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NONORIENTABLE MAXIMAL SURFACES IN THE LORENTZ-MINKOWSKI 3-SPACE

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Abstract. The geometry and topology of complete nonorientable maximal surfaces with lightlike singularities in the Lorentz-Minkowski 3-space are studied. Some topological congruence formulae for surfaces of this kind are obtained. As a consequence, some existence and uniqueness results for maximal Möbius strips and maximal Klein bottles with one end are proved.

Introduction. A maximal surface in the Lorentz-Minkowski 3-space $\boldsymbol{L}^3$ is a spacelike surface with zero mean curvature. Besides their mathematical interest, these surfaces have a significant importance in classical Relativity, Dynamic of Fluids, Cosmology, and so on (more information can be found for instance in [MT, Ki1, Ki2]).

Maximal surfaces in $\boldsymbol{L}^3$ share some properties with minimal surfaces in the Euclidean 3-space $\boldsymbol{R}^3$. Both families arise as solutions of variational problems: local maxima (minima) for the area functional in the Lorentzian (Euclidean) case. Like minimal surfaces in $\boldsymbol{R}^3$, maximal surfaces in $\boldsymbol{L}^3$ also admit a Weierstrass representation in terms of meromorphic data [Ko1, Ko2, Mc].

Calabi [C] proved that a complete maximal surface in $\boldsymbol{L}^3$ is necessarily a spacelike plane. Therefore, it is meaningless to consider global problems on maximal and *everywhere regular* surfaces in $\boldsymbol{L}^3$. However, physical and geometrical experience suggests to extend the global analysis to the wider family of complete maximal immersions with *singularities* (see [Ki1, Ki2]). A point of a maximal surface is said to be *singular* if the induced metric ds^2 degenerates at p. Throughout this paper, it will be always assumed that the complement of the singular set is a dense subset of the surface. Roughly speaking, there are two kinds of singular points: classical branch points and lightlike singular points or points with lightlike tangent planes (see [UY] for a good setting). Complete maximal surfaces with lightlike singularities and *no branch points* have given rise to an interesting theory (see for instance [FL, FLS, UY]). Following Umehara and Yamada [UY], this kind of surfaces will be called (complete) *maxfaces*. Generic singularities of maxfaces are classified in [FSUY].

Although the family of complete maxfaces is very vast, all previously known examples are orientable. Among them, we emphasize the Lorentzian catenoid described by O.

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Kobaysshi [Ko2], the Riemann type maximal examples exhibited by F. J. López, R. López and R. Souam [LLS], the high genus maxfaces produced by Umehara and Yamada [UY], the universal cover of the entire maximal graphs with conical singularities described by Fernandez, Lopez and Souam [FL, FLS] and Kim-Yang maximal examples [KY].

The purpose of this paper is to study the geometry and topology of complete *nonorientable* maxfaces in $\boldsymbol{L}^3$. It is interesting to notice that spacelike surfaces in $\boldsymbol{L}^3$ are orientable, and so the singular set of a nonorientable maxface is always non empty. We introduce the first basic examples of this kind of surfaces and obtain some natural characterization theorems. By definition, a nonorientable "Riemann surface" is a nonorientable surface endowed with an atlas whose transition maps are either holomorphic or antiholomorphic.

Like in the orientable case (see [UY]), a conformal complete nonorientable maxface $X : M \to \boldsymbol{L}^3$ is conformally equivalent to a compact nonorientable "Riemann surface" minus a finite set of points: $M = \overline{M} - \{p_1, \dots, p_n\}$. Furthermore, dX has a "meromorphic" extension to $\overline{M}$ and the ends have finite total curvature. The "Gauss map" N of M is well-defined on the complement of the singular set S of M, and takes values on $\boldsymbol{H}^2/\langle I \rangle$, where $\boldsymbol{H}^2$ is the Lorentzian sphere of radius -1 and $I : \boldsymbol{H}^2 \to \boldsymbol{H}^2$ is the antipodal map $I(p) = -p$. Since N is conformal, the composition $\hat{N} = p_s \circ N : M - S \to \boldsymbol{D} \equiv (\overline{\boldsymbol{C}} - \{|z| = 1\})/\langle A \rangle$ is conformal as well, where A is the complex involution $A(z) = 1/\bar{z}$ and p_s is, up to passing to the quotients, the Lorentzian stereographic projection. Furthermore, $\hat{N}$ extends meromorphically to $\overline{M}$ and satisfies that $|\hat{N}(p_i)| \neq 1$ and $\hat{N}(S) \subset \{|z| = 1\}$.

The immersion X behaves like a spacelike sublinear multigraph around each end p_i of M, and labeling $\mu_i \geq 1$ as the winding number of X at p_i, the following Jorge-Meeks type formula holds:

$$\deg(\hat{N}) = -\chi(\overline{M}) + \sum_{i=1}^{n} (\mu_i + 1)\,,$$

where $\deg(\hat{N})$ and $\chi(\overline{M})$ are the degree of $\hat{N}$ and the Euler characteristic of $\overline{M}$, respectively (see [Me, FL, FLS]).

The first part of the paper is devoted to prove the following topological congruence formulae:

THEOREM A. *If $X : M \to \boldsymbol{L}^3$ is a conformal complete nonorientable maxface with Gauss map $\hat{N}$, then*

(i) $\deg \hat{N}$ *is even and greater than or equal to* 4.

(ii) *If in addition X has embedded ends* (*that is to say*, $\mu_i = 1$ *for all* i), *then* $\chi(\overline{M})$ *is even.*

In the second part, we produce the first known examples of complete nonorientable maxfaces. To be more precise, we describe the moduli space of complete maxfaces with the topology of a Möbius strip and Gauss map of degree four, and construct two complete one-ended Klein bottles, named KB_1 and KB_2, with Gauss map of degree four as well. Both KB_1 and KB_2

contain the x_1- and x_2-axes, and therefore their symmetry group contains four elements. Finally, we prove the following characterization theorem:

THEOREM B. *KB_1 and KB_2 are the unique complete maxfaces with the topology of a one-ended Klein bottle, Gauss map of degree four and have at least four symmetries.*

The results in this work have been inspired by Meeks [Me], López [L1, L2] and López-Martín papers [LM1, LM2] about complete nonorientable minimal surfaces in $\boldsymbol{R}^3$.

1. Preliminaries. Throughout this paper, we denote by $\overline{\boldsymbol{C}}$ the Riemann sphere.

Let $\boldsymbol{L}^3$ be the three dimensional Lorentz-Minkowski space with the metric $\langle\ ,\ \rangle = dx_1^2 + dx_2^2 - dx_3^2$. Let M be a two dimensional manifold. An immersion $X : M \to \boldsymbol{L}^3$ is called *spacelike* if the induced metric on the immersed surface is positive definite. Using isothermal parameters, M can be naturally considered as a Riemann surface and X a conformal map. A conformal spacelike immersion $X : M \to \boldsymbol{L}^3$ is said to be *maximal* if X has vanishing mean curvature.

Let M be a Riemann surface, and let X_1, X_2, X_3 be three harmonic functions on M satisfying that

$$dX_1^2 + dX_2^2 - dX_3^2 = 0\,,$$

$$|dX_1|^2 + |dX_2|^2 + |dX_3|^2 > 0\,.$$

Then the map $X := (X_1, X_2, X_3) : M \to \boldsymbol{L}^3$ gives a conformal maximal immersion with no branch points and eventually lightlike singularities (i.e., points where the tangent plane is lightlike). The singularities correspond to the null set of $|dX_1|^2 + |dX_2|^2 - |dX_3|^2$.

If the nonsingular set $W = \{p \in M\,;\ (|dX_1|^2 + |dX_2|^2 - |dX_3|^2)(p) > 0\}$ is dense in M, X is said to be a *maxface* [UY].

We label ϕ_j as the holomorphic 1-form dX_j ($j = 1, 2, 3$), and call g as the meromorphic function $i\phi_3/(\phi_1 - i\phi_2)$. Up to a translation,

$$X = \operatorname{Re} \int (\phi_1, \phi_2, \phi_3)\,, \tag{1.1}$$

where

$$\phi_1 = \frac{i}{2}\left(\frac{1}{g} - g\right)\phi_3\,, \quad \phi_2 = \frac{1}{2}\left(\frac{1}{g} + g\right)\phi_3\,. \tag{1.2}$$

The induced metric ds^2 on M (which is positive definite on W) is given by

$$ds^2 = |\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 = \left(\frac{|\phi_3|}{2}\left(\frac{1}{|g|} - |g|\right)\right)^2. \tag{1.3}$$

The singular set can be rewritten as $\{p \in M\,;\ |g(p)| = 1\}$.

REMARK 1.1. Up to composing with the Lorentzian stereographic projection, g coincides with the Gauss map of X, and for this reason it will be called as the meromorphic Gauss map of X. For more details, see [Ko1].

Conversely, let M, g, ϕ_3 be a Riemann surface, a meromorphic function and a holomorphic 1-form on M, respectively, satisfying that the 1-forms ϕ_1 and ϕ_2 in equation (1.2) are holomorphic,

$$|\phi_1|^2 + |\phi_2|^2 + |\phi_3|^2 > 0\,, \quad \text{and} \tag{1.4}$$

$$\operatorname{Re}\int_\gamma (\phi_1, \phi_2, \phi_3) = (0, 0, 0) \quad \text{for all } \gamma \in H_1(M, \boldsymbol{Z})\,. \tag{1.5}$$

Then $X = \operatorname{Re}\int(\phi_1, \phi_2, \phi_3) : M \longrightarrow \boldsymbol{L}^3$ defines a maxface.

REMARK 1.2. (1) We call (M, g, ϕ_3) (or simply (g, ϕ_3)) as the Weierstrass data of X.

(2) The condition (1.4) is equivalent to

$$\left(\frac{|\phi_3|}{2}\left(\frac{1}{|g|} + |g|\right)\right)^2 > 0\,, \tag{1.6}$$

and simply means that X has no branch points.

(3) The condition (1.5) is the so called *period condition*, and guarantees that X is well-defined on M. This condition is equivalent to the following two equations:

$$\int_\gamma g\phi_3 + \overline{\int_\gamma \frac{\phi_3}{g}} = 0 \quad \text{for all } \gamma \in H_1(M, \boldsymbol{Z})\,, \tag{1.7}$$

$$\operatorname{Re}\int_\gamma \phi_3 = 0 \quad \text{for all } \gamma \in H_1(M, \boldsymbol{Z})\,. \tag{1.8}$$

(4) Since the coordinate functions of X are harmonic, the maximum principle implies that there exist no compact maxfaces with empty boundary.

The following notions of completeness and finite type for maxfaces can be found in [UY].

DEFINITION 1.3. A maxface $X : M \to \boldsymbol{L}^3$ is said to be *complete* (resp. of *finite type*) if there exist a compact set C and a symmetric $(0, 2)$-tensor T on M such that T vanishes on $M - C$ and $ds^2 + T$ is a complete (resp. finite total curvature) Riemannian metric.

PROPOSITION 1.4 ([UY, Proposition 4.5]). *Let $X : M \to \boldsymbol{L}^3$ be a complete maxface. Then there exist a compact Riemann surface $\overline{M}$ and finite number of points $p_1, \ldots, p_n \in \overline{M}$ so that M is biholomorphic to $\overline{M} - \{p_1, \ldots, p_n\}$. Moreover, the Weierstrass data g and ϕ_3 extend meromorphically to $\overline{M}$ and the limit normal vector at the ends is timelike.*

By definition, the genus of X is the genus of $\overline{M}$. The removed points $p_1, \ldots, p_n \in \overline{M}$ correspond to the *ends* of X (note that no end is accumulation point of the singular set).

THEOREM 1.5 ([UY, Theorem 4.6]). *Complete maxfaces are of finite type.*

It is not hard to see that any complete maxface $X : M \to \boldsymbol{L}^3$ is eventually a finite multigraph over any spacelike plane. Indeed, consider a spacelike plane $\Sigma \subset \boldsymbol{L}^3$ and let $p : \boldsymbol{L}^3 \to \Sigma$ denote the Lorentzian orthogonal projection on Σ. Then take a solid circular cylinder $C \subset \boldsymbol{L}^3$ orthogonal to Σ and containing all of the singularities of $X(M)$. By basic topological arguments $X^{-1}(C)$ is compact, and it is not hard to check that the map $p \circ X : M - X^{-1}(C) \to \Sigma - C$ is a proper local diffeomorphism (and so a covering) with finitely many sheets, proving our assertion. The converse is also true (see [FLS, FL] for more details).

Let μ_i denote the winding number (or multiplicity) of the multigraph X around p_i. It is not hard to check that $\mu_i = \max\{\mathrm{Ord}_{p_i}(\phi_j),\ j = 1, 2, 3\} - 1$, where $\mathrm{Ord}_{p_i}(\phi_j)$ is the pole order of ϕ_j at p_i (see, for instance [FL]). The following Jorge-Meeks type formula and Osserman-type inequality will be useful:

THEOREM 1.6 ([FL, UY]). *If $X : \overline{M} - \{p_1, \dots, p_n\} \to \boldsymbol{L}^3$ is a complete maxface with meromorphic Gauss map g, then*

$$2 \deg g = -\chi(\overline{M}) + \sum_{i=1}^{n} (\mu_i + 1)\,,$$

where $\chi(\overline{M})$ denotes the Euler characteristic of $\overline{M}$. In particular,

$$2 \deg g \geq -\chi(\overline{M}) + 2n\,. \tag{1.9}$$

Moreover, the equality holds if and only if X is an embedding around any end of M.

2. Nonorientable maxfaces. Let M' be a *nonorientable Riemann surface*, that is to say, a nonorientable surface endowed with an atlas whose transition maps are holomorphic or antiholomorphic. Let $\pi : M \to M'$ denote the orientable conformal double cover of M'.

DEFINITION 2.1. A conformal map $X' : M' \to \boldsymbol{L}^3$ is said to be a *nonorientable maxface* if the composition

$$X = X' \circ \pi : M \to \boldsymbol{L}^3$$

is a maxface. In addition, X' is said to be complete if X is complete.

REMARK 2.2. For any maxface $X : M \to \boldsymbol{L}^3$, regardless of whether M is orientable or nonorientable, there exists a real analytic normal vector field which is well-defined on M. See Section 5 of [KU] for more details.

Let $X' : M' \to \boldsymbol{L}^3$ be a nonorientable maxface, and let $I : M \to M$ denote the antiholomorphic order two deck transformation associated to the orientable double cover $\pi : M \to M'$. Since $X \circ I = X$, then $I^*(\phi_j) = \bar{\phi}_j$ ($j = 1, 2, 3$), or equivalently,

$$g \circ I = \frac{1}{\bar{g}} \quad \text{and} \quad I^*(\phi_3) = \bar{\phi}_3\,. \tag{2.1}$$

As a consequence, I leaves invariant the singular set $\{p \in M\ ;\ |g(p)| = 1\}$.

Conversely, if (g, ϕ_3) is the Weierstrass data of an orientable maxface $X : M \to \boldsymbol{L}^3$ and I is an antiholomorphic involution without fixed points in M satisfying (2.1), then the unique map $X' : M' = M/\langle I \rangle \to \boldsymbol{L}^3$ satisfying that $X = X' \circ \pi$ is a nonorientable maxface. We call (M, I, g, ϕ_3) as the Weierstrass data of $X' : M' \to \boldsymbol{L}^3$.

Assume that $X' : M' = M/\langle I \rangle \to \boldsymbol{L}^3$ is complete. Then I extends conformally to the compactification $\overline{M}$ of M and

$$M = \overline{M} - \{q_1, \dots, q_m, I(q_1), \dots, I(q_m)\},$$

where $q_1, \dots, q_m \in \overline{M}$. Consequently, $M' = \overline{M}' - \{\pi(q_1), \dots, \pi(q_m)\}$, where $\overline{M}' = \overline{M}/\langle I \rangle$ is a compact nonorientable conformal surface of genus $2 - \chi(\overline{M}') = 2 - (1/2)\chi(\overline{M})$. By definition, the genus of X' is the genus of M'.

2.1. Topological congruence formulae for nonorientable maxfaces. Let $X' : M' \to \boldsymbol{L}^3$ be a complete nonorientable maxface with Weierstrass data (M, I, g, ϕ_3), and label as $\pi : M \to M'$ as the orientable double cover of M'. Denote by $A : \overline{\boldsymbol{C}} \to \overline{\boldsymbol{C}}$ the complex conjugation $A(z) = 1/\bar{z}$, and consider the projection $p_0 : \overline{\boldsymbol{C}} \to \overline{\boldsymbol{D}} \equiv \overline{\boldsymbol{C}}/\langle A \rangle$.

DEFINITION 2.3. The unique conformal map $\hat{g} : M' \to \overline{\boldsymbol{C}}/\langle A \rangle$ satisfying that $\hat{g} \circ \pi = p_0 \circ g$ is said to be the Gauss map of X'.

By Proposition 1.4, if X' is complete then $\hat{g}$ extends conformally to the compatification $\overline{M}'$ of M'. Moreover, $\hat{g}$ has the same degree as $g : \overline{M} \to \overline{\boldsymbol{C}}$. The Jorge-Meeks type formula in Theorem 1.6 gives

$$\deg \hat{g} = -\chi(\overline{M}') + \sum_{i=1}^{m} (\mu_i + 1),$$

where μ_i is the multiplicity of X at q_i, hence the inequality (1.9) becomes:

$$\deg \hat{g} \geq -\chi(\overline{M}') + 2m, \tag{2.2}$$

where m is the number of ends of M'.

THEOREM 2.4. *If X' is complete then the degree of $\hat{g}$ is even.*

PROOF. Let $X' : M' \to \boldsymbol{L}^3$ be a complete nonorientable maxface with the Weierstrass data (M, I, g, ϕ_3). As in the previous section, let $\overline{M}$ and $\overline{M}'$ be the compactifications of M and M', respectively.

Consider a meromorphic function h on $\overline{M}$ such that $h \circ I = -1/\bar{h}$ (the existence of this kind of functions is well known, see [R]), and call $\hat{h} : \overline{M}' \to \boldsymbol{RP}^2$ as the unique conformal map making the following diagram commutative:

$$\begin{array}{ccc} \overline{M} & \xrightarrow{h} & \overline{\boldsymbol{C}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ \overline{M}' & \xrightarrow{\hat{h}} & \boldsymbol{RP}^2 . \end{array}$$

Here $\boldsymbol{RP}^2 = \overline{\boldsymbol{C}}/I_0$, where $I_0(z) = -1/\bar{z}$ is the antipodal map, and $\pi_0 : \overline{\boldsymbol{C}} \to \boldsymbol{RP}^2 = \overline{\boldsymbol{C}}/I_0$ is the natural projection. Since $\deg \pi = \deg \pi_0 = 2$, the degree of $\hat{h}$ is well-defined, and as a matter of fact $\deg \hat{h} = \deg h$.

On the other hand, Meeks [Me, Theorem 1] proved the following fact:

FACT 2.5 ([Me, Theorem 1]). *Let M_1 and M_2 be compact surfaces without boundary and let $f : M_1 \to M_2$ be a branched cover of M_2. If $\chi(M_2)$ is odd, then $\chi(M_1)$ and* $\deg f$ *are either both even or both odd. If $\chi(M_2)$ is even, then $\chi(M_1)$ is even.*

Therefore, we deduce that $\deg h = \deg \hat{h} \equiv \chi(\overline{M}') \pmod 2$.

Up to composing h with a suitable Möbius transformation of the form $L(z) = (z + a)/(\overline{a}z - 1)$, we can suppose that $h(p) \neq 0, \infty$ for all zero or pole p of g. Thus the meromorphic function $G : \overline{M} \to \overline{\boldsymbol{C}}$ defined by $G(z) = g(z)h(z)$ has

$$\deg G = \deg(gh) = \deg g + \deg h\,.$$

Since $G \circ I = (g \cdot h) \circ I = (g \circ I)(h \circ I) = (1/\bar{g})(-1/\bar{h}) = -1/\bar{G}$, Meeks result gives that $\deg G \equiv \chi(\overline{M}') \pmod 2$, and so $\deg(\hat{g}) = \deg g \equiv 0 \pmod 2$, proving the theorem. □

COROLLARY 2.6. *Let $X' : M' \to \boldsymbol{L}^3$ be a complete nonorientable maxface with embedded ends. Then X' has even genus.*

PROOF. Let (M, I, g, ϕ_3) be the Weierstrass data of $X' : M' \to \boldsymbol{L}^3$, and write $M = \overline{M} - \{q_1, \dots, q_m, I(q_1), \dots, I(q_m)\}$. Since the ends are embedded, Theorem 1.6 gives that $2 \deg g = -\chi(\overline{M}) + 2 \cdot (2m)$, hence $\chi(\overline{M}) \equiv 0 \pmod 4$ by Theorem 2.4, which completes the proof. □

COROLLARY 2.7. *Let $X' : M' \to \boldsymbol{L}^3$ be a complete nonorientable maxface. Then the Gauss map of X' has degree greater than or equal to* 4.

PROOF. Label (M, I, g, ϕ_3) as the Weierstrass data of X'.

If X' has genus greater than two, the corollary follows straightforwardly from equation (2.2) and Theorem 2.4.

Assume that X' has genus two, and reasoning by contradiction suppose that $\deg(\hat{g}) = 2$. By equation (2.2) and Theorem 2.4, X' has an unique embedded end. Furthermore, up to Lorentzian isometries we may assume that X' is asymptotic at infinity to either a horizontal plane or a horizontal upward half catenoid. In the first case, the third coordinate function of X' is bounded, hence constant by the maximum principle (recall that the double cover M is parabolic), which is absurd. In the second case, the third coordinate function of X' has an interior minimum, contradicting the maximum principle for harmonic functions as well.

Finally, suppose that X' has genus one, and as above suppose $\deg(\hat{g}) = 2$. Up to a conformal transformation, we may assume that $M = \boldsymbol{C} - \{0\}$ and $I(z) = -1/\bar{z}$. Up to a suitable Lorentzian rotation, we will also assume $g(0) = 0$ and $g(\infty) = \infty$. Moreover, recall that g and ϕ_3 satisfy (2.1) and (1.6) on M. Since $g \circ I = 1/\bar{g}$, up to a suitable conformal transformation and rotation around the x_3-axis, we have that $g = z(z - r)/(rz + 1)$, $r \in \boldsymbol{R}$. By equation (1.6) and the condition $I^*\phi_3 = \bar{\phi}_3$, we get that $\phi_3 = is(rz + 1)(z - r)z^{-2}dz$, $s \in \boldsymbol{R} - \{0\}$. A direct computation shows that (1.7) does not hold for a loop around $z = 0$, completing the proof. □

REMARK 2.8. A similar result does not hold in the orientable case. The Lorentzian catenoid is a complete maxface of genus zero and has degree one Gauss map. Moreover, there exist complete orientable one-ended genus one maxface with degree two Gauss map (see [UY]), and complete orientable two-ended genus one maxface with degree two Gauss map (see [KY]).

Theorem A in the introduction follows from Theorem 2.4 and Corollaries 2.6 and 2.7.

3. Maximal Möbius strips with low degree Gauss map. This section is devoted to describe the family of one-ended genus one nonorientable complete maxfaces with degree four Gauss map.

Let $X' : M' \to \boldsymbol{L}^3$ be a complete maxface with the topological type of a Möbius strip. Without loss of generality we can write $M' = \boldsymbol{RP}^2 - \{\pi_0(0)\}$, where $\pi_0 : \overline{\boldsymbol{C}} \to \boldsymbol{RP}^2 = \overline{\boldsymbol{C}}/\langle I_0 \rangle$ is the conformal universal cover and $I_0(z) = -1/\bar{z}$. Call $(M = \boldsymbol{C} - \{0\}, I_0, g, \phi_3)$ as the Weierstrass data of X', where g is a meromorphic function of even degree (see Theorem 2.4). We are going to deal only with the simplest case $\deg g = 4$. Up to a suitable Lorentzian rotation, we will assume that $g(0) = 0$ and $g(\infty) = \infty$.

LEMMA 3.1. *In the above setting, the branching number of g at* 0 *and* ∞ *is even.*

PROOF. Suppose that g has a branch point of order three at $z = 0$. After a rotation around the x_3-axis, we have that $g = z^4$ (recall that $g \circ I = 1/\bar{g}$). Since g has neither zeros nor poles on M, the same holds for ϕ_3 by (1.6). Taking into account that $I^*\phi_3 = \bar{\phi}_3$, we infer that $\phi_3 = i\,dz/z$, contradicting that ϕ_3 has no real periods on $\boldsymbol{C} - \{0\}$.

Assume now that g has a branch point of order one at $z = 0$. In this case and after a rotation around the x_3-axis, we can put

$$g = z^2 \frac{(rz - 1)(sz - 1)}{(z + \bar{r})(z + \bar{s})}$$

for some constants $r, s \in \boldsymbol{C} - \{0\}$, and so by (2.1) and (1.6)

$$\phi_3 = i\frac{(rz - 1)(z + \bar{r})(sz - 1)(z + \bar{s})}{z^3} dz\,.$$

A direct computation shows that (1.7) does not hold for a loop around $z = 0$, proving the Lemma. □

Suppose now that g has a branch point of order two at $z = 0$. Up to conformal transformations in $\boldsymbol{C} - \{0\}$ and rotations around the x_3-axis, we may set $g = z^3(rz - 1)/(z + r)$ for some real positive constant r. Reasoning as in the proof of Corollary 2.7, we get $\phi_3 = i(rz - 1)(z + r)z^{-2}dz$. Obviously $g\phi_3$ and ϕ_3/g have no residues at the ends, hence ϕ_1 and ϕ_2 have no real periods on $\boldsymbol{C} - \{0\}$. Moreover, ϕ_3 has no real periods if and only if $\int_\gamma \phi_3 = -2\pi(r^2 - 1) = 0$ for any loop γ winding once around $z = 0$, and so $r = 1$.

Clearly X is complete and its singular set is compact. Therefore, it induces a complete nonorientable maxface $X' : \boldsymbol{RP}^2 - \{\pi(0)\} \to \boldsymbol{L}^3$. See the left-hand side of Figure 3.2.

REMARK 3.2. For each $k \in \boldsymbol{N}$, the data $g = z^{2k+1}(z+1)/(z-1)$, $\phi_3 = i(z^2-1)z^{-2}dz$ on $\boldsymbol{C} - \{0\}$ determine a complete nonorientable maxface $X' : \boldsymbol{RP}^2 - \{\pi_0(0)\} \to \boldsymbol{L}^3$ with $\deg g = 2k + 2$.

FIGURE 3.1. Henneberg-type maximal surface.

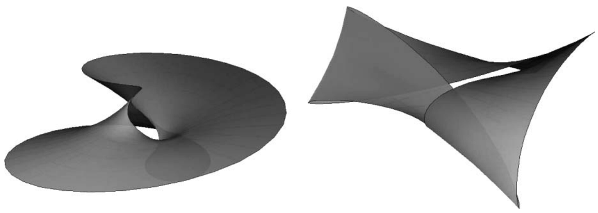

FIGURE 3.2. Maximal Möbius strips. Left: g has a branch point of order two at $z = 0$. Right: g has no branch points at the ends.

REMARK 3.3. If we set $g = z^2$ and $\phi_3 = i(z^2 - 1)z^{-2}dz$ on $\boldsymbol{C} - \{0\}$, we obtain a Henneberg-type maximal immersion $X' : \boldsymbol{RP}^2 - \{\pi_0(0)\} \to \boldsymbol{L}^3$ with singularities (see [ACM]). This X' is complete and has branch points at $z = \pm 1$, so it is not a maxface. See Figure 3.1.

Assume now that g has no branch points at the ends. As before, up to changes of coordinates and rotations around the x_3-axis, we may set

$$g = z\frac{(rz-1)(sz-1)(tz-1)}{(z+r)(z+\bar{s})(z+\bar{t})}$$

and

$$\phi_3 = i\frac{(rz-1)(z+r)(sz-1)(z+\bar{s})(tz-1)(z+\bar{t})}{z^4}dz$$

for some positive real constant r and constants $s, t \in \boldsymbol{C} - \{0\}$. Take a loop γ around $z = 0$. Then direct calculation gives that

$$\int_\gamma g\phi_3 + \overline{\int_\gamma \frac{\phi_3}{g}} = -4\pi(r^2 + s^2 + t^2 + 4rs + 4st + 4tr),$$

$$\frac{1}{2\pi}\int_\gamma \phi_3 = (r^2-1)\{(|s|^2-1)(|t|^2-1) - s\bar{t} - \bar{s}t\}$$
$$- r\{(|s|^2-1)(t+\bar{t}) + (|t|^2-1)(s+\bar{s})\}\,.$$

The arising moduli space of maxfaces is parameterized by the real analytic set of solutions of this system. For instance, the choice $r = 1$, $s = e^{2\pi i/3}$ and $t = e^{-2\pi i/3}$ provides a surface in this family with high symmetry. See the right-hand side of Figure 3.2.

4. Maximal Klein bottles with one end. In this section we construct complete maxfaces with the topology of a Klein bottle minus one point and the lowest Gauss map degree. Consider the genus one algebraic curve

$$\overline{M}_r = \left\{(z, w_r) \in \overline{\boldsymbol{C}}^2 \,;\, w_r^2 = z\frac{rz-1}{z+r}\right\}, \quad r \in \boldsymbol{R} - \{0\}\,,$$

and set $M_r = \overline{M}_r - \{(0,0), (\infty,\infty)\}$. Define

$$I_r : \overline{M}_r \longrightarrow \overline{M}_r\,, \quad I_r(z, w_r) = \left(-\frac{1}{\bar{z}}, -\frac{1}{\bar{w}_r}\right),$$

$$g_r = w_r\frac{z+1}{z-1}\,, \quad \phi_3 = i\frac{z^2-1}{z^2}dz\,,$$

and note that I_r has no fixed points, and g_r and ϕ_3 satisfy (1.6) and (2.1). See Table 4.1.

TABLE 4.1. The Divisors of the Weierstrass data.

(z, w_r)	$(-r,\infty)$	$(0,0)$	$(r^{-1},0)$	(∞,∞)	$(1,*)$	$(-1,*)$
g_r	∞^1	0^1	0^1	∞^1	∞^1	0^1
$g_r\phi_3$	—	∞^2	0^2	∞^4	—	0^2
ϕ_3	0^1	∞^3	0^1	∞^3	0^1	0^1
ϕ_3/g_r	0^2	∞^4	—	∞^2	0^2	—

THEOREM 4.1 (Existence). *There are exactly two real values $r_1, r_2 \in \boldsymbol{R} - \{0\}$ for which the maxface*

$$X_r : M_r \ni p \mapsto \operatorname{Re}\int^p \left(\frac{i}{2}\left(\frac{1}{g_r} - g_r\right), \frac{1}{2}\left(\frac{1}{g_r} + g_r\right), 1\right)\phi_3 \in \boldsymbol{L}^3$$

is well-defined and induces a one-ended maximal Klein bottle $X'_r : M_r/\langle I_r\rangle \to \boldsymbol{L}^3$.

Furthermore, the maxfaces X'_{r_1} and X'_{r_2} have Gauss map of degree four and four symmetries.

PROOF. In order to solve the arising period problem, we first observe that $\phi_3 = d(i(z^2+1)/z)$ is exact and (1.8) is satisfied. Moreover, $\phi_{1,r} = (i/2)(1/g_r - g_r)\phi_3$ and $\phi_{2,r} = (1/2)(1/g_r + g_r)\phi_3$ have no residues at the ends, hence it remains to check (1.7) for $\gamma \in H_1(\overline{M}_r, \boldsymbol{Z})$. Let c_1 and c_2 be two loops in $\boldsymbol{C} - \{0, -r, 1/r\}$ winding once around $[-r, 0]$

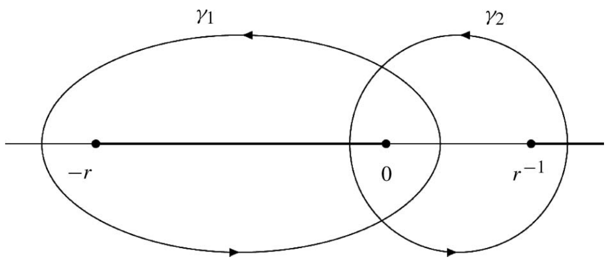

FIGURE 4.1. Projection to the z-plane of the loops γ_1 and γ_2.

and $[0, r^{-1}]$, respectively, and call γ_1 and γ_2 as their corresponding liftings via z to $\overline{M}_r$ (see Figure 4.1).

Let $(I_r)_* : H_1(\overline{M}_r, \boldsymbol{Z}) \to H_1(\overline{M}_r, \boldsymbol{Z})$ denote the group isomorphism induced by I_r. A straightforward computation gives that

$$(I_r)_*(\gamma_1) = -\gamma_1 \quad \text{and} \quad (I_r)_*(\gamma_2) = \gamma_2 . \tag{4.1}$$

For any $j, k \in \{1, 2\}$, we have

$$\int_{\gamma_j} \phi_{k,r} = \int_{(I_r)_*(\gamma_j)} I_r^*(\phi_{k,r}) = \int_{(I_r)_*(\gamma_j)} \overline{\phi_{k,r}}$$

and so

$$\int_{\gamma_j} \phi_{k,r} + \int_{\gamma_j} \overline{\phi_{k,r}} = \int_{(I_r)_*(\gamma_j)} \overline{\phi_{k,r}} + \int_{\gamma_j} \overline{\phi_{k,r}} .$$

Thus

$$2\,\mathrm{Re} \int_{\gamma_j} \phi_{k,r} = \int_{\gamma_j + (I_r)_*(\gamma_j)} \overline{\phi_{k,r}} = \int_{\gamma_j + (I_r)_*(\gamma_j)} \phi_{k,r} ,$$

and $X_r = \mathrm{Re} \int (\phi_{1,r}, \phi_{2,r}, \phi_3) : M_r \to \boldsymbol{L}^3$ is well-defined on M_r if and only if

$$\int_{\gamma_j + (I_r)_*(\gamma_j)} \phi_{k,r} = 0 \tag{4.2}$$

for all $j, k \in \{1, 2\}$.

LEMMA 4.2. *$X_r : M_r \to \boldsymbol{L}^3$ is well-defined on M_r if and only if*

$$\int_{\gamma_2} \frac{w_r (z+1)^2}{z^2} dz = 0 . \tag{4.3}$$

PROOF. By (4.1) and (4.2), X_r is well-defined if and only if

$$\int_{\gamma_2 + (I_r)_*(\gamma_2)} \phi_{k,r} = 0$$

holds for $k = 1, 2$. In other words, X_r is well-defined if and only if

$$\int_{\gamma_2} \left(\frac{1}{g_r} + g_r\right)\phi_3 = \int_{\gamma_2} \left(\frac{1}{g_r} - g_r\right)\phi_3 = 0$$

holds, that is to say,

$$\int_{\gamma_2} \frac{\phi_3}{g_r} = \int_{\gamma_2} g_r\phi_3 = 0$$

holds. However,

$$\int_{\gamma_2} \frac{\phi_3}{g_r} = \int_{(I_r)_*(\gamma_2)} I_r^*\left(\frac{\phi_3}{g_r}\right) = \int_{\gamma_2} \overline{g_r\phi_3}\,,$$

hence X_r is well-defined on M_r if and only if

$$\int_{\gamma_2} g_r\phi_3 = \int_{\gamma_2} \frac{w_r(z+1)^2}{z^2}dz = 0\,. \qquad \square$$

The period problem is equivalent to solve (4.3). To avoid divergent integrals we add the exact one-form dF, where

$$F = \frac{2w_r(z - 2r^3z^2 + r^2z(1+2z) - r(-1+2z+z^2))}{rz}\,,$$

getting

$$\frac{w_r(z+1)^2}{z^2}dz + dF = -\frac{2w_r(-1+z+r(2-3z+r(-4+4r+3z)))}{r+z}dz\,.$$

Since the right-hand side is a holomorphic differential on $M_r - \{(-r, \infty)\}$, the loop γ_2 can be collapsed over the interval $[0, r^{-1}]$ by Stokes theorem and X_r is well-defined if and only if

$$h(r) := \int_0^{r^{-1}} -\frac{2|w_r(z)|(-1+z+r(2-3z+r(-4+4r+3z)))}{r+z}dz = 0\,.$$

A straightforward computation gives that

$$h_+(0) := \lim_{r\to 0,\, r>0} h(r) = -\infty,\ h(+\infty) := \lim_{r\to+\infty} h(r) = -\pi\,,$$

$$h_-(0) := \lim_{r\to 0,\, r<0} h(r) = +\infty,\ h(-\infty) := \lim_{r\to-\infty} h(r) = +\pi\,.$$

Moreover,

$$h(1/2) = \int_0^2 \frac{2|w_{1/2}(z)|(2-z)}{1+2z}dz > 0 \quad \text{and} \quad h(1) = -\frac{4\Gamma(3/4)^2 + \Gamma(-3/4)\Gamma(5/4)}{\sqrt{2\pi}} < 0\,,$$

where Γ is the classical Gamma function. As a consequence, h has at least two roots in $(0, 1)$ (and X_r is well-defined at least for these two real values).

Let us show that h has exactly two real roots on $\boldsymbol{R} - \{0, 1\}$ (recall that $h(1) < 0$).

It is clear that

$$h'(r) = \frac{1}{2}\int_{\gamma_2} \frac{\partial}{\partial r}\left(\frac{w_r(z+1)^2}{z^2}\right) dz\,,$$

hence a direct computation gives that

$$h'(r) = \int_0^{r^{-1}} \frac{|w_r(z)|(1+z)^2(1+z^2)}{2z^2(r+z)(-1+rz)} dz\,. \tag{4.4}$$

Moreover,

$$\frac{w_r(1+z)^2(1+z^2)}{2z^2(r+z)(-1+rz)} dz + dH = -\frac{2w_r(-r+4r^2-z+3rz)}{r(r+z)} dz\,,$$

where

$$H = -\frac{w_r(r+2z-2rz-rz^2+4r^2z^2)}{r^2 z}\,.$$

Integrating by parts, we deduce that

$$h'(r) = \int_0^{r^{-1}} -\frac{2|w_r(z)|(-r+4r^2-z+3rz)}{r(r+z)} dz\,.$$

Now we rewrite $h(r)$ and $h'(r)$ as follows:

$$\begin{aligned} h(r) &= -2((3r^2-3r+1)A_1(r) + (r-1)(r^2+1)A_2(r))\,, \\ h'(r) &= -2\left(\frac{3r-1}{r}A_1(r) + rA_2(r)\right), \end{aligned}$$

where $A_i : \boldsymbol{R} - \{0\} \to \boldsymbol{R}_+$ $(i = 1, 2)$ are the positive functions given by

$$A_1(r) = \int_0^{r^{-1}} |w_r(z)| dz \quad \text{and} \quad A_2(r) = \int_0^{r^{-1}} \frac{|w_r(z)|}{z+r} dz\,.$$

If $h(r_0) = 0$, then

$$A_2(r_0) = -\frac{3r_0^2 - 3r_0 + 1}{(r_0-1)(r_0^2+1)} A_1(r_0)\,,$$

hence necessarily $r_0 < 1$. Therefore $h(r_0) = 0$ implies that

$$h'(r_0) = -2\left(\frac{3r_0-1}{r_0} - \frac{r(3r_0^2-3r_0+1)}{(r_0-1)(r_0^2+1)}\right) A_1(r_0) = q(r_0)\int_0^{r_0^{-1}} |w_{r_0}(z)| dz\,,$$

where $q : \boldsymbol{R} - \{0, 1\} \to \boldsymbol{R}$ is the rational function

$$q(r) = \frac{2(r^3-3r^2+4r-1)}{r(r-1)(r^2+1)}\,.$$

Basic algebra says that

$$s = 1 - \left(\frac{2}{3(-9+\sqrt{93})}\right)^{1/3} + \left(\frac{-9+\sqrt{93}}{18}\right)^{1/3} \approx 0.317672$$

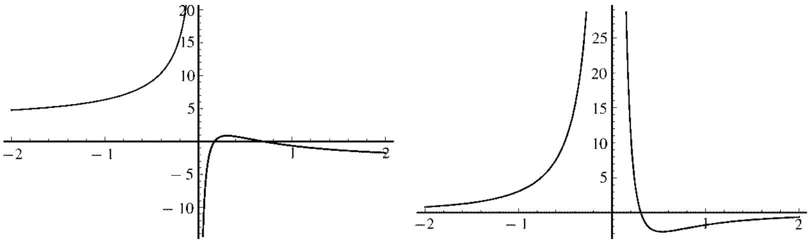

FIGURE 4.2. Left: The period function $h(r)$. $h(r) = 0$ when $r \approx 0.17137$ and $r \approx 0.691724$. Right: The derivative $h'(r)$ of $h(r)$.

is the unique real root of q in $\boldsymbol{R} - \{0, 1\}$, and an elementary analysis says that $q|_{(-\infty,0)} < 0$, $q|_{(0,s)} > 0$ and $q|_{(s,1)} < 0$.

Assume for a moment that h has a root in $(-\infty, 0)$. Since $h_-(0) = +\infty$ and $h(-\infty) > 0$, we can find $s_0 \in (-\infty, 0)$ such that $h(s_0) = 0$ and $h'(s_0) \geq 0$, contradicting that $q(s_0) < 0$. Therefore, the roots of h (at least two) lie in $A = (0, 1)$. Suppose that h has three real roots on A, and label $r_1 < r_2 < r_3$ as the three smallest real roots of h in A.

Since $h_+(0) = -\infty$, h must be increasing on $(r_1 - \varepsilon, r_1)$ for small ε and $h'(r_1) \geq 0$. This implies that $r_1 \leq s$.

Let us show that $r_2 \geq s$. If $r_1 = s$ then $r_2 > s$ and we are done. Suppose $r_1 < s$. In this case $h'(r_1) > 0$ and h must be positive in (r_1, r_2), hence h must be decreasing on $(r_2 - \varepsilon, r_2)$ for small ε and $h'(r_2) \leq 0$. This clearly implies that $r_2 \geq s$.

As a consequence, $r_3 > s$ and $h'(r_3) < 0$, which obviously contradicts that h increasing on $(r_3 - \varepsilon, r_3)$ for small ε and proves our assertion.

This proves that h has exactly two real roots r_1 and r_2 lying in $(0, 1)$.

Finally, observe that the transformations $T_0(z, w_r) = (z, -w_r)$, $T_1(z, w_r) = (\overline{z}, \overline{w_r})$ and $T_2 = T_1 \circ T_0$ on $\overline{M}_r$ induce the 180°-rotations about the x_3, x_1 and x_2 axes, respectively. This implies that the maxface X_r has four symmetries. □

The values r_1 and r_2 can be estimated using the Mathematica software, obtaining that $r_1 \approx 0.17137$ and $r_2 \approx 0.691724$. See Figure 4.2.

REMARK 4.3. The above argument is based on the construction of the López' minimal Klein bottle [L1]. The most significant difference is that in the Riemannian case the period problem has a unique solution.

The maximal Klein bottles exhibited in Theorem 4.1 can be characterized in terms of their symmetry:

THEOREM 4.4 (Uniqueness). *Let $X' : M' \to \boldsymbol{L}^3$ be a complete nonorientable maxface with genus two, one end and Gauss map of degree four. Assume that X' has at least four symmetries. Then X' is one of the examples constructed in Theorem* 4.1.

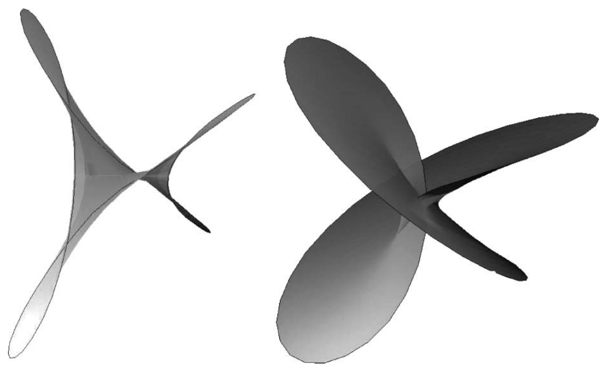

FIGURE 4.3. Maximal Klein Bottles with one end. $r \approx 0.17137$ in the left, and $r \approx 0.691724$ in the right.

PROOF. By definition, an intrinsic isometry $S : M' \to M'$ is said to be a symmetry of X' if there exists a Lorentzian isometry $\widetilde{S} : \boldsymbol{L}^3 \to \boldsymbol{L}^3$ such that $X \circ S = \widetilde{S} \circ X$. Symmetries of X' are conformal transformations and extend conformally to the compactification $\overline{M}'$ of M'. We call $\mathrm{Sym}(X')$ as the symmetry group of X'.

Let (M, I, g, ϕ_3) denote the Weierstrass data of $X' : M' \to \boldsymbol{L}^3$, and up to a Lorentzian isometry, suppose that $g(P) = 1/g(I(P)) = 0$. We know that $M = \overline{M} - \{P, I(P)\}$, where $\overline{M}$ is a conformal torus and $P \in \overline{M}$. As usual, label $\pi : \overline{M} \to \overline{M}'$ as the two sheeted orientable double cover of $\overline{M}'$ and $X = X' \circ \pi : M \to \boldsymbol{L}^3$ as the associated orientable maxface. For each $S \in \mathrm{Sym}(X')$, let $\hat{S} : \overline{M} \to \overline{M}$ denote the unique holomorphic lifting of S, that is to say, the unique orientation preserving transformation in $\overline{M}$ satisfying that $\pi \circ \hat{S} = S \circ \pi$. Obviously $\hat{S} \circ I = I \circ \hat{S}$. Write $\mathrm{Sym}_+(X) = \{\hat{S}\,;\ S \in \mathrm{Sym}(X')\}$ and observe that $\mathrm{Sym}_+(X)$ is a group isomorphic to $\mathrm{Sym}(X')$. Note that $\hat{S} \in \mathrm{Sym}_+(X)$ satisfies $\hat{S}(P) = P$ or $\hat{S}(P) = I(P)$.

Take an arbitrary $S \in \mathrm{Sym}(X')$, and let us show that $S^2 = \mathrm{Id}$.

Indeed, since $\{S^m\,;\ m \in \boldsymbol{Z}\}$ is a discrete group, there is $n \in \boldsymbol{N}$ such that $S^n = \mathrm{Id}$ and $S^j \neq \mathrm{Id}$, $j = 1, \dots, n-1$. Consider the orbit space $\overline{M}'/\langle S\rangle$ and the projection $\sigma : \overline{M}' \to \overline{M}'/\langle S\rangle$. By Riemann-Hurwitz formula, $0 = \chi(\overline{M}') = n\chi(\overline{M}'/\langle S\rangle) - V_S$, where V_S is the total branching number of σ. Since $S(\pi(P)) = \pi(P)$, we get $V_S \geq n-1$ and $0 \leq n\chi(\overline{M'}/\langle S\rangle) - n + 1$. This implies that $\chi(\overline{M'}/\langle S\rangle) = 1$ and $V_S = n$. Therefore, there exists $Q \in \overline{M'}$ and a divisor k of n such that $n - k = 1$. This is only possible when $n = k + 1 = 2$, proving our assertion.

As a consequence, $T^2 = \mathrm{Id}$ for all $T \in \mathrm{Sym}_+(X)$. Moreover, up to a rotation about the x_3-axis, $g \circ T \in \{\pm g, 1/g\}$ and $T^*(\phi_3) = \pm\phi_3$ for any $T \in \mathrm{Sym}_+(X)$. To check this, just take into account that $g \circ T = L \circ g$, where L is the Möbius transformation induced by the linear part of T (here we are identifying $\overline{\boldsymbol{C}} - \{|z| = 1\}$ with the Lorentzian sphere of radius -1 via the Lorentzian stereographic projection). The normalization $g(P) = 1/g(I(P)) = 0$ and the fact $T^2 = \mathrm{Id}$ show that $g \circ T \in \{\pm g, \theta/g\}$, $|\theta| = 1$, and so the desired statement.

Let us show that there exists $T_0 \in \mathrm{Sym}_+(X)$, $T_0 \neq \mathrm{Id}$, satisfying that $T_0(P) = P$. Indeed, since $\#\mathrm{Sym}_+(X) \geq 4$, we can find $T_1, T_2 \in \mathrm{Sym}_+(X) - \{\mathrm{Id}\}$ with $T_1 \neq T_2$. If $T_1(P) = T_2(P) = I(P)$ (otherwise we are done), it suffices to take $T_0 = T_1 \circ T_2$.

Consider a such T_0, and note that $T_0(I(P)) = I(P)$ as well, that is to say, T_0 has at least two fixed points. By the Riemann-Hurwitz formula

$$0 = \chi(\overline{M}) = 2\chi(\overline{M}/\langle T_0\rangle) - V \geq 2(\chi(\overline{M}/\langle T_0\rangle) - 1)\,,$$

where V is the number of fixed points of T_0. This clearly implies that $\chi(\overline{M}/\langle T_0\rangle) = 2$ and $V = 4$. In other words, $\chi(\overline{M}/\langle T_0\rangle) = \overline{\boldsymbol{C}}$ and T_0 has in fact four fixed points, namely $\{P, I(P), Q, I(Q)\}$.

Let $z : \overline{M} \to \overline{\boldsymbol{C}} \equiv \overline{M}/\langle T_0\rangle$ denote the natural two sheeted branched covering. Up to a conformal transformation, we will suppose that $z(P) = 1/z(I(P)) = 0$ and $r = z(Q) \in \boldsymbol{R} - \{0\}$. We infer that $z \circ I = \mu/\overline{z}$, and since I is an involution, then $\mu \in \boldsymbol{R} - \{0\}$. Up to the change $z \to \sqrt{|\mu|}z$, we can put $\mu^2 = 1$. We distinguish two cases: $z \circ I = 1/\overline{z}$ and $z \circ I = -1/\overline{z}$.

Case 1. $z \circ I = 1/\overline{z}$. Up to biholomorphisms, $\overline{M} = \{(z, v) \in \overline{\boldsymbol{C}}^2\,;\ v^2 = z(z - r)(rz - 1)\}$ and $T_0(z, v) = (z, -v)$. As $T_0 \circ I = I \circ T_0$ and I has no fixed points, we get $I(z, v) = (1/\overline{z}, \overline{v}/\overline{z}^2)$. Consider $T_1 \in \mathrm{Sym}_+(X) - \{\mathrm{Id}, T_0\}$ and note that $T_1(P) = I(P)$ (otherwise T_1 would be an holomorphic involution fixing P and $I(P)$, hence $T_1 = T_0$ which is absurd). Thus we get that $z \circ T_1 = \lambda/z$, and since T_1 leaves invariant the branch point set of z, $\lambda = 1$.

Let us determine g. Basic Algebraic Geometry says that g is a rational function of z and v. Moreover, we know that $g \circ I = 1/\overline{g}$ and $g \circ T_0 = \pm g$ (recall that $T_0(P) = P$ and so $(g \circ T_0)(P) = 0$).

Suppose for a moment that $g \circ T_0 = g$. In this case, $g = R(z)$ where $R(z)$ is a rational function of z. Up to rotations about the x_3-axis, it is easy to get $g = z(z - a)/(\overline{a}z - 1)$, $a \in \boldsymbol{C}$. Here we have taken into account that g has degree four, $g(0) = 0$ and $g \circ I = 1/\overline{g}$. Then the conditions (1.4) and $I^*(\phi_3) = \overline{\phi}_3$ imply that $\phi_3 = iA(z - a)(\overline{a}z - 1)(zv)^{-1}dz$, $A \in \boldsymbol{R} - \{0\}$ (up to scaling in $\boldsymbol{L}^3$, we may assume $A \in \{\pm 1\}$). Furthermore, $g \circ T_1 = \pm 1/g$ forces $a \in \boldsymbol{R}$. Let $\gamma \in H_1(\overline{M}, \boldsymbol{Z})$ denote the loop $z^{-1}([r, 1/r])$ and observe that $I_*(\gamma) = \gamma$, where $I_* : H_1(\overline{M}, \boldsymbol{Z}) \to H_1(\overline{M}, \boldsymbol{Z})$ is the isomorphism induced by I. By the same argument as in Lemma 4.2, X' is well defined if and only if

$$\int_\gamma \phi_3 g = 0\,.$$

However, $\phi_3 g = iA(z-a)^2 v^{-1}dz$ has non zero integral along $[r, 1/r]$, getting a contradiction.

Assume now that $g \circ T_0 = -g$. Then $g = R(z)v$, where $R(z)$ is a rational function of z. By reasoning as above, we get either

$$g = \frac{v(z - a)}{(z - r)(az - 1)} \quad \text{or} \quad g = \frac{v(z - a)}{(rz + 1)(az - 1)}\,,$$

and in any case $\phi_3 = i(z-a)(az-1)z^{-2}dz$, where $a \in \boldsymbol{R} - \{0, 1/r\}$. Since ϕ_3 has no real periods, its residue at $z = 0$ must be real, that is to say, $1 + a^2 = 0$, a contradiction.

Therefore, this case is impossible.

Case 2. $z \circ I = -1/\bar{z}$. By reasoning as above, we get $\overline{M} = \{(z, v) \in \overline{\boldsymbol{C}}^2 ;\ v^2 = z(z+r)(rz-1)\}$, $r \in \boldsymbol{R} - \{0\}$, $I(z, v) = (-1/\bar{z}, \pm\bar{v}/\bar{z}^2)$, $T_0(z, v) = (z, -v)$ and $T_1(z, v) = (-1/z, \pm v/z^2)$.

Suppose that $g \circ T_0 = g$ and $g = R(z)$, where $R(z)$ is a rational function of z. Up to a rotation about the x_3-axis, we get $g = z(z-a)/(az+1)$, $\phi_3 = A(z-a)(az+1)(zv)^{-1}dz$, $a \in \boldsymbol{R}$, $A \in \{\pm 1, \pm i\}$. Consider the interval $J \subset \boldsymbol{R}$ with endpoints in $\{0, -r, 1/r\}$ and such that $I_*(\gamma) = \gamma$, where $\gamma = z^{-1}(J)$. By reasoning as above, we get

$$\int_\gamma \phi_3 g \neq 0 ,$$

contradicting the period condition.

Assume now that $g \circ T_0 = -g$. As above, either

$$g = \frac{v(z+a)}{(z+r)(az-1)} \quad \text{or} \quad g = \frac{v(z+a)}{(rz-1)(az-1)} .$$

Up to relabeling $r = z(I(Q))$, we can deal only with the first case

$$g = \frac{v(z+a)}{(z+r)(az-1)} .$$

Then $\phi_3 = i(z-a)(az+1)z^{-2}dz$, where $a \in \boldsymbol{R} - \{r\}$. Moreover, the condition $g \circ I = 1/\bar{g}$ forces that $I(z, v) = (-1/\bar{z}, -\bar{v}/\bar{z}^2)$. Since ϕ_3 has no real periods, its residue at $z = 0$ vanishes and $a^2 = 1$ (up to the changes $z \to -z$ and $r \to -r$, we can put $a = 1$). These Weierstrass data correspond to the examples in Theorem 4.1, concluding the proof. $\square$

Theorem B in the introduction follows from Theorems 4.1 and 4.4.

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