 1$ , and thus  $\dim_{\mathbf{C}} C(S) = m + 1$ . Note that  $C(S)$  does not contain the apex.  $S$  is always identified with the real hypersurface  $\{r = 1\}$  in  $C(S)$ .

Putting  $\tilde{\xi} = J(r \frac{\partial}{\partial r})$ ,  $\tilde{\xi} - iJ\tilde{\xi}$  defines a holomorphic vector field on  $C(S)$ . The restriction of  $\tilde{\xi}$  to  $S \cong \{r = 1\}$ , which is tangent to  $S$ , is called the *Reeb vector field* of  $S$  and denoted by  $\xi$ . The Reeb vector field  $\xi$  is a Killing vector field on  $S$ , and thus generates a toral subgroup in the isometry group of  $S$ .

Let  $\eta$  be the dual 1-form to  $\xi$  using the Riemannian metric  $g$ . Then  $\eta$  can be expressed as

$$\eta = (i(\bar{\partial} - \partial) \log r)|_{r=1} = (2d^c \log r)|_{r=1}.$$

Put  $D = \text{Ker } \eta$  and call it the *contact bundle*. Then  $d\eta$  is non-degenerate on  $D$  and thus  $S$  becomes a contact manifold with the contact form  $\eta$ . The Reeb vector field  $\xi$  satisfies

$$i(\xi)\eta = 1 \quad \text{and} \quad i(\xi)d\eta = 0$$

where  $i(\xi)$  denotes the interior product, which are often used as the defining properties of the Reeb vector field for contact manifolds.

The Reeb vector field  $\xi$  generates a 1-dimensional foliation  $\mathcal{F}_\xi$ , called the characteristic foliation, on  $S$ . We also regard  $\mathcal{F}_\xi$  as the flow generated by  $\xi$  and call it the Reeb flow. Since the Reeb flow shares common local orbit spaces with the holomorphic flow generated by  $\tilde{\xi} - iJ\tilde{\xi}$  on  $C(S)$  and the latter has a natural transverse holomorphic structure, the characteristic foliation on  $S$  admits a transverse holomorphic structure. The tangent spaces to the local leaf spaces are naturally isomorphic to a fiber of  $D$ , and using this isomorphism and considering the symplectic form  $\frac{1}{2}d\eta$  on  $D$  we obtain a well-defined Kähler form on the local leaf spaces of  $\mathcal{F}_\xi$ . Though the Kähler forms are defined on the local leaf spaces, they are pulled back to  $S$  and glued together to give a global 2-form

$$(4) \quad \omega^T = \frac{1}{2}d\eta = d(d^c \log r|_{r=1}) = (dd^c \log r)|_{r=1}$$

on  $S$ , which we call the *transverse Kähler form*. Note that the clumsy constant  $1/2$  is necessary since

$$(5) \quad d\eta(X, Y) = 2g(\Phi X, Y)$$

for  $X, Y \in D_x$ ,  $x \in S$ , where  $\Phi$  is the natural complex structure on  $D$ . We call the collection of Kähler structures on local leaf spaces of  $\mathcal{F}_\xi$  the *transverse Kähler structure*.

Recall that a smooth differential form  $\alpha$  on  $S$  is basic if

$$i(\xi)\alpha = 0 \quad \text{and} \quad \mathcal{L}_\xi\alpha = 0$$

where  $\mathcal{L}_\xi$  denotes the Lie derivative by  $\xi$ . The basic forms are preserved by the exterior derivative  $d$  which decomposes into  $d = \partial_B + \bar{\partial}_B$ , and we can define basic cohomology groups and basic Dolbeault cohomology groups. We also have the transverse Chern-Weil theory and can define basic Chern classes for complex vector bundles with basic transition functions. As in the Kähler case, the basic first Chern class of the Reeb foliation  $c_1^B$  is represented by the  $1/2\pi$  times the transverse Ricci form  $\rho^T$ :

$$(6) \quad \rho^T = -i\partial_B\bar{\partial}_B \log \det (g_{i\bar{j}}^T)$$

where

$$\omega^T = i g_{i\bar{j}}^T dz^i \wedge d\bar{z}^{\bar{j}}$$

and  $z^1, \dots, z^m$  are local holomorphic coordinates on the local leaf space.

Now we turn to the study of Sasaki-Einstein manifolds. We start with the following fact.

**FACT 2.1.** *Let  $(S, g)$  be a  $(2m+1)$ -dimensional Sasaki manifold. The following three conditions are equivalent.*

- (a)  *$(S, g)$  is a Sasaki-Einstein manifold. The Einstein constant is necessarily  $2m$ .*
- (b)  *$(C(S), \bar{g})$  is a Ricci-flat Kähler manifold.*
- (c) *The local leaf spaces of the Reeb foliation have transverse Kähler-Einstein metrics with Einstein constant  $2m+2$ .*

See for proofs [2] or [3]. A typical example of Fact 2.1 is when

$$(C(S), S, \text{local leaf spaces}) = (\mathbf{C}^{m+1} - \{0\}, S^{2m+1}, \mathbf{C}P^m)$$

where  $S^{2m+1}$  is the standard sphere of dimension  $2m+1$ . Of course, the Reeb flow is induced by the standard  $S^1$ -action.

Suppose that  $S$  has an Einstein metric. Then by (c) of Fact 2.1 we have

$$(7) \quad \rho^T = (2m+2)\omega^T = (m+1)d\eta,$$

hence  $c_1^B > 0$ , i.e.,  $c_1^B$  is represented by a positive basic  $(1, 1)$ -form. Moreover under the natural homomorphism  $H_B^2(\mathcal{F}_\xi) \rightarrow H^2(S)$  of the basic cohomology group to the ordinary de Rham cohomology group, the basic first Chern class  $c_1^B$  is sent to the ordinary first Chern class  $c_1(D)$ , but by (7)

$$(8) \quad c_1(D) = (2m+2)\omega^T = (m+1)[d\eta] = 0.$$

Conversely if  $c_1^B > 0$  and  $c_1(D) = 0$  then  $c_1^B = \tau[d\eta]$  for some positive constant  $\tau$ . See Proposition 4.3 in [10] for the detail.

A Sasaki manifold  $(S, g)$  is said to be toric if the Kähler cone manifold  $(C(S), \bar{g})$  is toric, namely if  $(m+1)$ -dimensional compact torus  $G$  acts on  $(C(S), \bar{g})$  effectively as holomorphic isometries. Note that then  $G$  preserves  $\tilde{\xi}$  because  $G$  preserves  $r$  and the complex structure  $J$ . Since the one-parameter group of transformations generated by  $\tilde{\xi}$  acts on  $C(S)$  as holomorphic

isometries and since  $G$  already has the maximal dimension of possible torus actions on  $C(S)$ ,  $\tilde{\xi}$  is contained in the Lie algebra  $\mathfrak{g}$  of  $G$ . The action of  $G$  on  $C(S)$  naturally descends to an action on  $S$ .

**DEFINITION 2.2.** Let  $\mathfrak{g}^*$  be the dual of the Lie algebra  $\mathfrak{g}$  of the  $(m+1)$ -dimensional compact torus  $G$ . Let  $\mathbf{Z}_{\mathfrak{g}}$  be the integral lattice of  $\mathfrak{g}$ , that is, the kernel of the exponential map  $\exp : \mathfrak{g} \rightarrow G$ . A subset  $C \subset \mathfrak{g}^*$  is a rational polyhedral cone if there exists a finite set of vectors  $\lambda_i \in \mathbf{Z}_{\mathfrak{g}}$ ,  $1 \leq i \leq d$ , such that

$$C = \{y \in \mathfrak{g}^* ; \langle y, \lambda_i \rangle \geq 0 \text{ for } i = 1, \dots, d\},$$

where  $\langle \cdot, \cdot \rangle$  denotes the pairing of  $\mathfrak{g}^*$  and  $\mathfrak{g}$ . We assume that the set  $\{\lambda_i\}$  is minimal, that is, for any  $j$

$$C \neq \{y \in \mathfrak{g}^* ; \langle y, \lambda_i \rangle \geq 0 \text{ for all } i \neq j\}$$

and that each  $\lambda_i$  is primitive, i.e.,  $\lambda_i$  is not of the form  $\lambda_i = a\mu$  for an integer  $a \geq 2$  and  $\mu \in \mathbf{Z}_{\mathfrak{g}}$ . (Thus  $d$  is the number of codimension 1 faces if  $C$  has non-empty interior.) Under these two assumptions a rational polyhedral cone  $C$  with nonempty interior is good if the following condition holds. If

$$\{y \in C ; \langle y, \lambda_{i_j} \rangle = 0 \text{ for all } j = 1, \dots, k\}$$

is a non-empty face of  $C$  for some  $\{i_1, \dots, i_k\} \subset \{1, \dots, d\}$ , then  $\lambda_{i_1}, \dots, \lambda_{i_k}$  are linearly independent over  $\mathbf{Z}$  and

$$(9) \quad \left\{ \sum_{j=1}^k a_j \lambda_{i_j} ; a_j \in \mathbf{R} \right\} \cap \mathbf{Z}_{\mathfrak{g}} = \left\{ \sum_{j=1}^k m_j \lambda_{i_j} ; m_j \in \mathbf{Z} \right\}.$$

Given a good rational polyhedral cone  $C$  we can construct a *smooth* toric contact manifold whose moment map image is  $C$ .

**DEFINITION 2.3.** An  $(m+1)$ -dimensional toric diagram with height  $l$  is a collection of  $\lambda_i \in \mathbf{Z}^{m+1} \cong \mathbf{Z}_{\mathfrak{g}}$  satisfying (9) and  $\gamma \in \mathbf{Q}^{m+1} \cong (\mathbf{Q}_{\mathfrak{g}})^*$  such that

- (1)  $l$  is a positive integer such that  $l\gamma$  is a primitive element of the integer lattice  $\mathbf{Z}^{m+1} \cong \mathbf{Z}_{\mathfrak{g}}^*$ .
- (2)  $\langle \gamma, \lambda_i \rangle = -1$ .

We say that a good rational polyhedral cone  $C$  is associated with a toric diagram of height  $l$  if there exists a rational vector  $\gamma$  satisfying (1) and (2) above.

The reason why we use the terminology ‘‘height  $l$ ’’ is because, using a transformation by an element of  $SL(m+1, \mathbf{Z})$ , we may assume that

$$\gamma = \begin{pmatrix} -1/l \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and the first component of  $\lambda_i$  is equal to  $l$  for each  $i$ . The following theorem asserts that the condition of constant height in Definition 2.3 is a Calabi-Yau condition for  $C(S)$ .

**THEOREM 2.4** ([7]). *Let  $S$  be a compact toric Sasaki manifold with  $\dim S \geq 5$ . Then the following three conditions are equivalent.*

- (a)  $c_1^B > 0$  and  $c_1(D) = 0$ .
- (b) *The Sasaki manifold  $S$  is obtained from a toric diagram with height  $l$  for some positive integer  $l$  defined by  $\lambda_1, \dots, \lambda_d \in \mathfrak{g}$  and  $\gamma \in \mathfrak{g}^*$  and the Reeb field  $\xi \in \mathfrak{g}$  satisfies*

$$\langle \gamma, \xi \rangle = -m - 1 \quad \text{and} \quad \langle y, \xi \rangle > 0 \quad \text{for all } y \in C$$

where  $C = \{y \in \mathfrak{g}^* ; \langle y, \lambda_j \rangle \geq 0, j = 1, \dots, d\}$ .

- (c) *For some positive integer  $l$ , the  $l$ -th power  $K_{C(S)}^{\otimes l}$  of the canonical line bundle  $K_{C(S)}$  is trivial.*

The main theorem of [10] is the following.

**THEOREM 2.5** ([10]). *Suppose that we are given a compact toric Sasaki manifold satisfying one of the equivalent conditions in Theorem 2.4 so that we may assume that  $c_1^B = (2m + 2)[\omega^T]$  as basic cohomology classes. Then we can deform the Sasaki structure by varying the Reeb vector field and then performing a transverse Kähler deformation to obtain a Sasaki-Einstein metric.*

The proof of Theorem 2.5 is outlined as follows. Fixing a Reeb vector field we have a fixed transverse holomorphic structure. The first step is to prove that, for the fixed transverse holomorphic structure, we can deform the transverse Kähler structure in the form

$$(10) \quad \omega^T + dd^c \psi_1 = d(d^c(\log r + \psi_1)|_{r=1}) = \frac{1}{2}d((d^c \log(r^2 \exp(2\psi_1)))|_{r=1})$$

where  $\psi_1$  is a smooth basic function  $S$  such that the deformed transverse Kähler metric satisfies the Kähler Ricci soliton equation:

$$\rho^T - (2m + 2)\omega^T = \mathcal{L}_X \omega$$

for some ‘‘Hamiltonian-holomorphic vector field’’  $X$  in the sense of [10]. Note that this deformation deforms  $r^2$  into  $r^2 \exp(2\psi_1)$ . A transverse Kähler-Ricci soliton becomes a positive transverse Kähler-Einstein metric, i.e.,  $X = 0$ , if the invariant  $f_1$  which obstructs the existence of positive transverse Kähler-Einstein metric vanishes. Note that  $f_1$  depends only on the Reeb vector field  $\xi$ . The second step is then to show that there exists a unique Reeb vector field  $\xi'$  such that  $f_1$  vanishes; this idea is due to Martelli-Sparks-Yau [13]. For this  $\xi'$  we have a positive transverse Kähler-Einstein metric, that is a Sasaki-Einstein metric.

**DEFINITION 2.6.** A Sasaki metric  $g$  is said to be  $\eta$ -Einstein if there exist constants  $\lambda$  and  $\nu$  such that

$$\text{Ric}_g = \lambda g + \nu \eta \otimes \eta$$

where  $\text{Ric}_g$  denotes the Ricci curvature of  $g$ .

By elementary computations in Sasakian geometry, we always have  $\text{Ric}_g(\xi, \xi) = 2m$  on any Sasaki manifolds. This implies that  $\lambda + \nu = 2m$ . Let  $\text{Ric}^T$  denote the Ricci curvature of the local leaf space. Then again elementary computations show

$$(11) \quad \text{Ric}_g(\tilde{X}, \tilde{Y}) = (\text{Ric}^T - 2g^T)(X, Y)$$

where  $\tilde{X}, \tilde{Y} \in D$  are the lifts of tangent vectors  $X, Y$  of local leaf space. From this we see that the condition of  $\eta$ -Einstein metric is equivalent to

$$(12) \quad \text{Ric}^T = (\lambda + 2)g^T.$$

Given a Sasaki manifold with the Kähler cone metric  $\bar{g} = dr^2 + r^2g$ , we transform the Sasakian structure by deforming  $r$  into  $r' = r^a$  for a positive constant  $a$ . This transformation is called the  $D$ -homothetic transformation. Then the new Sasaki structure has

$$(13) \quad \eta' = d \log r^a = a\eta, \quad \xi' = \frac{1}{a}\xi,$$

$$(14) \quad g' = ag^T + a\eta \otimes a\eta = ag + (a^2 - a)\eta \otimes \eta.$$

Suppose that  $g$  is  $\eta$ -Einstein with  $\text{Ric}_g = \lambda g + \nu\eta \otimes \eta$ . Since the Ricci curvature of a Kähler manifold is invariant under homotheties, we have  $\text{Ric}'^T = \text{Ric}^T$ . From this and  $\text{Ric}_{g'}(\xi', \xi') = 2m$ , we have

$$(15) \quad \begin{aligned} \text{Ric}_{g'} &= \text{Ric}'^T - 2g'^T + 2m\eta' \otimes \eta' \\ &= \text{Ric}^T - 2ag^T + 2m\eta' \otimes \eta' \\ &= \text{Ric}|_{D \times D} + 2g^T - 2ag^T + 2m\eta' \otimes \eta' \\ &= \lambda g^T + 2g^T - 2ag^T + 2m\eta' \otimes \eta'. \end{aligned}$$

This shows that  $g'$  is  $\eta$ -Einstein with

$$(16) \quad \lambda' = \frac{\lambda + 2 - 2a}{a}.$$

In summary, we have

- Sasaki-Einstein metric is a special case of an  $\eta$ -Einstein metric with  $\lambda = 2m$  and  $\nu = 0$ ;
- a Sasaki metric is an  $\eta$ -Einstein metric with  $\lambda + 2 > 0$  if and only if its transverse Kähler metric is positive Kähler-Einstein with Einstein constant  $\lambda + 2$ ;
- under the  $D$ -homothetic transformation of an  $\eta$ -Einstein metric, we have a new  $\eta$ -Einstein metric with

$$(17) \quad \rho'^T = \rho^T, \quad \omega'^T = a\omega^T, \quad \rho'^T = (\lambda' + 2)\omega'^T = \frac{\lambda + 2}{a}\omega'^T,$$

and thus, for any positive constants  $\kappa$  and  $\kappa'$ , a transverse Kähler-Einstein metric with Einstein constant  $\kappa$  can be transformed by a  $D$ -homothetic transformation to a transverse Kähler-Einstein metric with Einstein constant  $\kappa'$ .

**3. Calabi ansatz for Sasakian  $\eta$ -Einstein manifolds.** Let  $(S, g)$  be a Sasakian  $\eta$ -Einstein manifold with  $\text{Ric}_g = \lambda g + \nu \eta \otimes \eta$  and with Kähler cone metric on  $C(S)$

$$\bar{g} = dr^2 + r^2 g .$$

Let  $\omega^T = \frac{1}{2}d\eta$  be the transverse Kähler form which gives positive Kähler-Einstein metrics on local leaf spaces with

$$(18) \quad \rho^T = \kappa \omega^T$$

where we have set

$$\kappa := \lambda + 2 .$$

As we work on  $C(S)$  it is convenient to lift  $\eta$  on  $S$  to  $C(S)$  by

$$(19) \quad \eta = 2d^c \log r .$$

We use the same notation  $\eta$  for this lifted one to  $C(S)$ . Then  $\omega^T$  is also lifted to  $C(S)$  by

$$(20) \quad \omega^T = dd^c \log r .$$

Again we use the same notation  $\omega^T$  for the lifted one to  $C(S)$ . The *Calabi ansatz* searches for a Kähler form on  $C(S)$  of the form

$$(21) \quad \omega = \omega^T + i \partial \bar{\partial} F(t)$$

where  $t = \log r$  and  $F$  is a smooth function of one variable on  $(t_1, t_2) \subset (-\infty, \infty)$ .

We set

$$(22) \quad \tau = F'(t) ,$$

$$(23) \quad \varphi(\tau) = F''(t) .$$

Since we require  $\omega$  to be a positive form and

$$(24) \quad \begin{aligned} i \partial \bar{\partial} F(t) &= i F''(t) \partial t \wedge \bar{\partial} t + i F'(t) \partial \bar{\partial} t \\ &= i \varphi(\tau) \partial t \wedge \bar{\partial} t + \tau \omega^T , \end{aligned}$$

we must have  $\varphi(\tau) > 0$ . We also require that the image of  $F'$  is an open interval  $(0, b)$  with  $b \leq \infty$ , i.e.,

$$(25) \quad \lim_{t \rightarrow t_1} F'(t) = 0, \quad \lim_{t \rightarrow t_2} F'(t) = b .$$

It follows from  $\varphi(\tau) > 0$  that  $F'$  is a diffeomorphism from  $(t_1, t_2)$  onto  $(0, b)$ . The function  $\varphi(\tau)$  is called the profile of the Calabi ansatz (21).

Conversely, given a positive function  $\varphi > 0$  on  $(0, b)$  such that

$$(26) \quad \lim_{\tau \rightarrow 0^+} \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)} = t_1, \quad \lim_{\tau \rightarrow b^-} \int_{\tau_0}^{\tau} \frac{dx}{\varphi(x)} = t_2,$$

we can recover the Calabi ansatz as follows. Fix  $\tau_0$  with  $0 < \tau_0 < b$ , and introduce a function  $\tau(t)$  by

$$(27) \quad t = \int_{\tau_0}^{\tau(t)} \frac{dx}{\varphi(x)},$$

and then  $F(t)$  by

$$(28) \quad F(t) = \int_{\tau_0}^{\tau(t)} \frac{x dx}{\varphi(x)}.$$

Putting  $t = \log r$ , we may regard  $\tau$  and  $F$  as functions on

$$(29) \quad C(S)_{(t_1, t_2)} := \{e^{t_1} < r < e^{t_2}\} \subset C(S).$$

Put

$$(30) \quad \begin{aligned} \omega_\varphi &:= \omega^T + dd^c F(t) \\ &= (1 + \tau) \omega^T + \varphi(\tau) i \partial t \wedge \bar{\partial} t \\ &= (1 + \tau) \omega^T + \varphi(\tau)^{-1} i \partial \tau \wedge \bar{\partial} \tau. \end{aligned}$$

As we assume  $\varphi > 0$  on  $(0, b)$ ,  $\omega_\varphi$  defines a Kähler form and have recovered Calabi ansatz.

REMARK 3.1. When  $S$  is the total space of the  $U(1)$ -bundle associated with an Hermitian line bundle  $(L, h) \rightarrow M$  such that  $i \partial \bar{\partial} \log h$  is a Kähler form on  $M$ , then we may regard the total space of  $L$  as a Hamiltonian  $U(1)$ -space with  $\tau = F'(t)$  the moment map. The Kähler potential  $F$  along the fiber is transformed under the Legendre transform into the symplectic potential  $G$  given by the symmetrical relation

$$(31) \quad G(\tau) + F(t) = \tau t.$$

It is easy to see that  $G''(\tau) = 1/\varphi(\tau)$ . It is well-known that, especially in the case of toric Kähler manifolds (cf. [1]), the scalar curvature has a less complicated expression if one uses the symplectic potential. This principle works in our situation as is seen in the computations below.

Next we compute the Ricci form  $\rho_\varphi$  and the scalar curvature  $\sigma_\varphi$  of  $\omega_\varphi$ . First we need to choose local holomorphic coordinates  $(z^0, z^1, \dots, z^m)$  on  $C(S)$ . We choose  $z^0$  to be the coordinate along the holomorphic Reeb flow, more precisely,

$$(32) \quad \frac{\partial}{\partial z^0} = \frac{1}{2} \left( r \frac{\partial}{\partial r} - i J \left( r \frac{\partial}{\partial r} \right) \right) = \frac{1}{2} \left( r \frac{\partial}{\partial r} - i \tilde{\xi} \right),$$

and  $z^1, \dots, z^m$  to be the pull-back of local holomorphic coordinates on the local leaf space. Then it is easy to check that

$$(33) \quad dz^0 = \frac{dr}{r} + i \eta,$$

and that

$$(34) \quad i dz^0 \wedge d\bar{z}^0 = 2 \frac{dr}{r} \wedge \eta.$$

Using these coordinates one can compute the volume form as

$$\begin{aligned}
\omega_\varphi^{m+1} &= (1 + \tau)^m (m + 1) \varphi(\tau) i \partial t \wedge \bar{\partial} t \wedge (\omega^T)^m \\
&= (1 + \tau)^m (m + 1) \varphi(\tau) dt \wedge d^c t \wedge (\omega^T)^m \\
(35) \quad &= (1 + \tau)^m (m + 1) \varphi(\tau) i \frac{dr}{r} \wedge \eta \wedge (\omega^T)^m \\
&= (1 + \tau)^m (m + 1) \varphi(\tau) \frac{i}{2} dz^0 \wedge d\bar{z}^0 \wedge (\omega^T)^m.
\end{aligned}$$

The Ricci form and the scalar curvature can be computed as

$$\begin{aligned}
(36) \quad \rho_\varphi &= \rho^T - i \partial \bar{\partial} \log((1 + \tau)^m \varphi(\tau)) \\
&= \kappa \omega^T - i \partial \bar{\partial} \log((1 + \tau)^m \varphi(\tau)),
\end{aligned}$$

$$\begin{aligned}
(37) \quad \sigma_\varphi &= \frac{\sigma^T}{1 + \tau} - \Delta_\varphi \log((1 + \tau)^m \varphi(\tau)) \\
&= \frac{m\kappa}{1 + \tau} - \Delta_\varphi \log((1 + \tau)^m \varphi(\tau)),
\end{aligned}$$

where  $\Delta_\varphi$  denotes the complex Laplacian with respect to  $\omega_\varphi$ .

Let  $u(\tau)$  be a smooth function of  $\tau$ . Then

$$\begin{aligned}
(38) \quad dd^c u(\tau) &= d\left(u'(\tau) \frac{d\tau}{dt} d^c t\right) \\
&= u'(\tau) \varphi(\tau) dd^c t + (u' \varphi)' \varphi dt \wedge d^c t \\
&= u'(\tau) \varphi(\tau) dd^c t + \frac{1}{\varphi} (u' \varphi)' d\tau \wedge d^c \tau.
\end{aligned}$$

Taking wedge product of this with

$$(39) \quad \omega_\varphi^m = (1 + \tau)^m (\omega^T)^m + m(1 + \tau)^{m-1} \wedge \varphi^{-1} d\tau \wedge d^c \tau$$

and comparing it with

$$(40) \quad \omega_\varphi^{m+1} = (1 + \tau)^m (m + 1) \varphi^{-1} d\tau \wedge d^c \tau \wedge (\omega^T)^m,$$

we obtain

$$(41) \quad \Delta_\varphi u = \frac{m}{1 + \tau} u' \varphi + (u' \varphi)'$$

Apply (41) with  $u = \log((1 + \tau)^m \varphi)$  and insert it to (37). Then we obtain

$$(42) \quad \sigma_\varphi = \frac{m\kappa}{1 + \tau} - \Delta_\varphi \log((1 + \tau)^m \varphi)$$

$$\begin{aligned}
(43) \quad &= \frac{m\kappa}{1 + \tau} - \frac{m}{1 + \tau} \varphi \frac{d}{d\tau} \log((1 + \tau)^m \varphi) \\
&\quad - \frac{d}{d\tau} \left( \varphi \frac{d}{d\tau} \log((1 + \tau)^m \varphi(\tau)) \right)
\end{aligned}$$

$$(44) \quad = \frac{m\kappa}{1 + \tau} - \frac{1}{(1 + \tau)^m} \frac{d}{d\tau} \left( (1 + \tau)^m \varphi \frac{d}{d\tau} \log((1 + \tau)^m \varphi) \right)$$

$$= \frac{m\kappa}{1+\tau} - \frac{1}{(1+\tau)^m} \frac{d^2}{d\tau^2} ((1+\tau)^m \varphi).$$

Setting  $\sigma_\varphi = c$  we get an ordinary differential equation

$$(45) \quad (\varphi(1+\tau)^m)'' = \left( \frac{m\kappa}{(1+\tau)} - c \right) (1+\tau)^m.$$

We can easily solve this equation as

$$(46) \quad \varphi(\tau) = \frac{\kappa}{m+1} (1+\tau) - \frac{c}{(m+1)(m+2)} (1+\tau)^2 + \frac{c_1\tau + c_2}{(1+\tau)^m}$$

where  $c_1$  and  $c_2$  are constants.

Now we take up the problem of completeness of the metrics obtained by Calabi ansatz starting from a compact  $\eta$ -Einstein metric. First define the function  $s(t)$  by

$$(47) \quad s(t) = \int_{\tau_0}^{\tau(t)} \frac{dx}{\sqrt{\varphi(x)}}.$$

Then

$$(48) \quad \frac{ds}{dt} = \frac{1}{\sqrt{\varphi(\tau)}} \frac{d\tau}{dt} = \sqrt{\varphi(\tau)}.$$

Thus  $s(x)$  gives the geodesic length along the  $t$ -direction with respect to the Kähler form  $\omega_\varphi$  of (30); recall  $t = \log r$ .

**PROPOSITION 3.2.** *Let  $\omega_\varphi$  be the Kähler form obtained by Calabi ansatz starting from a compact Sasaki manifold with an  $\eta$ -Einstein metric  $g$ . Then  $\omega_\varphi$  defines a complete metric and have noncompact ends towards the end points of  $I = (0, b)$  if and only if the following conditions are satisfied at the end points:*

- At  $\tau = 0$ ,  $\varphi$  vanishes at least to the second order.
- If  $b$  is finite then  $\varphi$  vanishes at  $\tau = b$  at least to the second order.
- If  $b = \infty$  then  $\varphi$  grows at most quadratically as  $\tau \rightarrow \infty$ .

**PROOF.** First consider at  $\tau = 0$ . By elementary calculus  $s(t) \rightarrow \infty$  as  $\tau \rightarrow 0$  if and only if  $\varphi$  vanishes at 0 at least to the second order. By the same reason, if  $b$  is finite then  $\varphi$  must vanish at  $\tau = b$  at least to the second order. Similarly if  $b = \infty$ ,  $s(t) \rightarrow \infty$  as  $\tau \rightarrow \infty$  if and only if  $\varphi$  grows at most quadratically.  $\square$

#### 4. Proofs of the theorems.

**PROOF OF THEOREM 1.1.** Going back to (46), assume that  $\varphi(0) = 0$  and  $\varphi'(0) = 0$ . Then  $c_1$  and  $c_2$  are determined as

$$(49) \quad c_2 = -\frac{\kappa}{m+1} + \frac{c}{(m+1)(m+2)}, \quad c_1 = -\kappa + \frac{c}{m+1}.$$

Thus we have

$$(50) \quad \begin{aligned} \varphi(\tau) = & \frac{\kappa(1+\tau)}{m+1} + \frac{1}{(1+\tau)^m} \left( -\kappa\tau - \frac{\kappa}{m+1} \right) - \frac{c(1+\tau)^2}{(m+1)(m+2)} \\ & + \frac{c}{(1+\tau)^m} \left( \frac{\tau}{m+1} + \frac{1}{(m+1)(m+2)} \right). \end{aligned}$$

In the case of this theorem we have  $\kappa = 2m + 2$ , and we may also assume that  $\kappa$  is any positive number by  $D$ -homothetic transformation. One can check that  $\varphi(\tau) > 0$  for all  $\tau > 0$  if  $c \leq 0$ , i.e.,  $b = \infty$ , that  $t_1 = -\infty$  for  $c \leq 0$ ,  $t_2 = \infty$  if  $c = 0$  and that  $t_2 < \infty$  if  $c < 0$ . Using Proposition 3.2 one sees that  $\omega_\varphi$  gives a complete metric. Let us see when the metric is Kähler-Einstein. Recall from (36) and (38) that

$$(51) \quad \rho_\varphi = \left( \kappa - \frac{m\varphi + (1+\tau)\varphi'}{1+\tau} \right) \omega^T - \left( \left( \frac{m\varphi}{1+\tau} \right)' + \varphi'' \right) \varphi dt \wedge d^c t.$$

From this and (30) we see that  $\rho_\varphi = \alpha\omega_\varphi$  if and only if

$$(52) \quad \kappa - \frac{m\varphi + (1+\tau)\varphi'}{1+\tau} = \alpha(1+\tau),$$

$$(53) \quad -\left( \frac{m\varphi}{1+\tau} + \varphi' \right)' = \alpha.$$

Using  $\varphi(0) = 0$  and  $\varphi'(0) = 0$  one obtains from (53)

$$(54) \quad -\left( \frac{m\varphi}{1+\tau} + \varphi' \right) = \alpha\tau.$$

Inserting (54) into (52) we have

$$(55) \quad \kappa = \alpha = \frac{c}{m+1}.$$

But this can not happen because  $c \leq 0$  and  $\kappa > 0$ . The last statement of Theorem 1.1 follows from Theorem 2.5. This completes the proof of Theorem 1.1.  $\square$

**PROOF OF THEOREM 1.3.** Consider the case (a), i.e.,  $\kappa = 0$ . Then

$$(56) \quad \varphi(\tau) = -\frac{c(1+\tau)^2}{(m+1)(m+2)} + \frac{c}{(1+\tau)^m} \left( \frac{\tau}{m+1} + \frac{1}{(m+1)(m+2)} \right).$$

If  $c < 0$  then  $\varphi(\tau) > 0$  for all  $\tau > 0$ , i.e.,  $b = \infty$ . Moreover  $t_1 = -\infty$  and  $t_2 < \infty$ . Proposition 3.2 shows that  $\omega_\varphi$  is complete. One sees from (55) that the metric is not Einstein because  $\kappa = 0$  and  $c < 0$ .

Next consider the case (b), i.e.,  $\kappa < 0$ . Since  $\varphi(0) = \varphi'(0) = 0$  and  $c < 0$ , (50) shows that there exists a sufficiently large  $-c'$  such that for all  $c < c'$ ,  $\varphi_c(\tau) > 0$  for all  $\tau > 0$ , where we write the function  $\varphi$  as  $\varphi_c$  for a specific value  $c$  in (50). Let  $c_0$  be the supremum of  $c'$  with such a property. Let  $b$  be the smallest  $\tau > 0$  such that  $\varphi_{c_0}(\tau) = 0$ . Obviously  $\varphi_{c_0}(b) = \varphi'_{c_0}(b) = 0$ . Then  $t_1 = -\infty$  and  $t_2 = \infty$  for this  $\varphi_{c_0}$ . Proposition 3.2 shows that this metric is complete. Now consider  $c < c_0$ . Then  $\varphi_c(\tau) > 0$  for all  $\tau > 0$ , i.e.,  $b = \infty$ .

One sees that  $t_1 = -\infty$ ,  $t_2 < \infty$  and that  $\omega_\varphi$  is complete by Proposition 3.2. This metric is Einstein if  $c = (m + 1)\kappa$  by (55). This completes the proof of Theorem 1.3.  $\square$

REMARK 4.1. A Sasakian structure is said to be positive (resp. negative) if  $[\rho^T] = \kappa[\omega^T]$  for some positive (resp. negative) real number  $\kappa$  as basic cohomology classes. A Sasakian structure is said to be null if  $[\rho^T] = 0$  as basic cohomology classes. This case is considered as the case when  $\kappa = 0$ . As in the Kähler case, one can consider in each case the problem of finding a Sasaki metric with  $\rho^T = \kappa\omega^T$ , i.e., an  $\eta$ -Einstein metric, by a transverse Kähler deformation. But the same proofs of the existence results of Kähler-Einstein metrics by Aubin and Yau in the negative and zero cases can be applied for the transverse geometry of Sasaki manifolds, and we obtain the following results. If the Sasakian structure is negative (resp. null), there exists an  $\eta$ -Einstein metric with  $\rho^T = \kappa\omega^T$  with  $\kappa$  negative (resp. zero). But there are many examples of negative or null Sasaki structures. For example the link of an isolated hypersurface singularity defined by a weighted homogeneous polynomials  $f$  has a negative Sasakian structure if  $|\mathbf{w}| - d < 0$  and null Sasakian structure if  $|\mathbf{w}| - d = 0$ , where  $d$  is the degree of the polynomial and  $\mathbf{w} = (w_0, \dots, w_n)$  is the weights, i.e.,

$$f(\lambda \cdot \mathbf{z}) = f(\lambda^{w_0} z_0, \dots, \lambda^{w_n} z_n) = \lambda^d f(z_0, \dots, z_n) = \lambda^d f(\mathbf{z}).$$

See [5] for more details.

**5. Construction of Ricci-flat metrics.** In this section we will try to construct a Ricci-flat Kähler metric on the total space of the canonical line bundle  $K_M$  of a toric Fano manifold  $M$  using Calabi ansatz. But when the Sasaki-Einstein structure is irregular, we find it difficult to see if the metric extends smoothly to the zero section. The results in this section are therefore constructions outside the zero section, just as Proposition 1.4, but when the Sasaki-Einstein structure is regular, the metrics obtained extend smoothly to the zero section.

Besides Proposition 1.4, we also prove the following results applying the Calabi ansatz to Sasakian  $\eta$ -Einstein manifolds.

PROPOSITION 5.1. *Let  $L$  be a holomorphic line bundle over a compact Kähler manifold  $M$  such that  $K_M = L^{\otimes p}$  for some positive integer  $p$ . Let  $k$  be a positive integer and suppose that the total space  $S$  of the  $U(1)$ -bundle associated with  $L^{\otimes k}$  satisfies*

$$(57) \quad \rho^T = \frac{2p}{k} \omega^T.$$

*Then we have the following.*

(a) *There exists a scalar-flat Kähler metric on the total space of  $L^{\otimes k}$  minus the zero section, which need not be complete near the zero section but is complete near  $r = \infty$ . This metric is Ricci-flat when  $k = p$ , that is,  $L^{\otimes k} = K_M$  and is asymptotic to the cone metric near the infinity.*

(b) *For any constant  $c < 0$ , there exists a constant  $\gamma > 0$  such that the disk bundle  $\{0 < r < \gamma\} \subset L^{\otimes k}$  admits a Kähler metric of constant scalar curvature  $c$ . This metric need not be complete near  $\{r = 0\}$  but is complete near  $\{r = \gamma\}$ . This metric is Kähler-Einstein if  $k > p$  and  $c = (m + 1)(2p/k - 2)$ .*

PROOF OF PROPOSITION 5.1. We are in the position of (46) with  $\kappa = 2p/k$ . In the case of regular Sasaki-Einstein manifolds, i.e.,  $U(1)$ -bundles associated with the canonical bundle of Kähler-Einstein manifolds, for  $\omega_\varphi$  to be complete we need to have  $\varphi(0) = 0$  and  $\varphi'(0) = 2$ . For this fact refer to for example [11]. We therefore assume the same conditions  $\varphi(0) = 0$  and  $\varphi'(0) = 2$ . From these conditions the constants  $c_1$  and  $c_2$  in (46) are determined as

$$(58) \quad c_2 = \frac{c}{(m+1)(m+2)} - \frac{\kappa}{m+1}, \quad c_1 = \frac{c}{m+1} + 2 - \kappa.$$

Hence  $\varphi(\tau)$  is given by

$$(59) \quad \begin{aligned} \varphi(\tau) = & \frac{\kappa}{m+1}(1+\tau) - \frac{(\kappa-2)(m+1)\tau + \kappa}{(1+\tau)^m(m+1)} \\ & + \frac{c}{(m+1)(m+2)} \left( \frac{(m+2)\tau + 1}{(1+\tau)^m} - (1+\tau)^2 \right). \end{aligned}$$

One easily checks that if  $c \leq 0$  then  $\varphi(\tau) > 0$  for all  $\tau > 0$ . Thus  $b = \infty$ , i.e.,  $I = (0, \infty)$ . Since  $\varphi(0) = 0$  and  $\varphi'(0) = 2$  we see from (27) that  $t_1 = -\infty$ . On the other hand, if  $c = 0$  then  $\varphi(\tau)$  has linear growth as  $\tau \rightarrow \infty$ . Thus  $t_2 = \infty$ . If  $c < 0$  then  $\varphi(\tau)$  has quadratic growth as  $\tau \rightarrow \infty$ . Thus  $t_2 < \infty$ . In either case,  $\omega_\varphi$  defines a complete metric near the infinity since the growth is at most quadratic.

In the case of  $c = 0$ ,  $\omega_\varphi$  defines a scalar-flat Kähler metric on the  $L$  minus the zero section since  $t_2 = \infty$ , and the metric is complete near the infinity. In the case of  $c < 0$ ,  $\omega_\varphi$  defines a Kähler metric of constant negative scalar curvature  $c$  on a disk bundle of  $L$  minus the zero section since  $t_2 < \infty$ .

This metric satisfies  $\rho_\varphi = \alpha\omega_\varphi$  if and only if (52) and (53) hold. From (53),  $\varphi(0) = 0$  and  $\varphi'(0) = 2$ , we see

$$(60) \quad -\left( \frac{m\varphi}{1+\tau} + \varphi' \right) = \alpha\tau - 2.$$

Inserting (60) into (52) and using  $\kappa = 2p/k$  we have

$$(61) \quad \alpha = \frac{2p}{k} - 2.$$

But since the scalar curvature is computed as

$$(62) \quad c = (m+1)\alpha$$

and since  $c \leq 0$ , the metric is Einstein if it happens that  $k \geq p$  and that

$$(63) \quad c = (m+1) \left( \frac{2p}{k} - 2 \right).$$

If  $c = 0$  this certainly happens when  $k = p$ . This completes the proof of Theorem 5.1.  $\square$

PROOF OF PROPOSITION 1.4. By Theorem 2.5,  $S$  admits an  $\eta$ -Einstein Sasaki metric with  $\rho^T = (2p/k)\omega^T$ . Hence we can apply Theorem 5.1. This completes the proof.  $\square$

PROPOSITION 5.2. *Let  $L$  be a holomorphic line bundle over a compact Kähler manifold  $M$  such that  $K_M = L^{\otimes p}$  for some positive integer  $p$ . Let  $k$  be a positive integer and suppose that the total space  $S$  of the  $U(1)$ -bundle associated with  $L^{\otimes k}$  has a Sasakian  $\eta$ -Einstein metric with*

$$(64) \quad \rho^T = \kappa \omega^T$$

for some non-positive constant  $\kappa$ . Then we have the following.

Case  $\kappa = 0$ : *There exists a scalar-flat Kähler metric on the total space of  $L^{\otimes k}$  minus the zero section. Further, for any constant  $c < 0$  there exists a constant  $\gamma > 0$  such that the disk bundle  $\{0 < r < \gamma\} \subset L^{\otimes k}$  admits a Kähler metric of constant scalar curvature  $c$ . This metric need not be complete along the zero section and complete near  $r = \gamma$ . This metric is Kähler-Einstein if  $c = -2(m+1)$ .*

Case  $\kappa < 0$ : *We have a negative constant  $c_0$  such that there exists a Kähler metric of negative constant scalar curvature on the total space of  $L^{\otimes k}$  minus the zero section and that for any  $c < c_0$  there exists a constant  $\gamma > 0$  such that the disk bundle  $\{0 < r < \gamma\} \subset L^{\otimes k}$  admits a Kähler metric of constant scalar curvature  $c$ . This metric need not be complete near the zero section and complete near  $r = \gamma$ . This metric is Kähler-Einstein if  $k > p$  and  $c = (m+1)(\kappa - 2)$ .*

PROOF. Consider the case  $\kappa = 0$ . By (59) we have

$$(65) \quad \varphi(\tau) = \frac{2\tau}{(1+\tau)^m} + \frac{c}{(m+1)(m+2)} \left( \frac{(m+2)\tau+1}{(1+\tau)^m} - (1+\tau)^2 \right).$$

As in the proof of Proposition 5.1, if  $c \leq 0$  we have  $b = \infty$  and  $t_1 = -\infty$ . If  $c = 0$  we have  $t_2 = \infty$ , and if  $c < 0$  we have  $t_2 < \infty$ . The rest of the proof goes as in the proof of Proposition 5.1. The metric is Einstein if and only if  $c = -2(m+1)$ .

Consider the case  $\kappa < 0$ . Let  $\varphi_c$  denote  $\varphi$  for a given  $c$ . (59) shows that, since  $\varphi(0) = 0$  and  $\varphi'(0) = 2$ , there exists a sufficiently large  $-c'$  such that for any  $c < c'$  we have  $\varphi_c(\tau) > 0$  for all  $\tau > 0$ . Let  $c_0$  be the supremum of  $c'$ 's with such a property. Let  $b$  be the smallest  $\tau > 0$  such that  $\varphi_{c_0}(\tau) = 0$ . Obviously  $\varphi'_{c_0}(b) = 0$ . Thus, as in Proposition 3.2,  $\omega_{\varphi_0}$  defines a complete metric. In this case  $t_1 = -\infty$  and  $t_2 = \infty$ , so the metric is defined on the whole total space of  $L^{\otimes k}$  minus the zero section. Take  $c$  such that  $c < c_0$ , then  $\varphi_c > 0$  for all  $\tau > 0$ , so  $b = \infty$ . Since  $\varphi_c$  grows quadratically as  $\tau \rightarrow \infty$ ,  $\omega_{\varphi_c}$  is complete by Proposition 3.2 except near the zero section. In this case  $t_2 < \infty$ . The metric is Einstein if and only if  $c = (m+1)(\kappa - 2)$ . That this metric is asymptotic to the cone metric will be proved in Lemma 5.3 below. This completes the proof.  $\square$

In the first version of arXiv:math/0703138 posted under a different title, the author claimed that the metric obtained in Proposition 1.4 extends smoothly to the zero section. But its proof is not correct because of the following reasons. First of all, it is proved in [10] that if the Reeb vector field corresponds to  $\xi \in \mathfrak{g}$  then the Kähler potential  $F_\xi^{\text{can}}$  is given by

[10, equation (61)]:

$$F_{\xi}^{\text{can}} = \frac{r^2}{2} = \frac{1}{2}l_{\xi}(y)$$

where  $y$  is the position vector on the moment cone. Here  $y$  is considered as a part of the action-angle coordinates with fixed symplectic form and varying complex structure. If we choose another Reeb vector field corresponding to  $\xi' \in \mathfrak{g}$ , then the new Kähler potential is given by

$$(66) \quad \frac{r'^2}{2} = \frac{1}{2}r^2 \frac{l_{\xi'}(y)}{l_{\xi}(y)} = \frac{1}{2}r^2 \frac{\langle \xi', y \rangle}{\langle \xi, y \rangle}.$$

But in the Abreu-Guillemin arguments, when the Kähler potential varies the complex structure varies, which means that the holomorphic coordinates change and also the  $i\partial\bar{\partial}$ -operator changes. As a result, to recover the new Kähler form we have to apply new  $i\partial'\bar{\partial}'$  to the new Kähler potential  $r'^2/2$ . This implies that in the logarithmic affine coordinates with fixed complex structure and varying symplectic structure, the two Kähler forms with varying Reeb vector fields are not related by

$$r'^2 = r^2 \exp \psi$$

with  $\psi \in C^{\infty}(S)$ , although  $r'^2/2$  and  $r^2/2$  are related in the action-angle coordinates by  $\langle \xi', y \rangle / \langle \xi, y \rangle \in C^{\infty}(S)$  as in (66). Second of all if two Sasaki structures are related in the standard holomorphic coordinates (with fixed complex structure and varying symplectic structure) by  $r' = r \exp \varphi$  with a basic function  $\varphi \in C^{\infty}(S)$ , then the two Sasaki structures share a common Reeb vector field. When we consider toric Sasaki manifolds all metrics are  $T^{m+1}$ -invariant, and if two Sasaki structures are related by  $r'^2 = r^2 \exp \psi$  with  $\psi \in C^{\infty}(S)$  then  $\psi$  is necessarily basic. This implies that if two Sasaki structures have different Reeb vector fields then the two Sasaki structures are not related by  $r'^2 = r^2 \exp \psi$  with  $\psi \in C^{\infty}(S)$ . Therefore, when the Futaki invariant of  $M$  is not zero, the Sasaki-Einstein structure on  $S$  is never bundle-adapted in the sense of the version 1 of arXiv:math/0703138.

LEMMA 5.3. *The Ricci-flat metric in (a) of Proposition 5.1 is asymptotic to the cone metric near  $r = \infty$ .*

PROOF. Putting in (59)  $\kappa = 2, c = 0$ , we have

$$\varphi(\tau) = \frac{2}{m+1} \left( (1+\tau) - \frac{1}{(1+\tau)^m} \right) = \frac{2}{m+1} \frac{(1+\tau)^{m+1} - 1}{(1+\tau)^m}.$$

Take  $\tau_0 = 2^{1/(m+1)} - 1$ . Then  $\tau(t)$  is obtained from (27) as

$$(67) \quad t = \int_{\tau_0}^{\tau(t)} \frac{dt}{\varphi(\tau)} = \frac{m+1}{2} \int_{\tau_0}^{\tau(t)} \frac{(\tau+1)^m}{(\tau+1)^{m+1} - 1} dt$$

$$(68) \quad = \frac{1}{2} \log((\tau(t)+1)^{m+1} - 1).$$

Since  $t = \log r$ , we have

$$\begin{aligned} e^{2t} &= r^2 = (\tau(t) + 1)^{m+1} - 1, \\ \tau(t) &= (e^{2t} + 1)^{1/(m+1)} - 1. \end{aligned}$$

$F(t)$  is obtained from (25) as

$$F(t) = \int_{\tau_0}^{\tau(t)} \frac{m+1}{2} \frac{(\tau+1)^m \tau}{(\tau+1)^{m+1} - 1} d\tau.$$

Hence we have

$$(69) \quad \tau = F'(t) = \frac{m+1}{2} \frac{(\tau(t)+1)^m \tau(t)}{(\tau(t)+1)^{m+1} - 1} \tau'(t)$$

$$(70) \quad = (e^{2t} + 1)^{1/(m+1)} - 1.$$

Let  $\zeta$  be the  $(m+1)$ -root of unity. Then, as a polynomial in  $x$ , we have

$$\begin{aligned} \sum_{j=0}^m \zeta^j \Pi_{i \neq j}(x - \zeta^i) &= - \sum_{j=0}^m (x - \zeta^j) \Pi_{i \neq j}(x - \zeta^i) + \sum_{j=0}^m x \Pi_{i \neq j}(x - \zeta^i) \\ &= -(m+1)(x^{m+1} - 1) + x(x^{m+1} - 1)' \\ &= -(m+1)(x^{m+1} - 1) + (m+1)x^{m+1} \\ &= m+1. \end{aligned}$$

Using this we get

$$\begin{aligned} &\frac{d}{dt} \left( (e^{2t} + 1)^{1/(m+1)} + \frac{1}{m+1} \sum_{j=0}^m \zeta^j \log((e^{2t} + 1)^{1/(m+1)} - \zeta^j) \right) \\ &= \frac{2}{m+1} e^{2t} (e^{2t} + 1)^{-m/(m+1)} \left( 1 + \frac{1}{m+1} \sum_{j=0}^m \frac{\zeta^j}{(e^{2t} + 1)^{1/(m+1)} - \zeta^j} \right) \\ &= \frac{2}{m+1} e^{2t} (e^{2t} + 1)^{-m/(m+1)} \left( 1 + \frac{1}{m+1} \frac{1}{e^{2t}} (m+1) \right) \\ &= \frac{2}{m+1} (1 + e^{2t})^{1/(m+1)} = \frac{2}{m+1} (F'(t) + 1). \end{aligned}$$

Thus Kähler potential is expressed up to the multiple of  $(m+1)/2$  as

$$(e^{2t} + 1)^{1/(m+1)} + \frac{1}{m+1} \sum_{j=0}^m \zeta^j \log((e^{2t} + 1)^{1/(m+1)} - \zeta^j).$$

Since we are using the  $\eta$ -Einstein metric such that  $\text{Ric}^T = 2\omega^T$ , to deform it into the Sasaki-Einstein metric such that  $\text{Ric}'^T = 2(m+1)\omega'^T$ , we need to perform  $D$ -homothetic transformation with

$$r = \tilde{r}^{m+1}.$$

Thus the Kähler potential of our Ricci-flat Kähler metric is given by

$$f = \left( (\tilde{r}^{2m+2} + 1)^{1/(m+1)} + \frac{1}{m+1} \sum_{j=0}^m \zeta^j \log((\tilde{r}^{2m+2} + 1)^{1/(m+1)} - \zeta^j) \right).$$

Putting  $s = 1/\tilde{r}$  and using l'Hôpital's theorem, we have

$$\begin{aligned} \lim_{\tilde{r} \rightarrow \infty} \frac{f - \tilde{r}^2}{\tilde{r}^\alpha} &= \lim_{\tilde{r} \rightarrow \infty} \frac{2(\tilde{r}^{2m+2} + 1)^{1/(m+1)} - 2\tilde{r}^2}{\alpha \tilde{r}^\alpha} \\ &= \lim_{s \rightarrow 0} \frac{2(1 + s^{2m+2})^{1/(m+1)} - 2}{\alpha s^{2-\alpha}}. \end{aligned}$$

Using the identity with a variable  $T$

$$(1 + T)^{1/(m+1)} = 1 + \frac{1}{m+1}T - \frac{m}{2(m+1)^2}T^2 + \dots$$

further, the limit above converges when  $\alpha = -2m$  as

$$\lim_{s \rightarrow 0} \frac{2/(m+1) s^{2m+2} - \dots}{\alpha s^{2-\alpha}} = -\frac{1}{m(m+1)}.$$

We finally get

$$f = \tilde{r}^2 - \frac{1}{m(m+1)}\tilde{r}^{-2m} + O(\tilde{r}^{-4m-2}).$$

This completes the proof of Lemma 5.3. □

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