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A TRANSFORMATION FORMULA FOR APPELL'S HYPERGEOMETRIC FUNCTION F_1 AND COMMON LIMITS OF TRIPLE SEQUENCES BY MEAN ITERATIONS

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Abstract. In this paper, we give a transformation formula for Appell's hypergeometric function F_1. As applications of this formula, we show that some common limits of triple sequences given by mean iterations of 3-terms can be expressed by F_1.

Introduction. It is known that the hypergeometric function

$$F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha, n)(\beta, n)}{(\gamma, n)(1, n)} z^n$$

satisfies the Gauss quadratic transformation formula:

$$(1+z)^{2\alpha} F\left(\alpha, \alpha - \beta + \frac{1}{2}, \beta + \frac{1}{2}; z^2\right) = F\left(\alpha, \beta, 2\beta; \frac{4z}{(1+z)^2}\right).$$

By substituting $b/a = (1-z)/(1+z)$, $\alpha = \beta = 1/2$ into this equality, we have

$$\frac{(a+b)/2}{F(1/2, 1/2, 1; 1 - (2\sqrt{ab}/(a+b))^2)} = \frac{a}{F(1/2, 1/2, 1; 1 - b^2/a^2)},$$

which means that $a/F(1/2, 1/2, 1; 1-b^2/a^2)$ is invariant under $(a, b) \mapsto ((a+b)/2, \sqrt{ab})$. This invariance implies that $a/F(1/2, 1/2, 1; 1 - b^2/a^2)$ coincides with the arithmetic-geometric mean of a and b. By using Goursat's list of transformation formulas in [3], we give a table of double sequences by mean iterations and expressions of their common limits by the hypergeometric function in [4]. It is shown in [7], [8] and [6] that transformation formulas of hypergeometric functions of multi variables imply expressions of common limits of multiple sequences by mean iterations. And these transformation formulas are extended to ones with a parameter in [9].

In this paper, we give a transformation formula for Appell's hypergeometric function $F_1(\alpha, \beta_1, \beta_2, \gamma; z_1, z_2)$ of two variables z_1, z_2 in Theorem 1.1. As applications of Theorem 1.1, we show that some common limits of triple sequences given by mean iterations of 3-terms

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can be expressed by F_1. Let (a_n, b_n, c_n) be a triple sequence with initial (a, b, c) given by the mean iteration of 3-terms:

$$(a_{n+1}, b_{n+1}, c_{n+1}) = \left(\frac{\sqrt{a_n}(\sqrt{b_n}+\sqrt{c_n})}{2}, \frac{\sqrt{b_n}(\sqrt{c_n}+\sqrt{a_n})}{2}, \frac{\sqrt{c_n}(\sqrt{a_n}+\sqrt{b_n})}{2} \right).$$

Theorem 2.2 states that its common limit can be expressed as

$$\frac{a}{F_1(1, 1/2, 1/2, 3/2; 1 - b/a, 1 - c/a)}.$$

For the case $b = c$, the triple sequence (a_n, b_n, c_n) reduces to a double sequence with initial (a, b) given as

$$(a_{n+1}, b_{n+1}) = \left(\sqrt{a_n b_n}, \frac{\sqrt{b_n}(\sqrt{a_n} + \sqrt{b_n})}{2} \right).$$

It is studied in [1], [2] and [4] that its common limit can be expressed as $a/F(1, 1, 3/2; 1 - b/a)$.

We also express common limits of modified triple sequences (a'_n, b'_n, c'_n) in Theorem 2.4.

1. Transformation formula. Appell's hypergeometric function F_1 of 2-variables z_1, z_2 with parameters $\alpha, \beta_1, \beta_2, \gamma$ is defined as

$$F_1(\alpha, \beta_1, \beta_2, \gamma; z) = \sum_{n_1, n_2 = 0}^{\infty} \frac{(\alpha, n_1 + n_2)(\beta_1, n_1)(\beta_2, n_2)}{(\gamma, n_1 + n_2)(1, n_1)(1, n_2)} z_1^{n_1} z_2^{n_2},$$

where $z = (z_1, z_2)$ satisfies $|z_j| < 1$ $(j = 1, 2)$, $\gamma \neq 0, -1, -2, \dots$ and $(\alpha, n) = \alpha(\alpha + 1) \cdots (\alpha + n - 1) = \Gamma(\alpha + n)/\Gamma(\alpha)$. This function admits an integral representation of Euler type:

$$\begin{aligned} &F_1(\alpha, \beta_1, \beta_2, \gamma; z) \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma - \alpha)} \int_0^1 t^{\alpha}(1 - t)^{\gamma - \alpha}(1 - z_1 t)^{-\beta_1}(1 - z_2 t)^{-\beta_2} \frac{dt}{t(1 - t)}. \end{aligned} \tag{1}$$

For properties of Appell's hypergeometric function F_1, refer to [5] and [10].

THEOREM 1.1. *We have a transformation formula for* F_1:

$$\begin{aligned} &(z_1 z_2)^{(1-p)/2} \left(\frac{z_1 + z_2}{2} \right)^p F_1\left(\frac{3+p}{4}, \frac{1+p}{4}, \frac{1+p}{4}, \frac{3+3p}{4}; 1 - z_1^2, 1 - z_2^2 \right) \\ &= F_1\left(p, \frac{1+p}{4}, \frac{1+p}{4}, \frac{3+3p}{4}; 1 - \frac{z_1(1+z_2)}{z_1 + z_2}, 1 - \frac{z_2(1+z_1)}{z_1 + z_2} \right), \end{aligned} \tag{2}$$

where (z_1, z_2) *is in a small neighbourhood of* $(1, 1)$ *and the values of* $(z_1 z_2)^{(1-p)/2}$ *and* $((z_1 + z_2)/2)^p$ *at* $(z_1, z_2) = (1, 1)$ *are* 1.

PROOF. Consider the following vector-valued functions

$${}^t\left(F_0, \frac{\partial F_0}{\partial z_1}, \frac{\partial F_0}{\partial z_2} \right), \quad {}^t\left(G_0, \frac{\partial G_0}{\partial z_1}, \frac{\partial G_0}{\partial z_2} \right),$$

where $F_0(z_1, z_2)$ and $G_0(z_1, z_2)$ are the left- and right-hand sides of (2), respectively. Each of them takes the value ${}^t(1, -p/6, -p/6)$ at $(z_1, z_2) = (1, 1)$ and satisfies an integrable Pfaffian system

$$dF(z) = (\Omega_1 dz_1 + \Omega_2 dz_2)F(z)\,,$$

where Ω_1 and Ω_2 are

$$\begin{pmatrix} 0 & 1 & 0 \\ \dfrac{p(1+p)z_2(1+z_1z_2)}{2z_1(1-z_1^2)(z_1+z_2)^2} & \dfrac{(1+p)((2z_1^2-1)(2z_1^2-z_2^2)-z_1^2z_2^2)}{2z_1(1-z_1^2)(z_1^2-z_2^2)}+\dfrac{2p}{z_1+z_2} & \dfrac{(1+p)z_2(1-z_2^2)}{2(1-z_1^2)(z_1^2-z_2^2)} \\ \dfrac{-p(1+p)}{2(z_1+z_2)^2} & \dfrac{z_1((1-p)z_1+2pz_2)}{2z_2(z_1^2-z_2^2)} & \dfrac{-z_2((1-p)z_2+2pz_1)}{2z_1(z_1^2-z_2^2)} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 & 1 \\ \dfrac{-p(1+p)}{2(z_1+z_2)^2} & \dfrac{z_1((1-p)z_1+2pz_2)}{2z_2(z_1^2-z_2^2)} & \dfrac{-z_2((1-p)z_2+2pz_1)}{2z_1(z_1^2-z_2^2)} \\ \dfrac{p(1+p)z_1(1+z_1z_2)}{2z_2(1-z_2^2)(z_1+z_2)^2} & \dfrac{-(1+p)z_1(1-z_1^2)}{2(1-z_2^2)(z_1^2-z_2^2)} & \dfrac{-(1+p)((2z_2^2-1)(2z_2^2-z_1^2)-z_1^2z_2^2)}{2z_2(1-z_2^2)(z_1^2-z_2^2)}+\dfrac{2p}{z_1+z_2} \end{pmatrix},$$

respectively. Thus we have $F_0(z_1, z_2) = G_0(z_1, z_2)$. For a way to get the connection matrix $\Omega_1 dz_1 + \Omega_2 dz_2$, refer to the proof of Proposition 1 in [6] and Section 4 in [9]. □

By putting $p = 1$ for the equality (2) in Theorem 1.1, we have the following.

COROLLARY 1.2. *For* (z_1, z_2) *in a small neighbourhood of* $(1, 1)$, *we have*

$$\begin{aligned} &\frac{z_1+z_2}{2}F_1\left(1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1-z_1^2, 1-z_2^2\right) \\ &= F_1\left(1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1-\frac{z_1(1+z_2)}{z_1+z_2}, 1-\frac{z_2(1+z_1)}{z_1+z_2}\right). \end{aligned}$$

2. Common limits of triple sequences. Let $\boldsymbol{R}_+^*$ be the multiplicative group of positive real numbers. We define a map $m : (\boldsymbol{R}_+^*)^3 \to (\boldsymbol{R}_+^*)^3$ by

$$\begin{aligned} m(x_1, x_2, x_3) &= (m_1(x_1, x_2, x_3), m_2(x_1, x_2, x_3), m_3(x_1, x_2, x_3)) \\ &= \left(\frac{\sqrt{x_1}(\sqrt{x_2}+\sqrt{x_3})}{2}, \frac{\sqrt{x_2}(\sqrt{x_3}+\sqrt{x_1})}{2}, \frac{\sqrt{x_3}(\sqrt{x_1}+\sqrt{x_2})}{2}\right). \end{aligned}$$

A triple sequence (a_n, b_n, c_n) is given by $(a_0, b_0, c_0) = (a, b, c)$, $a \geq b \geq c \geq 0$,

$$(a_{n+1}, b_{n+1}, c_{n+1}) = m(a_n, b_n, c_n)\,. \tag{3}$$

LEMMA 2.1. *The sequences* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *converge and satisfy*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n .$$

PROOF. If $a_n \geq b_n \geq c_n$ then

$$a_n - a_{n+1} = \frac{\sqrt{a_n}(\sqrt{a_n} - \sqrt{b_n} + \sqrt{a_n} - \sqrt{c_n})}{2} \geq 0 ,$$

$$a_{n+1} - b_{n+1} = \frac{\sqrt{c_n}(\sqrt{a_n} - \sqrt{b_n})}{2} \geq 0 , \quad b_{n+1} - c_{n+1} = \frac{\sqrt{b_n}(\sqrt{a_n} - \sqrt{b_n})}{2} \geq 0 ,$$

$$c_{n+1} - c_n = \frac{\sqrt{c_n}(\sqrt{a_n} - \sqrt{c_n} + \sqrt{b_n} - \sqrt{c_n})}{2} \geq 0 .$$

Thus we have

$$a \geq a_n \geq a_{n+1} \geq b_{n+1} \geq c_{n+1} \geq c_n \geq c .$$

Since the sequences $\{a_n\}$ and $\{c_n\}$ are bounded and monotonous, they converge. By

$$a_{n+1} - c_{n+1} = \frac{\sqrt{b_n}(\sqrt{a_n} - \sqrt{c_n})}{2} = \frac{\sqrt{b_n}}{\sqrt{a_n} + \sqrt{c_n}} \frac{a_n - c_n}{2} \leq \frac{1}{2}(a_n - c_n) ,$$

we have $\lim_{n\to\infty}(a_n - c_n) = 0$. Since $a_n \geq b_n \geq c_n$ for any $n \in \boldsymbol{N}$, $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ have a common limit. □

This common limit of the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ is denoted by $\mu(a, b, c)$.

THEOREM 2.2. *The common limit* $\mu(a, b, c)$ *of the triple sequence* (3) *can be expressed as*

$$\mu(a, b, c) = \frac{a}{F_1(1, 1/2, 1/2, 3/2; 1 - b/a, 1 - c/a)} .$$

PROOF. By putting $(z_1, z_2) = (\sqrt{b_n/a_n}, \sqrt{c_n/a_n})$ for Corollary 1.2, we have

$$\frac{\sqrt{b_n} + \sqrt{c_n}}{2\sqrt{a_n}} F\left(\frac{b_n}{a_n}, \frac{c_n}{a_n}\right) = F\left(\frac{\sqrt{b_n}(\sqrt{a_n} + \sqrt{c_n})}{\sqrt{a_n}(\sqrt{b_n} + \sqrt{c_n})}, \frac{\sqrt{c_n}(\sqrt{a_n} + \sqrt{b_n})}{\sqrt{a_n}(\sqrt{b_n} + \sqrt{c_n})}\right)$$

where $F(z_1, z_2)$ denotes $F_1(1, 1/2, 1/2, 3/2, 1 - z_1, 1 - z_2)$. This equality implies

$$\frac{a_n}{F\,(b_n/a_n, c_n/a_n)} = \cdots = \frac{a_1}{F\,(b_1/a_1, c_1/a_1)} = \frac{a_0}{F(b_0/a_0, c_0/a_0)} .$$

Since

$$\lim_{n\to\infty} a_n = \mu(a, b, c) , \quad \lim_{n\to\infty} \left(\frac{b_n}{a_n}, \frac{c_n}{a_n}\right) = (1, 1) , \quad F(1, 1) = 1 ,$$

the sequence $a_n/F(b_n/a_n, c_n/a_n)$ converges to $\mu(a, b, c)$ as $n \to \infty$. □

REMARK 2.3. It is known that the arithmetic-geometric mean of a and b can be expressed by an elliptic integral. The common limit $\mu(a, b, c)$ of the triple sequence (3) can be expressed by an incomplete elliptic integral, since we have

$$F_1\left(1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z_1, z_2\right) = \frac{1}{2}\int_0^1 \frac{dt}{\sqrt{(1-t)(1-z_1 t)(1-z_2 t)}} .$$

Let $m^{(r)}$ be a map from $(\boldsymbol{R}_+^*)^3$ to $(\boldsymbol{R}_+^*)^3$ given by

$$m^{(r)}(x_1, x_2, x_3) = (m_1^{(r)}(x_1, x_2, x_3), m_2^{(r)}(x_1, x_2, x_3), m_3^{(r)}(x_1, x_2, x_3))\,,$$

where $r \in \boldsymbol{R}_+^*$ and

$$m_i^{(r)}(x_1, x_2, x_3) = \sqrt[r]{m_i(x_1^r, x_2^r, x_3^r)}, \quad i=1,2,3\,.$$

We give a triple sequence (a'_n, b'_n, c'_n) by $(a'_0, b'_0, c'_0) = (a, b, c)$, $a \geq b \geq c \geq 0$,

$$(a'_{n+1}, b'_{n+1}, c'_{n+1}) = m^{(r)}(a'_n, b'_n, c'_n)\,. \tag{4}$$

Note that the triple sequence (a'_n, b'_n, c'_n) for $r = 1$ is equal to (a_n, b_n, c_n) in (3) and that (a'_n, b'_n, c'_n) for $r = 2$ is given as

$$(a'_{n+1}, b'_{n+1}, c'_{n+1}) = \left(\sqrt{\frac{a'_n(b'_n+c'_n)}{2}}, \sqrt{\frac{b'_n(c'_n+a'_n)}{2}}, \sqrt{\frac{c'_n(a'_n+b'_n)}{2}}\right).$$

THEOREM 2.4. *The triple sequence (a'_n, b'_n, c'_n) has a common limit. This value $\mu^{(r)}(a, b, c)$ can be expressed as*

$$\mu^{(r)}(a, b, c) = \frac{a}{\sqrt[r]{F_1(1, 1/2, 1/2, 3/2; 1 - b^r/a^r, 1 - c^r/a^r)}}\,.$$

In particular, $\mu^{(r)}(a, b, c)$ for $r = 2$ is given as

$$\mu^{(2)}(a, b, c) = \frac{a}{\sqrt{F_1(1, 1/2, 1/2, 3/2; 1 - b^2/a^2, 1 - c^2/a^2)}}\,.$$

PROOF. By argument similar to the proof of Lemma 2.1, we can easily show that (a'_n, b'_n, c'_n) has a common limit. By putting $(z_1, z_2) = ((b'_n/a'_n)^{r/2}, (c'_n/a'_n)^{r/2})$ for Corollary 1.2, we have

$$\frac{a'_n}{\sqrt[r]{F((b'_n/a'_n)^r, (c'_n/a'_n)^r)}} = \cdots = \frac{a'_1}{\sqrt[r]{F((b'_1/a'_1)^r, (c'_1/a'_1)^r)}} = \frac{a'_0}{\sqrt[r]{F((b'_0/a'_0)^r, (c'_0/a'_0)^r)}}\,.$$

For these equalities, consider the limit as $n \to \infty$. □

COROLLARY 2.5. *We have an infinite product expression*:

$$F_1\left(1, \frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1 - z_1^r, 1 - z_2^r\right) = \prod_{n=0}^{\infty} \left(\frac{a'_n}{a'_{n+1}}\right)^r,$$

where $0 < z_2 \leq z_1 \leq 1$, $r \in \boldsymbol{R}_+^*$, and the triple sequence (a'_n, b'_n, c'_n) is given in (4) with initial $(a, b, c) = (1, z_1, z_2)$.

PROOF. The infinite product $\prod_{n=0}^{\infty}(a'_n/a'_{n+1})$ converges to $a/\mu^{(r)}(a, b, c)$. Theorem 2.4 implies this corollary. □

For the case $b = c$, the triple sequences (a_n, b_n, c_n) and (a'_n, b'_n, c'_n) for $r = 2$ reduce to the double sequences with initial (a, b) given as

$$(a_{n+1}, b_{n+1}) = \left(\sqrt{a_n b_n}, \frac{\sqrt{b_n}(\sqrt{a_n} + \sqrt{b_n})}{2}\right), \quad (a'_{n+1}, b'_{n+1}) = \left(\sqrt{a'_n b'_n}, \sqrt{\frac{b'_n(a'_n + b'_n)}{2}}\right),$$

respectively. It is shown in [4] that their common limits $\mu(a,b)$ and $\mu^{(2)}(a,b)$ can be expressed as $a/F(1,1,3/2;1-b/a)$ and $a/\sqrt{F(1,1,3/2;1-b^2/a^2)}$, respectively. Refer also to [1] and [2]. Note that these expression can be obtained by Theorems 2.2 and 2.4 together with the integral representation (1).

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