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THE INTERSECTION OF TWO REAL FORMS IN THE COMPLEX HYPERQUADRIC

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Abstract. We show that, in the complex hyperquadric, the intersection of two real forms, which are certain totally geodesic Lagrangian submanifolds, is an antipodal set whose cardinality attains the smaller 2-number of the two real forms. As a corollary of the result, we know that any real form in the complex hyperquadric is a globally tight Lagrangian submanifold.

1. Introduction. Let $\bar{M}$ be a Hermitian symmetric space. A submanifold M is called a *real form* of $\bar{M}$, if there exists an involutive anti-holomorphic isometry σ of $\bar{M}$ satisfying

$$M = \{x \in \bar{M} \,;\, \sigma(x) = x\}\,.$$

Any real form M is a totally geodesic Lagrangian submanifold of $\bar{M}$, which follows from Leung [8] or Takeuchi [12, Lemma 1.1].

The complex hyperquadric $Q_n(\boldsymbol{C})$ is defined by

$$Q_n(\boldsymbol{C}) = \{[z_1, \dots, z_{n+2}] \in \boldsymbol{C}P^{n+1} \,;\, z_1^2 + \cdots + z_{n+2}^2 = 0\}\,,$$

and $Q_n(\boldsymbol{C})$ has the Kähler structure induced from the standard Kähler structure of the complex projective space $\boldsymbol{C}P^{n+1}$. It is known that $Q_n(\boldsymbol{C})$ is holomorphically isometric to the Hermitian symmetric space $SO(n+2)/SO(2) \times SO(n)$, which is the Grassmann manifold $\tilde{G}_2(\boldsymbol{R}^{n+2})$ consisting of all oriented linear subspaces of dimension 2 in $\boldsymbol{R}^{n+2}$. We also regard $\tilde{G}_2(\boldsymbol{R}^{n+2})$ as a submanifold in the exterior product $\bigwedge^2 \boldsymbol{R}^{n+2}$ in a natural way, because it is convenient to represent points of $\tilde{G}_2(\boldsymbol{R}^{n+2})$ by elements in $\bigwedge^2 \boldsymbol{R}^{n+2}$. We take an orthonormal basis $u_1, u_2, e_1, \dots, e_n$ of $\boldsymbol{R}^{n+2}$. For $0 \le k \le n$, we define a submanifold $S^{k,n-k}$ of $\tilde{G}_2(\boldsymbol{R}^{n+2})$ by

$$S^{k,n-k} = S^k(\boldsymbol{R}u_1 + \boldsymbol{R}e_1 + \cdots + \boldsymbol{R}e_k) \wedge S^{n-k}(\boldsymbol{R}u_2 + \boldsymbol{R}e_{k+1} + \cdots + \boldsymbol{R}e_n)\,,$$

where $S^m(V)$ is the unit hypersphere of dimension m in a real Euclidean space V of dimension $m+1$. This expression implies that $S^{k,n-k}$ is isometric to $(S^k \times S^{n-k})/\boldsymbol{Z}_2$. Leung [8] and Takeuchi [12] classified real forms of Hermitian symmetric spaces of compact type. We say that two submanifolds in $\tilde{G}_2(\boldsymbol{R}^{n+2})$ are congruent, if one is transformed to the other by the action of $SO(n+2)$. By the classification of Leung and Takeuchi, we can see that any real

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form in $\tilde{G}_2(\boldsymbol{R}^{n+2})$ is congruent to $S^{k,n-k}$ for a k with $0 \leq k \leq [n/2]$. This also follows from the classification of totally geodesic submanifolds of $\tilde{G}_2(\boldsymbol{R}^{n+2})$ obtained by Chen and Nagano [1].

A subset S in a Riemannian symmetric space M is called an *antipodal set*, if the geodesic symmetry s_x fixes every point of S for every point x of S. The 2*-number* $\#_2 M$ of M is the supremum of the cardinalities of antipodal sets of M, which was introduced by Chen and Nagano [2] and is known to be finite. We call an antipodal set in M *great* if its cardinality attains $\#_2 M$. Takeuchi [13] proved that if M is a symmetric R-space, then

$$\#_2 M = \dim H_*(M, \boldsymbol{Z}_2)\,, \tag{1}$$

where $H_*(M, \boldsymbol{Z}_2)$ denotes the homology group of M with coefficient $\boldsymbol{Z}_2$. We note that any real form of Hermitian symmetric spaces of compact type is a symmetric R-space, which is shown in [12].

We explicitly describe the intersection of two real forms of the complex hyperquadric in the following theorem.

THEOREM 1.1. *Let k and l be integers satisfying $0 \leq k \leq l \leq [n/2]$. Let L_1 be a real form of $\tilde{G}_2(\boldsymbol{R}^{n+2})$ congruent to $S^{k,n-k}$ and L_2 a real form of $\tilde{G}_2(\boldsymbol{R}^{n+2})$ congruent to $S^{l,n-l}$. If L_1 and L_2 intersect transversally, then $L_1 \cap L_2$ is congruent to*

$$\{\pm u_1 \wedge u_2, \pm e_1 \wedge e_2, \ldots, \pm e_{2k-1} \wedge e_{2k}\}\,,$$

which is an antipodal set of L_1 and L_2. In particular, $L_1 \cap L_2$ is a great antipodal set of L_1. Moreover, if $k = l = [n/2]$, $L_1 \cap L_2$ is a great antipodal set of $\tilde{G}_2(\boldsymbol{R}^{n+2})$.

REMARK 1.2. In the complex projective space $\boldsymbol{C}P^n$, any real form is congruent to the real projective space $\boldsymbol{R}P^n$ naturally embedded in $\boldsymbol{C}P^n$. Howard essentially showed the following fact in [4, pp. 26–27]. If two real forms L_1 and L_2 of $\boldsymbol{C}P^n$ intersect transversally, then there exists a unitary basis $u_1, \ldots, u_{n+1}$ of $\boldsymbol{C}^{n+1}$ satisfying

$$L_1 \cap L_2 = \{\boldsymbol{C}u_1, \ldots, \boldsymbol{C}u_{n+1}\}\,.$$

In particular $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2, because $\#_2 \boldsymbol{R}P^n = n + 1$. Thus Theorem 1.1 is a generalization of this fact. In this case, $L_1 \cap L_2$ is also a great antipodal set of $\boldsymbol{C}P^n$, because $\#_2 \boldsymbol{C}P^n = n + 1$.

In the proof, Howard showed that the intersection of two real forms in $\boldsymbol{C}P^n$ is not empty by a result of Frankel [3] and the positivity of the sectional curvature of $\boldsymbol{C}P^n$. Although the sectional curvature in our case is nonnegative, the argument of Frankel is still useful. See Lemma 3.1.

Oh [9] introduced the notion of global tightness of Lagrangian submanifolds in a Hermitian symmetric space. We call a Lagrangian submanifold L of a Hermitian symmetric space M *globally tight*, if L satisfies

$$\#(L \cap g \cdot L) = \dim H_*(L, \boldsymbol{Z}_2)$$

for any isometry g of M with property that L intersects $g \cdot L$ transversally. Considering the case where $k = l$ in Theorem 1.1, we obtain the following corollary from (1).

COROLLARY 1.3. *Any real form of the complex hyperquadric is a globally tight Lagrangian submanifold.*

REMARK 1.4. $Q_1(\boldsymbol{C}) = \boldsymbol{C}P^1 = S^2$ and its real form is the great circle, so its global tightness is well known. $Q_2(\boldsymbol{C}) = \boldsymbol{C}P^1 \times \boldsymbol{C}P^1 = S^2 \times S^2$ and its real forms $S^{0,2}$ and $S^{1,1}$ are globally tight, which Iriyeh and Sakai [5] proved in a different way. Recently, they also proved that $S^{0,n}$ and $S^{1,n-1}$ are globally tight in $Q_n(\boldsymbol{C})$.

REMARK 1.5. Makiko Sumi Tanaka and the author recently generalized Theorem 1.1 and obtained the following results in [14]. Let M be a Hermitian symmetric space of compact type. If two real forms L_1 and L_2 of M intersect transversally, then $L_1 \cap L_2$ is an antipodal set of L_1 and L_2. Moreover, if L_1 and L_2 are congruent, then $L_1 \cap L_2$ is a great antipodal set of L_1 and L_2. As a corollary of this result, we know that any real form in the Hermitian symmetric spaces of compact type is a globally tight Lagrangian submanifold. The cardinalities $\#(L_1 \cap L_2)$ of any two real forms L_1 and L_2 in the irreducible Hermitian symmetric spaces of comapct type are determined.

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2. The cut locus and the fixed point set of the geodesic symmetry. In this section, we review the results of Sakai [11] on the cut loci of compact symmetric spaces and of Chen and Nagano [1] on the fixed point set of the geodesic symmetry of the complex hyperquadric.

For a compact Riemannian manifold X and a point $p \in X$, we denote by $C_p(X)$ and $\tilde{C}_p(X)$ the cut locus and the tangent cut locus of X with respect to p.

THEOREM 2.1 (Sakai [11]). *Let $M = G/K$ be a compact Riemannian symmetric space with Riemannian symmetric pair (G, K). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of the Lie algebra $\mathfrak{g}$ of G. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{m}$ and denote by A the maximal torus of M corresponding to $\mathfrak{a}$. The following equalities hold.*

$$\tilde{C}_o(A) = \mathfrak{a} \cap \tilde{C}_o(M)\,, \qquad \tilde{C}_o(M) = \bigcup_{k \in K} \mathrm{Ad}(k)\tilde{C}_o(A)\,.$$

LEMMA 2.2. *Let $M_1 = G_1/K_1$, $M_2 = G_2/K_2$ be compact Riemannian symmetric spaces with symmetric pairs (G_1, K_1), (G_2, K_2). We assume that M_1 is a totally geodesic submanifold in M_2 and that $G_1 \subset G_2$, $K_1 \subset K_2$. Let $\mathfrak{g}_i = \mathfrak{k}_i + \mathfrak{m}_i$ be the canonical decompositions of the Lie algebras $\mathfrak{g}_i$ of G_i. We take maximal abelian subspaces $\mathfrak{a}_i$ of $\mathfrak{m}_i$ satisfying $\mathfrak{a}_1 \subset \mathfrak{a}_2$ and denote by A_i the maximal torus of M_i corresponding to $\mathfrak{a}_i$. If*

$$\tilde{C}_o(A_1) = \mathfrak{a}_1 \cap \tilde{C}_o(A_2) \tag{2}$$

holds, then

$$\tilde{C}_o(M_1) = \mathfrak{m}_1 \cap \tilde{C}_o(M_2) \tag{3}$$

holds and any shortest geodesic in M_1 is also shortest in M_2. In particular, if M_1 and M_2 have a same rank, then $A_1 = A_2$ and (2) *hold, thus* (3) *holds.*

PROOF. Theorem 2.1 and the assumption (2) imply

$$\begin{aligned}\tilde{C}_o(M_1) &= \bigcup_{k\in K_1} \mathrm{Ad}(k)\tilde{C}_o(A_1) = \bigcup_{k\in K_1} \mathrm{Ad}(k)(\mathfrak{a}_1 \cap \tilde{C}_o(A_2)) \\ &\subset \mathfrak{m}_1 \cap \bigcup_{k\in K_1} \mathrm{Ad}(k)\tilde{C}_o(A_2) \subset \mathfrak{m}_1 \cap \tilde{C}_o(M_2)\,.\end{aligned}$$

In order to prove the other inclusion, we take $X \in \mathfrak{m}_1 \cap \tilde{C}_o(M_2)$. There exists $k \in K_1$ satisfying $\mathrm{Ad}(k)X \in \mathfrak{a}_1$. Hence we have $\mathrm{Ad}(k)X \in \mathfrak{a}_1 \cap \mathrm{Ad}(k)\tilde{C}_o(M_2)$ and

$$\mathfrak{a}_1 \cap \mathrm{Ad}(k)\tilde{C}_o(M_2) = \mathfrak{a}_1 \cap \tilde{C}_o(M_2) = \tilde{C}_o(A_1)$$

by Theorem 2.1 and the assumption (2). Thus we obtain $X \in \mathrm{Ad}(k)^{-1}\tilde{C}_o(A_1)$ and

$$\mathfrak{m}_1 \cap \tilde{C}_o(M_2) \subset \bigcup_{k\in K_1} \mathrm{Ad}(k)\tilde{C}_o(A_1) = \tilde{C}_o(M_1)\,.$$

Therefore (3) holds. (3) implies that any shortest geodesic in M_1 is also shortest in M_2.

If M_1 and M_2 have a same rank, then $\dim A_1 = \dim A_2$ and $A_1 \subset A_2$. Thus $A_1 = A_2$, which implies (2). □

Using the results mentioned above, we can express the cut locus of $\tilde{G}_2(\boldsymbol{R}^{n+2})$. In this case we regard $u_1 \wedge u_2$ as the origin o of $\tilde{G}_2(\boldsymbol{R}^{n+2})$. Let

$$\begin{aligned}S^{1,1} &= S^1(\boldsymbol{R}u_1 + \boldsymbol{R}e_1) \wedge S^1(\boldsymbol{R}u_2 + \boldsymbol{R}e_2) \\ &= \{(\cos\theta_1 u_1 + \sin\theta_1 e_1) \wedge (\cos\theta_2 u_2 + \sin\theta_2 e_2);\, \theta_1, \theta_2 \in \boldsymbol{R}\}\,.\end{aligned}$$

This is a maximal torus of $\tilde{G}_2(\boldsymbol{R}^{n+2})$. We can see

$$\begin{aligned}&\{(\theta_1, \theta_2) \in \boldsymbol{R}^2;\, (\cos\theta_1 u_1 + \sin\theta_1 e_1) \wedge (\cos\theta_2 u_2 + \sin\theta_2 e_2) = u_1 \wedge u_2\} \\ &\quad = \{(\theta_1, \theta_2) \in (\pi\boldsymbol{Z})^2;\, \theta_1 + \theta_2 \in 2\pi\boldsymbol{Z}\} = \boldsymbol{Z}(\pi, \pi) + \boldsymbol{Z}(\pi, -\pi)\,.\end{aligned}$$

We identify the tangent space of $S^{1,1}$ at the origin with the coordinate plane consisting of (θ_1, θ_2). The tangent cut locus $\tilde{C}_o(S^{1,1})$ is the square of apexes $(\pi, 0)$, $(0, \pi)$, $(-\pi, 0)$ and $(0, -\pi)$. The region defined by $0 < \theta_2 < \theta_1$ is a Weyl chamber in the case where $n \geq 3$, while the region defined by $0 < \theta_1$ and $-\theta_1 < \theta_2 < \theta_1$ is a Weyl chamber in the case where $n = 2$. We set $P_1 = (\pi, 0)$, $P_2 = (\pi/2, \pi/2)$ and $P_3 = (\pi/2, -\pi/2)$. We denote by $\overline{XY}$ the segment joining X and Y. Considering the action of the Weyl group, we have

$$\begin{aligned}\tilde{C}_o(\tilde{G}_2(\boldsymbol{R}^{n+2})) &= \bigcup_{k\in SO(2)\times SO(n)} \mathrm{Ad}(k)(\overline{P_1P_2}) \quad (n \geq 3)\,, \\ \tilde{C}_o(\tilde{G}_2(\boldsymbol{R}^4)) &= \bigcup_{k\in SO(2)\times SO(2)} \mathrm{Ad}(k)(\overline{P_1P_2} \cup \overline{P_1P_3})\,.\end{aligned}$$

Next we express the fixed point set $F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o)$ of the geodesic symmetry s_o. The reflection $1_{\boldsymbol{R}u_1+\boldsymbol{R}u_2} - 1_{\boldsymbol{R}e_1+\cdots+\boldsymbol{R}e_n}$ with respect to $\boldsymbol{R}u_1 + \boldsymbol{R}u_2$ induces s_o. We can get

$$F(S^{1,1}, s_o) = \{\pm u_1 \wedge u_2, \pm e_1 \wedge e_2\}\,.$$

For $z = x_1 \wedge x_2 \in \tilde{G}_2(\boldsymbol{R}^{n+2})$, we denote $\bar{z} = -x_1 \wedge x_2$. We set $p_i = \mathrm{Exp}_o(P_i)$. The above fixed point set is expressed as follows:

$$F(S^{1,1}, s_o) = \{o, \bar{o}, p_2, \bar{p}_2\}$$

and $\bar{o} = p_1$, $\bar{p}_2 = p_3$ hold. We obtain

$$\begin{aligned} F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o) &= \bigcup_{k \in SO(2)\times SO(n)} kF(S^{1,1}, s_o) \\ &= \{\pm u_1 \wedge u_2\} \cup \tilde{G}_2(\boldsymbol{R}e_1 + \cdots + \boldsymbol{R}e_n) \\ &= \{o, \bar{o}\} \cup \tilde{G}_2(\boldsymbol{R}^n)\,. \end{aligned}$$

Here $SO(2) \times SO(n)$ is also the isotropy subgroup at $\bar{o}$. From this we can see that

$$\begin{aligned} C_{\bar{o}}(\tilde{G}_2(\boldsymbol{R}^{n+2})) &= \bigcup_{k \in SO(2)\times SO(n)} k\mathrm{Exp}_o(\overline{0P_2}) \quad (n \geq 3)\,, \\ C_{\bar{o}}(\tilde{G}_2(\boldsymbol{R}^{4})) &= \bigcup_{k \in SO(2)\times SO(2)} k\mathrm{Exp}_o(\overline{0P_2} \cup \overline{0P_3})\,. \end{aligned}$$

Since $s_o = s_{\bar{o}}$, we have $F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_{\bar{o}}) = F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o)$.

3. Proof of the main theorem. First we prove the existence of the intersection of two Lagrangian submanifolds under a condition weaker than that of Theorem 1.1.

LEMMA 3.1. *Let M be a compact Kähler manifold with positive holomorphic sectional curvature. If L_1 and L_2 are totally geodesic compact Lagrangian submanifolds in M, then $L_1 \cap L_2 \neq \emptyset$.*

PROOF. We suppose $L_1 \cap L_2 = \emptyset$. We join L_1 and L_2 by a shortest geodesic $c(s)$ $(0 \leq s \leq d(L_1, L_2))$. Since M is Kähler, the complex structure J of M is parallel. The velocity $c'(s)$ is parallel along $c(s)$, so $Jc'(s)$ is a parallel normal vector field along $c(s)$. The shortest property of $c(s)$ implies that $Jc'(s)$ are tangent to L_1 and L_2 at the end points, because L_1 and L_2 are Lagrangian. The parallel normal vector field $Jc'(s)$ generates a variation $c_t(s) = \mathrm{Exp}_{c(s)}(tJc'(s))$ of $c(s)$, each curve c_t of which joins L_1 and L_2, because L_1 and L_2 are totally geodesic. Its first variation of the length functional $\mathcal{L}$ vanishes, and by the second variation formula we have

$$\begin{aligned} &\left.\frac{d^2\mathcal{L}(c_t)}{dt^2}\right|_{t=0} \\ &\quad = \int_0^{d(L_1,L_2)} \{\langle \nabla_{\partial/\partial s} Jc'(s), \nabla_{\partial/\partial s} Jc'(s)\rangle - \langle R(Jc'(s), c'(s))c'(s), Jc'(s)\rangle\} ds \end{aligned}$$

$$
\begin{aligned}
&= -\int_0^{d(L_1,L_2)} \langle R(Jc'(s), c'(s))c'(s), Jc'(s)\rangle ds \\
&< 0\,,
\end{aligned}
$$

since $\nabla_{\partial/\partial s} Jc'(s) \equiv 0$ and $\langle R(Jc'(s), c'(s))c'(s), Jc'(s)\rangle$ is the holomorphic sectional curvature of $c'(s)$, which is positive by the assumption. This contradicts the shortest property of $c(s)$. Therefore $L_1 \cap L_2 \neq \emptyset$. □

REMARK 3.2. The method used in the above proof is due to Frankel [3]. Sakai [10] and Itoh [6] used this method to prove the existence of the fixed point of a certain transformation of Kähler manifolds with positive holomorphic sectionanl curvature. Kenmotsu and Xia [7] also used it to prove the existence of the intersection of two submanifolds in certain situations different from ours.

We prepare the following lemmas in order to prove Theorem 1.1.

LEMMA 3.3. *Let L be a real form through o in $\tilde{G}_2(\boldsymbol{R}^{n+2})$. If L is congruent to $S^{0,n}$, then*

$$L \cap F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o) = \{o, \bar{o}\}\,.$$

If L is congruent to $S^{k,n-k}$ $(1 \leq k \leq [n/2])$, then

$$L \cap F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o) = \{o, \bar{o}\} \cup L'\,,$$

where L' is a real form congruent to $S^{k-1,n-k-1}$ in $\tilde{G}_2(\boldsymbol{R}^n)$.

PROOF. Even if the isotropy subgroup at o acts on L, the conclusions of the lemma do not change. So we can suppose that $L = S^{k,n-k}$.

According to the description of $F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o)$ obtained in the previous section, we can get

$$
\begin{aligned}
&S^{0,n} \cap F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o) = \{o, \bar{o}\}\,, \\
&S^{k,n-k} \cap F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o) \\
&\quad = \{o, \bar{o}\} \cup S^{k-1}(\boldsymbol{R}e_1 + \cdots + \boldsymbol{R}e_k) \wedge S^{n-k-1}(\boldsymbol{R}e_{k+1} + \cdots + \boldsymbol{R}e_n) \\
&\quad = \{o, \bar{o}\} \cup S^{k-1,n-k-1}\,,
\end{aligned}
$$

which complete the proof of the lemma. □

LEMMA 3.4. *If L is a real form through o in $\tilde{G}_2(\boldsymbol{R}^{n+2})$, then we have*

$$\tilde{C}_o(L) = T_oL \cap \tilde{C}_o(\tilde{G}_2(\boldsymbol{R}^{n+2}))\,.$$

In particular, any shortest geodesic in L is also a shortest geodesic in $\tilde{G}_2(\boldsymbol{R}^{n+2})$.

PROOF. Similarly to the proof of Lemma 3.3, we can suppose that $L = S^{k,n-k}$. In the case where $L = S^{0,n}$, the closed geodesic $u_1 \wedge S^1(\boldsymbol{R}u_2 + \boldsymbol{R}e_2)$ is a maximal torus of $S^{0,n}$ and its tangent space $\{(0, \theta_2); \theta_2 \in \boldsymbol{R}\}$ satisfies the condition (2) of Lemma 2.2 by the description of $\tilde{C}_o(S^{1,1})$ obtained in the previous section. Hence the assertions of Lemma 3.4 hold in this case. In the case where $L = S^{k,n-k}$ $(1 \leq k \leq [n/2])$, the ranks of $S^{k,n-k}$ and $\tilde{G}_2(\boldsymbol{R}^{n+2})$ are equal to two, hence the assertions hold by Lemma 2.2. $\square$

LEMMA 3.5. *If two real forms L_1 and L_2 through o in $\tilde{G}_2(\boldsymbol{R}^{n+2})$ intersect transversally, then*

$$L_1 \cap L_2 \subset F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o)\,.$$

PROOF. We first prove

$$L_1 \cap L_2 - \{o\} \subset C_o(\tilde{G}_2(\boldsymbol{R}^{n+2}))\,. \tag{4}$$

We suppose there exists $x \in L_1 \cap L_2 - \{o\}$ satisfying $x \notin C_o(\tilde{G}_2(\boldsymbol{R}^{n+2}))$. Lemma 3.4 implies $x \notin C_o(L_i)$, so there exists a unique shortest geodesic c_i joining o and x in each L_i. These c_1 and c_2 are also the shortest geodesics in $\tilde{G}_2(\boldsymbol{R}^{n+2})$ because of Lemma 3.4. Therefore, we have $c_1 = c_2$, which contradicts the assumption that L_1 and L_2 intersect transversally. Hence we have proved (4).

Lemma 3.3 implies $\{o, \bar{o}\} \subset L_1 \cap L_2$. We can apply (4) to $\bar{o}$ and obtain

$$L_1 \cap L_2 - \{\bar{o}\} \subset C_{\bar{o}}(\tilde{G}_2(\boldsymbol{R}^{n+2}))\,.$$

In the case where $n \geq 3$, the inside of $\tilde{C}_o(S^{1,1})$ includes $\overline{0P_2} - \{P_2\}$. Hence the orbit of $\overline{0P_2} - \{P_2\}$ under the action of $SO(2) \times SO(n)$ does not intersect $\tilde{C}_o(\tilde{G}_2(\boldsymbol{R}^{n+2}))$ and

$$\begin{aligned} L_1 \cap L_2 - \{o, \bar{o}\} &\subset C_o(\tilde{G}_2(\boldsymbol{R}^{n+2})) \cap C_{\bar{o}}(\tilde{G}_2(\boldsymbol{R}^{n+2})) \\ &= \bigcup_{k \in SO(2) \times SO(n)} k\mathrm{Exp}_o(P_2) = \tilde{G}_2(\boldsymbol{R}^n)\,. \end{aligned}$$

In the case where $n = 2$, similarly

$$\begin{aligned} L_1 \cap L_2 - \{o, \bar{o}\} &\subset C_o(\tilde{G}_2(\boldsymbol{R}^4)) \cap C_{\bar{o}}(\tilde{G}_2(\boldsymbol{R}^4)) \\ &= \bigcup_{k \in SO(2) \times SO(2)} k\mathrm{Exp}_o(\{P_2, P_3\}) = \{p_2, \bar{p}_2\} \\ &= \tilde{G}_2(\boldsymbol{R}^n)\,. \end{aligned}$$

Therefore $L_1 \cap L_2 \subset F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o)$. $\square$

PROOF OF THEOREM 1.1. Since the holomorphic sectional curvatures of $\tilde{G}_2(\boldsymbol{R})$ are positive, $L_1 \cap L_2 \neq \emptyset$ by Lemma 3.1. Moreover, we can suppose that $o \in L_1 \cap L_2$. We prove the first assertion of the theorem by induction on k. If L_1 is congruent to $S^{0,n}$, Lemmas 3.3 and 3.5 imply $L_1 \cap L_2 = \{o, \bar{o}\}$, which is an antipodal set in L_1 and L_2. This equality holds even if $n = 1$. Thus we have the first assertion of the theorem in the case where $k = 0$. If L_1 is congruent to $S^{k,n-k}$ $(1 \leq k \leq [n/2])$, Lemma 3.3 implies

$$L_1 \cap F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o) = \{o, \bar{o}\} \cup L_1'\,,$$

where L_1' is a real form congruent to $S^{k-1,n-k-1}$ in $\tilde{G}_2(\boldsymbol{R}^n)$ and

$$L_2 \cap F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o) = \{o, \bar{o}\} \cup L_2'\,,$$

where L_2' is a real form congruent to $S^{l-1,n-l-1}$ in $\tilde{G}_2(\boldsymbol{R}^n)$. By the assumption of the induction, $L_1' \cap L_2'$ is congruent to

$$\{\pm e_1 \wedge e_2, \dots, \pm e_{2k-1} \wedge e_{2k}\},$$

which is an antipodal set in L_1' and L_2'. Since $L_1 \cap L_2 \subset F(\tilde{G}_2(\boldsymbol{R}^{n+2}), s_o)$ by Lemma 3.5, $L_1 \cap L_2$ is congruent to

$$\{\pm u_1 \wedge u_2, \pm e_1 \wedge e_2, \dots, \pm e_{2k-1} \wedge e_{2k}\},$$

which is an antipodal set in L_1 and L_2. [2, Proposition 3.12] and [2, Theorem 4.3] imply $\#_2 S^{k,n-k} = 2k + 2$ and $\#_2 \tilde{G}_2(\boldsymbol{R}^{n+2}) = 2[n/2] + 2$, which complete the proof of the theorem. □

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