Tohoku Math. J.
59 (2007), 423–440

DISCRETE TOMOGRAPHY AND HODGE CYCLES

FUMIO HAZAMA

(Received March 10, 2006, revised March 26, 2007)

Abstract. We study a problem in discrete tomography on the free abelian group of rank n through the theory of distributions on the n-dimensional torus, and show that there is an intimate connection between the problem and the study of the Hodge cycles on abelian varieties of CM-type. This connection enables us to apply our results in tomography to obtain several infinite families of abelian varieties for which the Hodge conjecture hold.

1. Introduction. The purpose of this paper is to generalize the theory developed in [2], which concerns discrete tomography by *hook-shape windows*, in order to investigate tomography by arbitrary windows in $\boldsymbol{Z}^n$, and show that the latter is closely connected with the study of Hodge cycles on abelian varieties with complex multiplication by abelian CM-fields.

We give below a rough description of the problem of our main concern in the case of windows in $\boldsymbol{Z}^2$. Let $\boldsymbol{A} = (\boldsymbol{C})^{\boldsymbol{Z}^2}$ denote the set of $\boldsymbol{C}$-valued functions on $\boldsymbol{Z}^2$. We write its element in the form $(\boldsymbol{a}_{(i,j)})_{(i,j)\in\boldsymbol{Z}^2}$ with $\boldsymbol{a}_{(i,j)} \in \boldsymbol{C}$, and call it simply an *array*. An array with finite support is called a *window*, and the set of windows is denoted by $\boldsymbol{W}$. For any window $\boldsymbol{t} = (\boldsymbol{t}_{(i,j)})$ and for any array $\boldsymbol{a} = (\boldsymbol{a}_{(i,j)})$, let $d_{\boldsymbol{t}}(\boldsymbol{a}) = \sum_{(i,j)\in\boldsymbol{Z}^2} \boldsymbol{t}_{(i,j)}\boldsymbol{a}_{(i,j)}$ and call it the *degree of* $\boldsymbol{a}$ *with respect to* $\boldsymbol{t}$. The main object of our study in this paper is the set

$$\boldsymbol{A}_{\boldsymbol{t}}^0 = \{(\boldsymbol{a}_{(i,j)})_{(i,j)\in\boldsymbol{Z}^2} \in \boldsymbol{A};\ \max\{|\boldsymbol{a}_{(i,j)}|\} < \infty \ \text{ and } \ d_{\boldsymbol{t}+(\alpha,\beta)}(\boldsymbol{a}) = 0 \text{ for any } (\alpha,\beta) \in \boldsymbol{Z}^2\}$$

of bounded arrays of degree zero with respect to every translation of $\boldsymbol{t}$. Inspired by Nivat's works [8, 9], we investigated in [2] the structure of $\boldsymbol{A}_{H_n}^0$ (denoted by $A_{H_n}(0)_{\text{bounded}}$ in the notation there), when the window is the characteristic function of an *n-hook* $H_n = \{(0,0), (1,0), \dots, (n-1,0), (0,1)\} \subset \boldsymbol{Z}^2$. We found in [2] that the theory of distributions provides us with a natural and unified viewpoint for the study of the structure of $\boldsymbol{A}_{H_n}^0$.

The main purpose of the present article is to show that the theory also permits us to understand the structure of $\boldsymbol{A}_{\boldsymbol{t}}^0$ for any window $\boldsymbol{t}$ too. In the course of our study we recognize an important role played by the *characteristic polynomial* $m_{\boldsymbol{t}}(z,w) = \sum_{(i,j)\in\boldsymbol{Z}^2} t_{(i,j)} z^i w^j \in \boldsymbol{Z}[z, z^{-1}, w, w^{-1}]$ of a window $\boldsymbol{t} = (\boldsymbol{t}_{(i,j)}) \in \boldsymbol{W}$ and its *star* $m_{\boldsymbol{t}}^*(z,w) = m_{\boldsymbol{t}}(z^{-1}, w^{-1})$. We will see that the intersection $V(m_{\boldsymbol{t}}^*) \cap \boldsymbol{T}^2$ of the zero locus of $m_{\boldsymbol{t}}^*$ with the self-product of the unit circle $\boldsymbol{T}$ controls the structure of $\boldsymbol{A}_{\boldsymbol{t}}^0$. Furthermore, through the behavior of $V(m_{\boldsymbol{t}}^*)$, we can analyze the structure of $\boldsymbol{A}_{\boldsymbol{t}}^0$ for every window $\boldsymbol{t}$ and obtain a dimension formula for $\boldsymbol{A}_{\boldsymbol{t}}^0$. As an amusing consequence, we can prove in a few lines that every bounded discrete harmonic function on $\boldsymbol{Z}^n$ is constant (Proposition 5.1, Remark 5.1.1 (2)).

2000 *Mathematics Subject Classification.* Primary 39A12; Secondary 14C30.

On the other hand, our study of tomography will reveal unexpectedly a close connection between the structure of $\boldsymbol{A}_{\boldsymbol{t}}^0$ and that of the Hodge rings of abelian varieties with complex multiplication by abelian CM-field. Roughly speaking, given a finite subset $S \subset \boldsymbol{Z}^n$, we construct from S an infinite family $\mathrm{AV}(S)$ of abelian varieties, and relate the structures of the Hodge rings of the abelian varieties in $\mathrm{AV}(S)$ to $V(m_S^*) \cap \boldsymbol{T}^n$. This connection enables us to reduce the study of the Hodge rings to that of the zero locus of the characteristic polynomial. By applying our theory to the simplest finite subset $\boldsymbol{O} = \{(0, \dots, 0)\} \subset \boldsymbol{Z}^n$, for example, we see that every abelian variety in $\mathrm{AV}(\boldsymbol{O})$ is simple and satisfies the Hodge conjecture (Proposition 6.7).

The plan of this paper is as follows. In Section 2, we introduce some notation and formulate the basic problems of our concern. In Section 3 we generalize some results in [2] and obtain a dimension formula for $\boldsymbol{A}_{\boldsymbol{t}}^0$ for an arbitrary window $\boldsymbol{t}$. A crucial role is played by the theory of distributions on $\boldsymbol{T}^n$ and their Fourier transforms. In Section 4 we investigate the periodicity of arrays in $\boldsymbol{A}_{\boldsymbol{t}}^0$ and give a characterization for $\boldsymbol{A}_{\boldsymbol{t}}^0$ to contain a multiply periodic array. Section 5 examines several examples and shows how to apply the general results to investigate concrete examples of windows. In Section 6 we reveal an intimate connection between discrete tomography and the study of the Hodge rings of abelian varieties with complex multiplication by abelian CM-field.

The author would like to take this opportunity to thank Professor Sadao Sato for helpful conversations and suggestions.

2. Problem setting. In this section we introduce some notation and formulate the basic problems of our concern.

Let $\boldsymbol{A} = (\boldsymbol{C})^{\boldsymbol{Z}^n}$ denote the set of $\boldsymbol{C}$-valued functions on $\boldsymbol{Z}^n$. We write its element in the form $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{i}})$ where $\boldsymbol{i} = (i_1, \dots, i_n) \in \boldsymbol{Z}^n$ and $\boldsymbol{a}_{\boldsymbol{i}} \in \boldsymbol{C}$. We call an element of $\boldsymbol{A}$ simply an *array*. When there exists a positive constant C such that $|\boldsymbol{a}_{\boldsymbol{i}}| < C$ for any $\boldsymbol{i} \in \boldsymbol{Z}^n$, the array is said to be *bounded*. We denote the set of bounded arrays by $\boldsymbol{A}^0$. For any array $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{i}})$, let $\operatorname{supp} \boldsymbol{a} = \{\boldsymbol{i} \in \boldsymbol{Z}^n; \boldsymbol{a}_{\boldsymbol{i}} \neq 0\} \subset \boldsymbol{Z}^n$ and call it the *support* of $\boldsymbol{a}$. An array with finite support is called a *window*, and the set of windows is denoted by $\boldsymbol{W}$. For any window $\boldsymbol{t} = (\boldsymbol{t}_{\boldsymbol{i}})$ and for any array $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{i}})$, let $d_{\boldsymbol{t}}(\boldsymbol{a}) = \sum_{\boldsymbol{i} \in \boldsymbol{Z}^n} \boldsymbol{t}_{\boldsymbol{i}} \boldsymbol{a}_{\boldsymbol{i}}$ and call it the *degree of* $\boldsymbol{a}$ *with respect to* $\boldsymbol{t}$. Furthermore, let

$$\boldsymbol{A}_{\boldsymbol{t}}^0 = \{\boldsymbol{a} \in \boldsymbol{A}^0;\ d_{\boldsymbol{t}+\boldsymbol{p}}(\boldsymbol{a}) = 0 \ \text{ for any } \ \boldsymbol{p} \in \boldsymbol{Z}^n\},$$

the set of bounded arrays of degree zero with respect to every translation of $\boldsymbol{t}$. Here the translated window $\boldsymbol{t} + \boldsymbol{p}$ is defined by $(\boldsymbol{t} + \boldsymbol{p})_{\boldsymbol{i}} = \boldsymbol{t}_{\boldsymbol{i}-\boldsymbol{p}}, \boldsymbol{i} \in \boldsymbol{Z}^n$. The main problems we study in this paper are the following:

(2.a) *Find a condition for finite-dimensionality of* $\boldsymbol{A}_{\boldsymbol{t}}^0$.

(2.b) *Find an explicit formula for the dimension of* $\boldsymbol{A}_{\boldsymbol{t}}^0$.

(2.c) *Find a condition under which* $\boldsymbol{A}_{\boldsymbol{t}}^0$ *contains a multiply periodic array.*

3. Dimension formula for $\boldsymbol{A}_{\boldsymbol{t}}^0$. In this section we investigate the problems (2.a) and (2.b) by appealing to the theory of *pseudomeasures* on the n-dimensional torus.

In order to formulate our result, we introduce some notation. For any window $\boldsymbol{t} = (\boldsymbol{t_i}) \in \boldsymbol{W}$, let $m_{\boldsymbol{t}}(z) = \sum_{\boldsymbol{i}\in \boldsymbol{Z}^n} \boldsymbol{t_i} \boldsymbol{z^i} \in \boldsymbol{C}[z_1, z_1^{-1}, \dots, z_n, z_n^1]$, where $\boldsymbol{z} = (z_1, \dots, z_n)$ and $\boldsymbol{z^i} = z_1^{i_1} \cdots z_n^{i_n}$. We call it the *characteristic polynomial* of $\boldsymbol{t}$. Let $\boldsymbol{T} = \{z \in \boldsymbol{C}; |z| = 1\}$ and let $\iota : \boldsymbol{T}^n \to \boldsymbol{T}^n$ denote the automorphism of $\boldsymbol{T}^n$ defined by $\iota(z_1, \dots, z_n) = (z_1^{-1}, \dots, z_n^{-1})$. We put $m_{\boldsymbol{t}}^* = \iota^*(m_{\boldsymbol{t}})$ so that $m_{\boldsymbol{t}}^*(z) = m_{\boldsymbol{t}}(\iota(z))$. For any subset $X \subset \boldsymbol{C}^n$, we denote the zero locus $\{z \in X; m_{\boldsymbol{t}}(z) = 0\}$ by $V_X(m_{\boldsymbol{t}})$. Let $\boldsymbol{P}$ denote the set of pseudomeasures on $\boldsymbol{T}^n$ (see [1]). For any pseudomeasure S we denote its Fourier transform by $\hat{S}$, which belongs by definition to $\ell^\infty(\boldsymbol{Z}^n)$. Note that if we put $f_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{1\le j\le n} e^{-ik_j x_j}$, where $\boldsymbol{k} = (k_1, \dots, k_n) \in \boldsymbol{Z}^n$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in \boldsymbol{R}^n$, then the equality $(f_{\boldsymbol{k}} S)^\wedge(\boldsymbol{p}) = \hat{S}(\boldsymbol{p} + \boldsymbol{k})$ holds for any $\boldsymbol{p} = (p_1, \dots, p_n) \in \boldsymbol{Z}^n$. Therefore we see that

$$(m_{\boldsymbol{t}}^* S)^\wedge(\boldsymbol{p}) = \sum_{\boldsymbol{k}\in \boldsymbol{Z}^n} \boldsymbol{t_k} \hat{S}(\boldsymbol{p} + \boldsymbol{k}) \text{ holds for any } \boldsymbol{p} \in \boldsymbol{Z}^n . \tag{3.1}$$

Now let $\boldsymbol{a} = (\boldsymbol{a_i}) \in \boldsymbol{A}_{\boldsymbol{t}}^0$ and let A denote the Fourier transform of $\boldsymbol{a}$. Note that A is a pseudomeasure, since $(\boldsymbol{a_i}) \in \boldsymbol{A}^0 = \ell^\infty(\boldsymbol{Z}^n)$. Furthermore, it follows from (3.1) that

$$(m_{\boldsymbol{t}}^* A)^\wedge(\boldsymbol{p}) = \sum_{\boldsymbol{k}\in \boldsymbol{Z}^n} \boldsymbol{t_k} \boldsymbol{a_{p+k}} ,$$

which is equal to zero for any $\boldsymbol{p} \in \boldsymbol{Z}^n$, since $\boldsymbol{a} = (\boldsymbol{a_i}) \in \boldsymbol{A}_{\boldsymbol{t}}^0$. Hence, by the injectivity of Fourier transform, we have $m_{\boldsymbol{t}}^* A = 0$. Thus we obtain the following

PROPOSITION 3.1. *Notation being as above, we have* $\operatorname{supp} A \subset V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$.

Recall that a pseudomeasure with a finite support is a measure ([1, 12.33]). Thus if we assume that $\#(V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)) < \infty$, then the Fourier transform A of an arbitrary array $\boldsymbol{a} \in \boldsymbol{A}_{\boldsymbol{t}}^0$ is expressed as $A = \sum_{\alpha\in V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)} c_{\boldsymbol{\alpha}} \delta_{\boldsymbol{\alpha}}$ for some $c_{\boldsymbol{\alpha}} \in \boldsymbol{C}$, where $\delta_{\boldsymbol{\alpha}}$ denotes the Dirac δ-function placed at $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \boldsymbol{T}^n$. Conversely, if we assume that $\boldsymbol{\alpha} \in V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$, then we can show that the Fourier transform $\hat{\delta}_{\boldsymbol{\alpha}}$ belongs to $\boldsymbol{A}_{\boldsymbol{t}}^0$ as follows. Let $\alpha_j = e^{ia_j}$, $1 \le j \le n$. Then we see that

$$\hat{\delta}_{\boldsymbol{\alpha}}(\boldsymbol{p}) = \delta_{\boldsymbol{\alpha}}(e^{-ip_1x_1}, \dots, e^{-ip_nx_n}) = e^{-ip_1a_1} \cdots e^{-ip_na_n} = \alpha_1^{-p_1} \cdots \alpha_n^{-p_n} .$$

Therefore, if we put $\boldsymbol{a^\alpha} = (\boldsymbol{a_i^\alpha}) = \hat{\delta}_{\boldsymbol{\alpha}} \in \ell^\infty(\boldsymbol{Z}^n)$, then

$$d_{\boldsymbol{t+p}}(\boldsymbol{a^\alpha}) = \sum_{\boldsymbol{i}\in \boldsymbol{Z}^n} \boldsymbol{t_{i-p}} \boldsymbol{a_i^\alpha} = \sum_{\boldsymbol{i}\in \boldsymbol{Z}^n} \boldsymbol{t_i} \boldsymbol{\alpha}^{-\boldsymbol{i}-\boldsymbol{p}} = \boldsymbol{\alpha}^{-\boldsymbol{p}} m_{\boldsymbol{t}}^*(\boldsymbol{\alpha}) = 0 ,$$

since we are assuming that $\boldsymbol{\alpha} \in V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$. Thus we obtain the following.

THEOREM 3.2. *Suppose that $V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$ is a finite set. Then for any window $\boldsymbol{t}$, the space $\boldsymbol{A}_{\boldsymbol{t}}^0$ is isomorphic through the Fourier transform to the space $\langle \delta_{\boldsymbol{\alpha}}; \boldsymbol{\alpha} \in V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)\rangle_{\boldsymbol{C}}$ spanned by the Dirac δ-functions placed at $\boldsymbol{\alpha} \in V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$. In particular, we have*

$$\dim_{\boldsymbol{C}} \boldsymbol{A}_{\boldsymbol{t}}^0 = \#(V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)) .$$

The following proposition deals with the case when $V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$ is infinite.

PROPOSITION 3.3. *When $V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$ is infinite, the space $\boldsymbol{A}_{\boldsymbol{t}}^0$ is infinite-dimensional.*

PROOF. Assume that $V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$ is infinite, and take mutually distinct elements $z_k \in V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$, $k \in \boldsymbol{Z}_{\geq 0}$. We show that the array $\boldsymbol{a}_k = (z_k^{-\boldsymbol{i}})_{\boldsymbol{i}\in\boldsymbol{Z}^n}$ belongs to $\boldsymbol{A}_{\boldsymbol{t}}^0$ for any $k \in \boldsymbol{Z}_{\geq 0}$, and they are linearly independent. We may assume that *the first coordinates of* $z_k, k \in \boldsymbol{Z}_{\geq 0}$, *are mutually distinct*, since at least one of $p_j(\{z_k; k \in \boldsymbol{Z}\})$, $1 \leq j \leq n$, where p_j denotes the projection to the j-th coordinate, must be an infinite subset of $\boldsymbol{T}$. Note that each array $\boldsymbol{a}_k$, $k \in \boldsymbol{Z}_{\geq 0}$, is bounded, since $z_k \in \boldsymbol{T}^n$. Furthermore we can compute the degree of $\boldsymbol{a}_k$ with respect to the translated windows $\boldsymbol{t}+\boldsymbol{p}, \boldsymbol{p} \in \boldsymbol{Z}^n$, as follows:

$$d_{\boldsymbol{t}+\boldsymbol{p}}(\boldsymbol{a}_k) = \sum_{\boldsymbol{i}\in\boldsymbol{Z}^n} \boldsymbol{t}_{\boldsymbol{i}-\boldsymbol{p}} z_k^{-\boldsymbol{i}} \sum_{\boldsymbol{i}\in\boldsymbol{Z}^n} \boldsymbol{t}_{\boldsymbol{i}} z_k^{-\boldsymbol{i}-\boldsymbol{p}} = z_k^{-\boldsymbol{p}} \sum_{\boldsymbol{i}\in\boldsymbol{Z}^n} \boldsymbol{t}_{\boldsymbol{i}} z_k^{-\boldsymbol{i}} = z_k^{-\boldsymbol{p}} m_{\boldsymbol{t}}^*(z_k) = 0\,, \tag{3.2}$$

by the assumption $z_k \in V_{\boldsymbol{T}^n}(m_{\boldsymbol{t}}^*)$, $k \in \boldsymbol{Z}$. Therefore, $\boldsymbol{a}_k \in \boldsymbol{A}_{\boldsymbol{t}}^0$ for any k. Furthermore we can show that the arrays $\boldsymbol{a}_k$, $k \in \boldsymbol{Z}_{\geq 0}$, are linearly independent as follows. Let $z_k^1 = p_1(z_k)$. Then we see that $(\boldsymbol{a}_k)_{(i,0,\dots,0)} = (z_k^1)^{-i}$, and hence the arrays $\boldsymbol{a}_k$ restrict to the sequences $((\boldsymbol{a}_k)_{\boldsymbol{i}})_{\boldsymbol{i}\in\boldsymbol{Z}\times\{(0,\dots,0)\}} = ((z_k^1)^{-i})_{i\in\boldsymbol{Z}}$, which are linearly independent. This completes the proof of Proposition 3.3.

4. Periodicity of arrays in $\boldsymbol{A}_{\boldsymbol{t}}^0$. In this section we give a simple criterion for $\boldsymbol{A}_{\boldsymbol{t}}^0$ to contain a multiply periodic array.

Let $\mu_n \subset \boldsymbol{T}$ denote the set of the n-th roots of unity and let $\mu_\infty = \bigcup_{n\geq 1}\mu_n$. Let $\zeta_n = e^{2\pi i/n} \in \mu_n$. An array $\boldsymbol{a} = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\boldsymbol{Z}^n} \in \boldsymbol{A}$ is said to be *n-ply periodic*, if there exists a nonzero $\boldsymbol{c} = (c_1,\dots,c_n) \in \boldsymbol{Z}_{\geq 1}^n$ such that $\boldsymbol{a}_{\boldsymbol{i}} = \boldsymbol{a}_{\boldsymbol{i}+\boldsymbol{c}}$ holds for any $\boldsymbol{i} \in \boldsymbol{Z}^n$. The following theorem provides us with a criterion for periodicity:

THEOREM 4.1. *For any window $\boldsymbol{t}$, there exists a nonzero n-ply periodic array in $\boldsymbol{A}_{\boldsymbol{t}}^0$ if and only if $V_{\mu_\infty^n}(m_{\boldsymbol{t}}) \neq \emptyset$.*

REMARK. The condition $V_{\mu_\infty^n}(m_{\boldsymbol{t}}) \neq \emptyset$ is equivalent to $V_{\mu_\infty^n}(m_{\boldsymbol{t}}^*) \neq \emptyset$, since ι restricts to a bijection on μ_∞^n.

PROOF. If-part: Suppose that $V_{\mu_\infty^n}(m_{\boldsymbol{t}}) \neq \emptyset$. Take any $z_0 \in V_{\mu_\infty^n}(m_{\boldsymbol{t}}^*)$ and let $\boldsymbol{a}_0 = (z_0^{-\boldsymbol{i}})_{\boldsymbol{i}\in\boldsymbol{Z}^n}$. One can check easily that it is n-ply periodic with period $(o_1,\dots,o_n)$, where o_j, $j \in [1,n]$, denotes the order of $p_j(z_0)$, and it belongs to $\boldsymbol{A}_{\boldsymbol{t}}^0$ as is seen in (3.2).

Only-If part: Suppose that $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{i}})_{\boldsymbol{i}\in\boldsymbol{Z}^n} \in \boldsymbol{A}_{\boldsymbol{t}}^0$ is a nonzero n-ply periodic array with period $\boldsymbol{c} = (c_1,\dots,c_n) \in \boldsymbol{Z}_{\geq 1}^n$. Let $\zeta_{\boldsymbol{c}} = (\zeta_{c_1},\dots,\zeta_{c_n})$. For any $\boldsymbol{d},\boldsymbol{e} \in \boldsymbol{Z}^n$, we let $[\boldsymbol{d},\boldsymbol{e}] = \prod_{1\leq j\leq n}[d_j,e_j] \subset \boldsymbol{Z}^n$ and let $\boldsymbol{d}*\boldsymbol{e} = (d_1e_1,\dots,d_ne_n) \in \boldsymbol{Z}^n$. Let $\boldsymbol{0} = (0,\dots,0)$, $\boldsymbol{1} = (1,\dots,1) \in \boldsymbol{Z}^n$. For any $\boldsymbol{\alpha} \in [\boldsymbol{0},\boldsymbol{c}-\boldsymbol{1}]$ and $\boldsymbol{i} \in \boldsymbol{Z}^n$, let

$$\boldsymbol{b}_{\boldsymbol{i}}^{\boldsymbol{\alpha}} = \sum_{\boldsymbol{k}\in[\boldsymbol{i},\boldsymbol{i}+\boldsymbol{c}-\boldsymbol{1}]} \zeta_{\boldsymbol{c}}^{(\boldsymbol{k}-\boldsymbol{i})*\boldsymbol{\alpha}} \boldsymbol{a}_{\boldsymbol{k}}\,, \tag{4.1}$$

and put $\boldsymbol{b}^{\boldsymbol{\alpha}} = (\boldsymbol{b}_{\boldsymbol{i}}^{\boldsymbol{\alpha}})_{\boldsymbol{i}\in\boldsymbol{Z}^n} \in \boldsymbol{A}$. It is evident that the array $\boldsymbol{b}^{\boldsymbol{\alpha}}$ is bounded. Since $\boldsymbol{A}_{\boldsymbol{t}}^0$ is a $\boldsymbol{C}$-vector space, all of these arrays $\boldsymbol{b}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in [\boldsymbol{0},\boldsymbol{c}-\boldsymbol{1}]$, belong to $\boldsymbol{A}_{\boldsymbol{t}}^0$. Furthermore we have the following

LEMMA 4.1.1. *At least one of $\boldsymbol{b}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} \in [\boldsymbol{0},\boldsymbol{c}-\boldsymbol{1}]$, is a nonzero array.*

PROOF. It follows from (4.1) that

$$\boldsymbol{b}_{\boldsymbol{0}}^{\alpha} = \sum_{\boldsymbol{k}\in[\boldsymbol{0},\boldsymbol{c}-\boldsymbol{1}]} \zeta_{\boldsymbol{c}}^{\boldsymbol{k}*\alpha} \boldsymbol{a}_{\boldsymbol{k}}\,,$$

This equality can be regarded as giving a linear transformation which sends $(\boldsymbol{a}_{\boldsymbol{k}})_{\boldsymbol{k}\in[\boldsymbol{0},\boldsymbol{c}-\boldsymbol{1}]} \in \boldsymbol{C}^{[0,c_1-1]} \otimes \cdots \otimes \boldsymbol{C}^{[0,c_n-1]}$ to $(\boldsymbol{b}_{\boldsymbol{0}}^{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in[\boldsymbol{0},\boldsymbol{c}-\boldsymbol{1}]} \in \boldsymbol{C}^{[0,c_1-1]} \otimes \cdots \otimes \boldsymbol{C}^{[0,c_n-1]}$ through the tensor product of the matrices $(\zeta_{c_j}^{k_j\alpha_j})_{(k_j,\alpha_j)\in[0,c_j-1]\times[0,c_j-1]} \in \mathrm{End}(\boldsymbol{C}^{c_j})$, $1 \leq j \leq n$, of van der Monde-type. Therefore if $\boldsymbol{a}$ is nonzero array, then $(\boldsymbol{a}_{\boldsymbol{k}})_{\boldsymbol{k}\in[\boldsymbol{0},\boldsymbol{c}-\boldsymbol{1}]}$ is nonzero by the periodicity, which implies that at least one of $\boldsymbol{b}^{\boldsymbol{\alpha}}$ does not vanish. This completes the proof of Lemma 4.1.1.

Moreover we notice the following:

LEMMA 4.1.2. *For any* $\boldsymbol{i} \in \boldsymbol{Z}^n$, *we have* $\zeta_{c_j}^{\alpha_j}\boldsymbol{b}_{\boldsymbol{i}}^{\boldsymbol{\alpha}} = \boldsymbol{b}_{\boldsymbol{i}-\boldsymbol{e}_j}^{\boldsymbol{\alpha}}$ *for any* $j \in [1,n]$, *where* $\boldsymbol{e}_j$ *denotes the* j*-th standard basis of* $\boldsymbol{Z}^n$.

PROOF OF LEMMA 4.1.2. This is a consequence of the periodicity of $\boldsymbol{a}$, since for any $j \in [1,n]$ we have

$$\begin{aligned}\zeta_{c_j}^{\alpha_j}\left(\sum_{i_j\leq k_j\leq i_j+(c_j-1)} \zeta_{c_j}^{(k_j-i_j)\alpha_j}\boldsymbol{a}_{\boldsymbol{k}}\right) &= \sum_{i_j\leq k_j\leq i_j+(c_j-1)} \zeta_{c_j}^{(k_j+1-i_j)\alpha_j}\boldsymbol{a}_{\boldsymbol{k}} \\ &= \sum_{i_j\leq k_j\leq i_j+(c_j-1)} \zeta_{c_j}^{(k_j-i_j)\alpha_j}\boldsymbol{a}_{\boldsymbol{k}-\boldsymbol{e}_j}\,.\end{aligned}$$

This finishes the proof of Lemma 4.1.2.

REMARK. This lemma expresses in concrete terms the spectral decomposition of the periodic arrays.

Now going back to the proof of Theorem 4.1, we take any nonzero $\boldsymbol{b}^{\boldsymbol{\alpha}}$ whose existence is assured by Lemma 4.1.1. It follows from Lemma 4.1.2 that none of the entries of $\boldsymbol{b}^{\boldsymbol{\alpha}}$ vanishes. Furthermore, the same lemma shows that $\boldsymbol{b}^{\boldsymbol{\alpha}} = (\boldsymbol{b}_{\boldsymbol{0}}^{\boldsymbol{\alpha}}\boldsymbol{\zeta}_{\boldsymbol{c}}^{-\boldsymbol{k}*\boldsymbol{\alpha}})_{\boldsymbol{k}\in\boldsymbol{Z}^n} = \boldsymbol{b}_{\boldsymbol{0}}^{\boldsymbol{\alpha}}(\boldsymbol{\zeta}_{\boldsymbol{c}}^{-\boldsymbol{k}*\boldsymbol{\alpha}})_{\boldsymbol{k}\in\boldsymbol{Z}^n}$, and hence the array $(\boldsymbol{\zeta}_{\boldsymbol{c}}^{-\boldsymbol{k}*\boldsymbol{\alpha}})_{\boldsymbol{k}\in\boldsymbol{Z}^n}$ belongs to $\boldsymbol{A}_{\boldsymbol{t}}^0$. Therefore the argument employed when we showed (3.2) implies again that $m_{\boldsymbol{t}}(\boldsymbol{\zeta}_{\boldsymbol{c}}^{-\boldsymbol{\alpha}}) = 0$, and hence $V_{\mu_\infty^n}(m_{\boldsymbol{t}}) \neq \emptyset$. This completes the proof of Theorem 4.1.

We see from the proof above that we can restate the content of the theorem in more precise form.

COROLLARY (of the proof). *For any window* $\boldsymbol{t}$, *there exists a nonzero n-ply periodic array with period* $(c_1,\ldots,c_n)$ *in* $\boldsymbol{A}_{\boldsymbol{t}}^0$ *if and only if* $V_{\mu_{c_1}\times\cdots\times\mu_{c_n}}(m_{\boldsymbol{t}}) \neq \emptyset$.

When a window $\boldsymbol{t}$ is defined over $\boldsymbol{Q}$, namely when $\boldsymbol{t} \in (\boldsymbol{Q})^{\boldsymbol{Z}^n}$, Theorem 4.1 provides us with a stronger result. Let $\boldsymbol{A}_{\boldsymbol{t}}^0(\boldsymbol{Z})$ denote the subset of $\boldsymbol{A}_{\boldsymbol{t}}^0$ consisting of $\boldsymbol{Z}$-valued arrays.

PROPOSITION 4.2. *For any window $\boldsymbol{t}$ defined over $\boldsymbol{Q}$, there exists a nonzero n-ply periodic array in $\boldsymbol{A}_{\boldsymbol{t}}^0(\boldsymbol{Z})$ if and only if $V_{\mu_\infty^n}(m_{\boldsymbol{t}}) \neq \emptyset$. More precisely, there exists a nonzero n-ply periodic array with period $(c_1, \dots, c_n)$ in $\boldsymbol{A}_{\boldsymbol{t}}^0(\boldsymbol{Z})$ if and only if $V_{\mu_{c_1} \times \cdots \times \mu_{c_n}}(m_{\boldsymbol{t}}) \neq \emptyset$.*

PROOF. It suffices to construct a nonzero periodic array with period $(c_1, \dots, c_n)$ in $\boldsymbol{A}_{\boldsymbol{t}}^0(\boldsymbol{Z})$ under the hypothesis that $V_{\mu_{c_1} \times \cdots \times \mu_{c_n}}(m_{\boldsymbol{t}}) \neq \emptyset$. Taking any element $z_0 \in V_{\mu_{c_1} \times \cdots \times \mu_{c_n}}(m_{\boldsymbol{t}})$, let $\boldsymbol{a} = (\boldsymbol{a}_{\boldsymbol{i}})_{\boldsymbol{i} \in \boldsymbol{Z}^n}$ with $\boldsymbol{a}_{\boldsymbol{i}} = z_0^{\boldsymbol{i}}$. Note that $\boldsymbol{a}$ is periodic with period $(c_1, \dots, c_n)$. Let K be the finite extension of $\boldsymbol{Q}$ obtained by adjoining the coordinates of z_0, and let $G = \mathrm{Gal}(K/\boldsymbol{Q})$. Since $\boldsymbol{t}$ is defined over $\boldsymbol{Q}$, every Galois conjugate $\boldsymbol{a}^\sigma = (\boldsymbol{a}_{\boldsymbol{i}}^\sigma)$, $\sigma \in G$, of $\boldsymbol{a}$ belongs to $\boldsymbol{A}_{\boldsymbol{t}}^0$ too, and has the same period as $\boldsymbol{a}$. Therefore their sum $\boldsymbol{b} = \sum_{\sigma \in G} \boldsymbol{a}^\sigma$, which is periodic with period $(c_1, \dots, c_n)$ as the sum of such arrays, belongs to $\boldsymbol{A}_{\boldsymbol{t}}^0(\boldsymbol{Z})$, since the entries of $\boldsymbol{a}$ are algebraic integers in K. Furthermore, it is not equal to the zero array, since $\boldsymbol{b}_{\boldsymbol{0}} = \sum_{\sigma \in G} 1 = \#(G) \neq 0$. This completes the proof of Proposition 4.2.

5. Applications. In this section, we apply Theorem 3.2 and Theorem 4.1 to determine the structure of $\boldsymbol{A}_{\boldsymbol{t}}^0$ for some examples of 2-dimensional windows.

We specify below a window by displaying its nonzero entries placed at the underlying lattice points. Note that the structure of $\boldsymbol{A}_{\boldsymbol{t}}^0$ remains invariant by the very definition wherever we translate the window by the elements of $\boldsymbol{Z}^2$.

5.1. Window $\boldsymbol{t}_{\text{harmonic}}$:

	1	
1	–4	1
	1	

("-4" is placed at the origin. See Remark 5.1.1 (1) below for the reason why we call it *harmonic*.) The characteristic polynomial is given by $m_{\boldsymbol{t}_{\text{harmonic}}} = w + (z - 4 + z^{-1}) + w^{-1}$. Let $(z_0, w_0) \in V_{\boldsymbol{T}^2}(m^*_{\boldsymbol{t}_{\text{harmonic}}})$. Then we have

$$m^*_{\boldsymbol{t}_{\text{harmonic}}}(z_0, w_0) = m_{\boldsymbol{t}_{\text{harmonic}}}(z_0^{-1}, w_0^{-1}) = w_0^{-1} + (z_0^{-1} - 4 + z_0) + w_0 = 0\,,$$

and hence $w_0 + z_0 + z_0^{-1} + w_0^{-1} = 4$. This is possible only if $z_0 = w_0 = 1$, since $(z_0, w_0) \in \boldsymbol{T}^2$. Hence we see that $V_{\boldsymbol{T}^2}(m^*_{\boldsymbol{t}_{\text{harmonic}}}) = \{(1, 1)\}$ and $\dim \boldsymbol{A}^0_{\boldsymbol{t}_{\text{harmonic}}} = \#(V_{\boldsymbol{T}^2}(m^*_{\boldsymbol{t}_{\text{harmonic}}})) = 1$ by Theorem 3.2. On the other hand, it is clear that the all-one array $\boldsymbol{1}$ belongs to $\boldsymbol{A}^0_{\boldsymbol{t}_{\text{harmonic}}}$. Hence we obtain the following.

PROPOSITION 5.1. *For the window $\boldsymbol{t}_{\text{harmonic}}$, we have $\boldsymbol{A}^0_{\boldsymbol{t}_{\text{harmonic}}} = \{c.\boldsymbol{1};\, c \in \boldsymbol{C}\}$.*

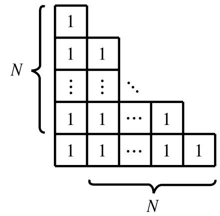
REMARK 5.1.1. (1) Note that an array $\boldsymbol{a} = (\boldsymbol{a}_{(i,j)})_{(i,j) \in \boldsymbol{Z}^2}$ belongs to $\boldsymbol{A}^0_{\boldsymbol{t}_{\text{harmonic}}}$ if and only if it is bounded and $\boldsymbol{a}_{(i,j)} = (\boldsymbol{a}_{(i+1,j)} + \boldsymbol{a}_{(i,j+1)} + \boldsymbol{a}_{(i-1,j)} + \boldsymbol{a}_{(i,j-1)})/4$ for any $(i, j) \in \boldsymbol{Z}^2$. Hence it gives rise to a *discrete harmonic function* on the lattice $\boldsymbol{Z}^2$. Thus Proposition 5.1 says that any bounded discrete harmonic function on $\boldsymbol{Z}^2$ must be constant.

(2) One can generalize the proposition to the n-dimensional window $\boldsymbol{t}^n_{\text{harmonic}}$ defined by

$$(\boldsymbol{t}^n_{\text{harmonic}})_{\boldsymbol{i}} = \begin{cases} -2^n\,, & \text{if } \boldsymbol{i} = 0\,, \\ 1\,, & \text{if } \sum_{1\le j\le n} |i_j| = 1\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Thus any bounded discrete harmonic function on $\boldsymbol{Z}^n$ must be constant.

5.2. Window $\boldsymbol{t}_{\text{stairs}}(N)$ $(N \ge 1)$:



Precisely speaking, we define $\boldsymbol{t}_{\text{stairs}}(N) = (\boldsymbol{t}_{(i,j)})$ by

$$\boldsymbol{t}_{(i,j)} = \begin{cases} 1\,, & \text{if } 0 \le i, j, i+j \le N\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

The characteristic polynomial is given by

$$m_{\boldsymbol{t}_{\text{stairs}}(N)} = w^N + (1+z)w^{N-1} + \cdots + (1+z+\cdots+z^{N-1})w + (1+z+\cdots+z^N)\,.$$

Note that $m_{\boldsymbol{t}_{\text{stairs}}(N)}$ is symmetric in z and w, reflecting the symmetry of the figure. Since

$$\begin{aligned}(1-w)m_{\boldsymbol{t}_{\text{stairs}}(N)}(1,w) &= (1-w)(w^N + 2w^{N-1} + \cdots + Nw + (N+1)) \\ &= (N+1) - (w + w^2 + \cdots + w^{N+1})\,,\end{aligned}$$

we see that if $(1, w_0) \in V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\text{stairs}}(N)})$, then w_0 is necessarily equal to one. Noting that $m_{\boldsymbol{t}_{\text{stairs}}(N)}(1,1) = (N+1)(N+2)/2 \ne 0$, we see by symmetry that if $(z_0, w_0) \in V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\text{stairs}}(N)})$, then neither z_0 nor w_0 are equal to one. Furthermore, since

$$(5.1)\qquad \begin{aligned}(1-z)m_{\boldsymbol{t}_{\text{stairs}}(N)}(z,w) &= (1-z)w^N + (1-z^2)w^{N-1} + \cdots + (1-z^{N+1}) \\ &= (w^N + w^{N-1} + \cdots + 1) - z(w^N + zw^{N-1} + \cdots + z^N)\,,\end{aligned}$$

a similar argument shows that if $(z_0, w_0) \in V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\text{stairs}}(N)})$, then $z_0 \ne w_0$. These considerations lead us to the following.

PROPOSITION 5.2. *Let $R^*_n = \mu_n - \{1\}$, the set of nontrivial n-th roots of unity, and let Δ_n denote the diagonal of $R^*_n \times R^*_n$. Then we have*

$$(5.2)\qquad V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\text{stairs}}(N)}) = (R^*_{N+1} \times R^*_{N+1} - \Delta_{N+1}) \cup (R^*_{N+2} \times R^*_{N+2} - \Delta_{N+2})\,.$$

PROOF. Let $(z_0, w_0) \in V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\text{stairs}}(N)})$. Since we already know that $z_0, w_0 \neq 1$, and $z_0 \neq w_0$, it follows from (5.1) that

$$(1 - z_0)m_{\boldsymbol{t}_{\text{stairs}}(N)}(z_0, w_0) = \frac{1 - w_0^{N+1}}{1 - w_0} - z_0^{N+1}\frac{1 - (w_0/z_0)^{N+1}}{1 - (w_0/z_0)} = 0\,.$$

Letting $z_0 = e^{i\theta}$, $w_0 = e^{i\varphi}$, we have the equality

$$\text{(5.3)} \qquad \frac{\sin(N+1)\varphi/2}{\sin\varphi/2}e^{iN\varphi/2} - e^{i(N+1)\theta}\frac{\sin(N+1)(\varphi-\theta)/2}{\sin(\varphi-\theta)/2}e^{iN(\varphi-\theta)/2} = 0\,.$$

When $\sin(N+1)\varphi/2 = 0$, it follows from this equality that $\sin(N+1)(\varphi-\theta)/2 = 0$. Hence we have $z^{N+1} = w^{N+1} = 1$. On the other hand, when $\sin(N+1)\varphi/2 \neq 0$, the equality (5.3) implies that $e^{i(N+1)\theta} \cdot e^{iN(\varphi-\theta)/2}/e^{iN\varphi/2} \in \boldsymbol{R}$, namely $e^{i(N+2)\theta/2} \in \boldsymbol{R}$, and hence $z_0 \in R_{N+2}$. By symmetry we have $w_0 \in R_{N+2}$. Hence we see that

$$\text{(5.4)} \qquad (z_0, w_0) \in (R^*_{N+1} \times R^*_{N+1} - \Delta_{N+1}) \cup (R^*_{N+2} \times R^*_{N+2} - \Delta_{N+2})\,.$$

Conversely, assume that (5.4) holds. When $(z_0, w_0) \in (R^*_{N+1} \times R^*_{N+1} - \Delta_{N+1})$, $m_{\boldsymbol{t}_{\text{stairs}}(N)}(z_0, w_0)$ vanishes trivially. On the other hand, if $(z_0, w_0) \in (R^*_{N+2} \times R^*_{N+2} - \Delta_{N+2})$, then

$$\begin{aligned}(1 - z_0)m_{\boldsymbol{t}_{\text{stairs}}(N)}(z_0, w_0) &= \frac{1 - w_0^{N+1}}{1 - w_0} - z_0^{N+1}\frac{1 - (w_0/z_0)^{N+1}}{1 - (w_0/z_0)} \\ &= \frac{1 - w_0^{-1}}{1 - w_0} - z_0^{-1}\frac{1 - (w_0/z_0)^{-1}}{1 - (w_0/z_0)} \\ &= w_0^{-1}\frac{w_0 - 1}{1 - w_0} - w_0^{-1}\frac{w_0 - z_0}{z_0 - w_0} = 0\,,\end{aligned}$$

and hence $(z_0, w_0) \in V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\text{stairs}}(N)})$. This completes the proof of Proposition 5.2.

Note that the right hand side of (5.2) is stable under $\iota : \boldsymbol{T}^2 \to \boldsymbol{T}^2$. Hence we see that

$$\text{(5.5)} \qquad V_{\boldsymbol{T}^2}(m^*_{\boldsymbol{t}_{\text{stairs}}(N)}) = (R^*_{N+1} \times R^*_{N+1} - \Delta_{N+1}) \cup (R^*_{N+2} \times R^*_{N+2} - \Delta_{N+2})$$

holds too. Since $\#(R^*_n \times R^*_n - \Delta_n) = (n-1)^2 - (n-1) = (n-1)(n-2)$ for any n and $(R^*_{N+1} \times R^*_{N+1} - \Delta_{N+1}) \cap (R^*_{N+2} \times R^*_{N+2} - \Delta_{N+2}) = \emptyset$, the equality (5.5) together with Theorem 3.2 implies the following.

COROLLARY 5.2.1. $\dim A^0_{\boldsymbol{t}_{\text{stairs}}(N)} = 2N^2$ *for any* $N \geq 1$.

In order to deal with the periodicity of arrays in $A^0_{\boldsymbol{t}_{\text{stairs}}(N)}$, we note that the equalities

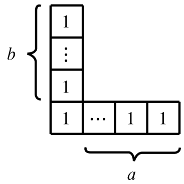
$$V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\text{stairs}}(N)}) = V_{\mu^2_\infty}(m_{\boldsymbol{t}_{\text{stairs}}(N)}) = (V_{\mu^2_{N+1}}(m_{\boldsymbol{t}_{\text{stairs}}(N)})) \cup (V_{\mu^2_{N+2}}(m_{\boldsymbol{t}_{\text{stairs}}(N)}))$$

hold by Proposition 5.2. Therefore we obtain the following corollary by Theorem 4.1:

COROLLARY 5.2.2. *Every array in* $A^0_{\boldsymbol{t}_{\text{stairs}}(N)}$ *is doubly periodic with period* $((N+1)(N+2), (N+1)(N+2))$.

REMARK. When $N = 1$, then the window $\boldsymbol{t}_{\mathrm{stairs}}(1)$ coincides with the 2-hook H_2 investigated in our previous paper [2], and Corollary 5.2.1 gives the correct dimension (= 2) of $\boldsymbol{A}^0_{H_2}$.

5.3. Window $\boldsymbol{t}_{\mathrm{leg}}(a, b)$:



The characteristic polynomial $m_{\boldsymbol{t}_{\mathrm{leg}}(a,b)}(z, w)$ is given by

$$m_{\boldsymbol{t}_{\mathrm{log}}(a,b)}(z, w) = w^b + w^{b-1} + \cdots + w + (1 + z + \cdots + z^a)\,.$$

Its zero locus is determined as follows.

PROPOSITION 5.3. *$V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\mathrm{leg}}(a,b)}) = (R^*_a \times R^*_{b+1}) \cup (R^*_{a+1} \times R^*_b) \cup \Delta'_{a+b+1}$, where $\Delta'_{a+b+1} = \{(z, w) \in R^*_{a+b+1} \times R^*_{a+b+1}; zw = 1\}$.*

PROOF. By symmetry, we may assume that $a \geq b$. Let $(z_0, w_0) \in V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\mathrm{leg}}(a,b)})$ and let $z_0 = e^{i\theta}$, $w_0 = e^{i\varphi}$. Then $z_0 \neq 1$, since any sum of b $(< a + 1)$ elements of $\boldsymbol{T}$ cannot make $a + 1$. On the other hand, if $w_0 = 1$, then

$$\begin{aligned} m_{\boldsymbol{t}_{\mathrm{leg}}(a,b)}(z_0, 1) &= b + (1 + z_0 + \cdots + z_0^a) \\ &= b + \frac{1 - z_0^{a+1}}{1 - z_0} = b + \frac{\sin((a+1)\theta/2)}{\sin(\theta/2)} e^{ia\theta/2} \\ &= 0\,, \end{aligned}$$

and hence $a\theta \in 2\pi\boldsymbol{Z}$, which implies $z_0^a = 1$. This in turn implies $b + (1 + z_0 + \cdots + z_0^a) = b + 1 = 0$, which is impossible. Thus we see that neither z_0 nor w_0 are equal to one. Hence we have

$$m_{\boldsymbol{t}_{\mathrm{leg}}(a,b)}(z_0, w_0) = w_0 \frac{1 - w_0^b}{1 - w_0} + \frac{1 - z_0^{a+1}}{1 - z_0} = 0\,. \tag{5.6}$$

This gives us the equality

$$\frac{\sin(b\varphi/2)}{\sin(\varphi/2)} + \frac{\sin((a+1)\theta/2)}{\sin(\theta/2)} e^{i(a\theta-(b+1)\varphi)/2} = 0\,. \tag{5.7}$$

First we consider the case $\sin(b\varphi/2) = 0$. It follows that $\sin((a + 1)/2) = 0$, and hence

$$z_0^{a+1} = w_0^b = 1\,. \tag{5.8}$$

Next we consider the case $\sin(b\varphi/2) \neq 0$. This implies through (5.7) that $a\theta-(b+1)\varphi \in 2\pi\boldsymbol{Z}$, and hence

$$w_0^{b+1} = z_0^a\,. \tag{5.9}$$

Therefore, it follows from (5.6) that

$$1 - z_0^{a+1} - w_0 z_0 + w_0 z_0^{a+1} - w_0^{b+1} + z_0 w_0^{b+1} = 0\,.$$

Inserting (5.9) into this, we see that

$$\begin{aligned}1 - z_0^{a+1} - w_0 z_0 + w_0 z_0^{a+1} - z_0^a + z_0^{a+1} &= 1 - w_0 z_0 + w_0 z_0^{a+1} - z_0^a \\ &= (1 - w_0 z_0)(1 - z_0^a) = 0\,,\end{aligned}$$

and hence $w_0 = 1/z_0$ or $z_0^a = 1$. When $w_0 = 1/z_0$, (5.9) gives us the equality $z_0^{a+b+1} = 1$ and

$$z_0^b m_{\boldsymbol{t}_{\mathrm{leg}}(a,b)}(z_0, w_0) = 1 + z_0 + \cdots + z_0^{b-1} + z_0^b(1 + z_0 + \cdots + z_0^a) = 0\,,$$

which implies that $m_{\boldsymbol{t}_{\mathrm{leg}}(a,b)}(z_0, w_0) = 0$. When $z_0^a = 1$, (5.9) gives us the equality $w_0^{b+1} = 1$. Hence, taking (5.8) into account, we see that $(z_0, w_0) \in \Delta'_{a+b+1} \cup R_a^* \times R_{b+1}^* \cup R_{a+1}^* \times R_b^*$. Since the converse inclusion can be checked easily, this completes the proof of Proposition 5.3.

Note that the pairwise intersections of three subsets $R_a^* \times R_{b+1}^*$, $R_{a+1}^* \times R_b^*$, and Δ'_{a+b+1} are computed to be

$$(R_a^* \times R_{b+1}^*) \cap \Delta'_{a+b+1} = \Delta'_{(a,b+1)}\,, \quad (R_{a+1}^* \times R_b^*) \cap \Delta'_{a+b+1} = \Delta'_{(a+1,b)}\,,$$
$$(R_a^* \times R_{b+1}^*) \cap (R_{a+1}^* \times R_b^*) = \emptyset\,.$$

Hence we see that

$$\begin{aligned}&\#((R_a^* \times R_{b+1}^*) \cup (R_{a+1}^* \times R_b^*) \cup \Delta'_{a+b+1}) \\ &\quad = \#(R_a^* \times R_{b+1}^*) + \#(R_{a+1}^* \times R_b^*) + \#(\Delta'_{a+b+1}) - \#(\Delta'_{(a,b+1)}) - \#(\Delta'_{(a+1,b)}) \\ &\quad = (a-1)b + a(b-1) + (a+b) - ((a, b+1) - 1) - ((a+1, b) - 1) \\ &\quad = 2(ab+1) - (a, b+1) - (a+1, b)\,.\end{aligned}$$

Furthermore, note that these three subsets are stable under $\iota : \boldsymbol{T}^2 \to \boldsymbol{T}^2$. Hence the proposition together with Theorem 3.2 implies the following dimension formula.

COROLLARY 5.3.1. *For any pair (a, b) of positive integers, we have*

$$\dim \boldsymbol{A}^0_{\boldsymbol{t}_{\mathrm{leg}}(a,b)} = 2(ab+1) - (a, b+1) - (a+1, b)\,. \tag{5.10}$$

As for the periodicity, Proposition 5.3 implies through Theorem 4.1 the following:

COROLLARY 5.3.2. *Every array in $\boldsymbol{A}^0_{\boldsymbol{t}_{\mathrm{leg}}(a,b)}$ is doubly periodic.*

REMARK. When $b = 1$, the window $\boldsymbol{t}_{\mathrm{leg}}(a, 1)$ coincides with H_{a+1}, called $(a+1)$-*hook* and investigated in [2]. Theorem 6.8 in that paper gives the dimension formula

$$\dim \boldsymbol{A}^0_{H_{a+1}} = \begin{cases} 2a\,, & \text{if } a \text{ is odd}\,, \\ 2a-1\,, & \text{if } a \text{ is even}\,. \end{cases} \tag{5.11}$$

On the other hand, the formula (5.10) with $b = 1$ provides us with

$$\dim \boldsymbol{A}^0_{\boldsymbol{t}_{\mathrm{leg}}(a,1)} = 2(a+1) - (a, 2) - (a+1, 1) = 2a + 1 - (a, 2)\,,$$

which is readily seen to coincide with (5.11).

5.4. Window $\boldsymbol{t}_{\mathrm{cross}}$:

	1	
1	1	1
	1	

The characteristic polynomial $m_{\boldsymbol{t}_{\mathrm{cross}}}(z, w)$ is given by

$$m_{\boldsymbol{t}_{\mathrm{cross}}}(z, w) = w + (z + 1 + z^{-1}) + w^{-1}\,.$$

Hence $V_{\boldsymbol{T}^2}(m_{\boldsymbol{t}_{\mathrm{cross}}}) = V_{\boldsymbol{T}^2}(m^*_{\boldsymbol{t}_{\mathrm{cross}}}) = \{(z, w) \in \boldsymbol{T}^2;\ (w + w^{-1}) + (z + z^{-1}) = -1\}$. Since $z + z^{-1}$ (resp. $w + w^{-1}$) takes any real values between -2 and 2 on $\boldsymbol{T}$, we see that the set $\{(z, w) \in \boldsymbol{T}^2;\ (w + w^{-1}) + (z + z^{-1}) = -1\}$ is infinite in contrast to the previous examples. Therefore the space $\boldsymbol{A}^0_{\boldsymbol{t}_{\mathrm{cross}}}$ is of infinite dimension by Proposition 3.3. One can show, however, that the subspace $\boldsymbol{A}^{0,\mathrm{periodic}}_{\boldsymbol{t}_{\mathrm{cross}}}$ of $\boldsymbol{A}^0_{\boldsymbol{t}_{\mathrm{cross}}}$ consisting of doubly periodic arrays is finite-dimensional. Indeed, it follows from the main theorem of [11] that

$$V_{\mu^2_\infty}(m_{\boldsymbol{t}_{\mathrm{cross}}}) = \{(-1, \zeta_6^{\pm 1}), (\zeta_6^{\pm 1}, -1), (\pm i, \zeta_3^{\pm 1}), (\zeta_3^{\pm 1}, \pm i), (\zeta_5^{\pm 1}, \zeta_5^{\pm 2}), (\zeta_5^{\pm 2}, \zeta_5^{\pm 1})\}\,,$$

and hence we see from Theorem 3.2 and Theorem 4.1 that $\dim \boldsymbol{A}^{0,\mathrm{periodic}}_{\boldsymbol{t}_{\mathrm{cross}}} = 20$.

6. Application to the study of Hodge cycles. In this section we recall the definition of nondegeneracy of an abelian variety of CM-type, and review certain examples of stably nondegenerate abelian varieties. Thereafter we show that our results in discrete tomography play an important role in the study of the ring of Hodge cycles on abelian varieties of CM-type.

Recall that an abelian variety A of CM-type is said to be *stably nondegenerate* if there are no nondivisorial Hodge cycles on A as well as on any of its self-products [3]. If A is not stably nondegenerate, then it is said to be *stably degenerate*. In particular, if A is stably nondegenerate, then the Hodge conjecture holds for any A^n, $n \geq 1$. For example, the following abelian varieties of CM-type are known to be stably nondegenerate:

(i) The jacobian variety of the hyperelliptic curve $y^2 = x^p - 1$ for an arbitrary odd prime p ([6]).

(ii) Certain factors of the jacobian variety of the Fermat curve $x^m + y^m = z^m$ ([12]).

(iii) The jacobian variety of the Catalan curve $y^q = z^p - 1$ for arbitrary pair of distinct odd primes p, q ([4]).

(See [7] for more examples of stably nondegenerate abelian varieties as well as stably degenerate ones.) The common feature of these investigations is to fix as a frame the Galois group of the abelian CM-field of an abelian variety A in question, to find its CM-type (which will be seen later in this section to correspond to a window in our sense), and then to show that the rank of the Hodge group of A is as large as possible. Roughly speaking, our strategy in this paper enables one to argue in reverse order. Namely, we fix a window and find (an infinite family of) appropriate frames (= the Galois groups) into which it can be fit (=stably nondegenerate). Furthermore, given a window, we can determine completely the set of abelian Galois groups for which the corresponding abelian varieties are stably nondegenerate.

For any n-tuple $\boldsymbol{c} = (c_1, \ldots, c_n)$ of integers ≥ 2, we consider a CM-field $K_{\boldsymbol{c}}$ such that the Galois group $G(K_{\boldsymbol{c}}/\boldsymbol{Q})$ is isomorphic to the abelian group $G_{\boldsymbol{c}} = \boldsymbol{Z}/2\boldsymbol{Z} \times H_{\boldsymbol{c}}$, where $H_{\boldsymbol{c}} = \prod_{1 \leq j \leq n} \boldsymbol{Z}/c_j\boldsymbol{Z}$, and the complex conjugation ρ corresponds to $(1, 0, \ldots, 0) \in G_{\boldsymbol{c}}$. A subset $T \subset G_{\boldsymbol{c}}$ is called a *CM-type* if

$$G_{\boldsymbol{c}} = T \coprod \rho(T) \quad \text{(disjoint sum)}.$$

Let $\boldsymbol{G_c} = \boldsymbol{Z}[G_{\boldsymbol{c}}]$ and let $\boldsymbol{H_c} = \boldsymbol{Z}[H_{\boldsymbol{c}}]$, the latter being regarded as a subring of $\boldsymbol{G_c}$ through the natural inclusion map. Furthermore we put

$$\boldsymbol{G}_{\boldsymbol{c}}^{\geq 0} = \left\{ \sum_{g \in G_{\boldsymbol{c}}} c_g.g \in \boldsymbol{G_c};\, c_g \geq 0 \ \text{ for any } g \in G_{\boldsymbol{c}} \right\}.$$

We will write the group operation on $G_{\boldsymbol{c}}$ multiplicatively in order to tell it from the addition in the group ring. Through this convention any element $g_1 \in G_{\boldsymbol{c}}$ acts as an automorphism of $\boldsymbol{G_c}$ by the rule $g_1(\sum_{g \in G_{\boldsymbol{c}}} c_g.g) = \sum_{g \in G_{\boldsymbol{c}}} c_g.g_1 g$. Let $p : \boldsymbol{G_c} \to \boldsymbol{H_c}$ denote the projection defined by $p(\sum_{g \in G_{\boldsymbol{c}}} c_g.g) = \sum_{g \in H_{\boldsymbol{c}}} c_g.g$. For any subset $S \subset G_{\boldsymbol{c}}$, let $[S] = \sum_{s \in S} s \in \boldsymbol{G_c}$. We define a linear map $\varphi : \boldsymbol{G_c} \to \boldsymbol{H_c}$ by

$$\varphi(\boldsymbol{v}) = p(\boldsymbol{v} - \rho\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{G_c},$$

and let $\psi : \boldsymbol{H_c} \to \boldsymbol{G_c}$ be defined by

$$\psi\left(\sum_{h \in H_{\boldsymbol{c}}} d_h.h \right) = \sum_{\substack{h \in H_{\boldsymbol{c}}, \\ d_h > 0}} d_h.(0, h) + \sum_{\substack{h \in H_{\boldsymbol{c}}, \\ d_h < 0}} (-d_h).(1, h).$$

Note that φ is $H_{\boldsymbol{c}}$-equivariant in the sense that $\varphi(h\boldsymbol{v}) = h\varphi(\boldsymbol{v})$. Note further that the image of φ is contained in

$$(\boldsymbol{G}_{\boldsymbol{c}}^{\geq 0})_{\text{nondiv}} = \left\{ \sum_{g \in G_{\boldsymbol{c}}} c_g.g \in \boldsymbol{G}_{\boldsymbol{c}}^{\geq 0};\ c_g c_{\rho g} = 0 \ \text{ for any } g \in G_{\boldsymbol{c}} \right\},$$

and the two maps $\varphi|_{(\boldsymbol{G}_{\boldsymbol{c}}^{\geq 0})_{\text{nondiv}}} : (\boldsymbol{G}_{\boldsymbol{c}}^{\geq 0})_{\text{nondiv}} \to \boldsymbol{H_c}$ and $\varphi : \boldsymbol{H_c} \to (\boldsymbol{G}_{\boldsymbol{c}}^{\geq 0})_{\text{nondiv}}$ are inverse to each other. (We will see below that each element of $(\boldsymbol{G}_{\boldsymbol{c}}^{\geq 0})_{\text{nondiv}}$ gives rise to a nondivisorial Hodge cycle on a certain abelian variety constructed from these data.) We introduce a natural $\boldsymbol{Z}$-valued pairing $\langle\ ,\ \rangle_{\boldsymbol{G_c}}$ by $\langle \sum_{g \in G_{\boldsymbol{c}}} c_g.g, \sum_{g \in G_{\boldsymbol{c}}} d_g.g \rangle_{\boldsymbol{G_c}} = \sum_{g \in G_{\boldsymbol{c}}} c_g d_g$, and $\langle\ ,\ \rangle_{\boldsymbol{H_c}}$ by a similar formula. Furthermore, for any $\boldsymbol{v} \in \boldsymbol{G_c}$ (resp. $\boldsymbol{w} \in \boldsymbol{H_c}$), we let $(\boldsymbol{v})^{\perp}_{\boldsymbol{G_c}} = \{\boldsymbol{v}' \in \boldsymbol{G_c};\ \langle \boldsymbol{v}', \boldsymbol{v} \rangle_{\boldsymbol{G_c}} = 0\}$ (resp. $(\boldsymbol{w})^{\perp}_{\boldsymbol{H_c}} = \{\boldsymbol{w}' \in \boldsymbol{H_c};\ \langle \boldsymbol{w}', \boldsymbol{w} \rangle_{\boldsymbol{H_c}} = 0\}$). We show the following:

LEMMA 6.1. *Let T be a CM-type. Then for any* $\boldsymbol{v} \in \boldsymbol{G_c}$ *we have*

$$\langle \boldsymbol{v}, [T] - \rho[T] \rangle_{\boldsymbol{G_c}} = \langle \varphi(\boldsymbol{v}), \varphi([T]) \rangle_{\boldsymbol{H_c}}. \tag{6.1}$$

In particular, we have $\boldsymbol{v} \in ([T] - \rho[T])^{\perp}_{\boldsymbol{G_c}}$ *if and only if* $\varphi(\boldsymbol{v}) \in (\varphi([T]))^{\perp}_{\boldsymbol{H_c}}$.

PROOF OF LEMMA 6.1. For any $\boldsymbol{v} \in \boldsymbol{G_c}$, let $\boldsymbol{v}_0 = p(\boldsymbol{v})$, $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{v}_0$ so that $\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1$. Note that $\varphi(\boldsymbol{v}) = \boldsymbol{v}_0 - \rho\boldsymbol{v}_1$, since $\mathrm{supp}(\boldsymbol{v}_0) \subset H_{\boldsymbol{c}}$, $\mathrm{supp}(\boldsymbol{v}_1) \subset \rho H_{\boldsymbol{c}}$. Similarly, let $T_0 = T \cap H_{\boldsymbol{c}}$, $T_1 = T \cap \rho H_{\boldsymbol{c}}$ so that $[T] = [T_0] + [T_1]$ and $[T] - \rho[T] = [T_0] + [T_1] - \rho[T_0] - \rho[T_1]$. Therefore, the left hand side of (6.1) is computed as

$$\begin{aligned}\langle \boldsymbol{v}, [T] - \rho[T]\rangle_{\boldsymbol{G_c}} &= \langle \boldsymbol{v}_0 + \boldsymbol{v}_1, [T_0] + [T_1] - \rho[T_0] - \rho[T_1]\rangle_{\boldsymbol{G_c}} \\ &= \langle \boldsymbol{v}_0, [T_0]\rangle_{\boldsymbol{G_c}} - \langle \boldsymbol{v}_0, \rho[T_1]\rangle_{\boldsymbol{G_c}} + \langle \boldsymbol{v}_1, [T_1]\rangle_{\boldsymbol{G_c}} - \langle \boldsymbol{v}_1, \rho[T_0]\rangle_{\boldsymbol{G_c}}\,,\end{aligned}$$

since $\boldsymbol{H_c}$ and $\rho\boldsymbol{H_c}$ are orthogonal to each other with respect to the pairing $\langle\ ,\ \rangle_{\boldsymbol{G_c}}$. On the other hand, the right hand side of (6.1) is computed as

$$\begin{aligned}\langle \varphi(\boldsymbol{v}), \varphi([T])\rangle_{\boldsymbol{H_c}} &= \langle \boldsymbol{v}_0 - \rho\boldsymbol{v}_1, [T_0] - \rho[T_1]\rangle_{\boldsymbol{H_c}} \\ &= \langle \boldsymbol{v}_0, [T_0]\rangle_{\boldsymbol{H_c}} - \langle \rho\boldsymbol{v}_1, [T_0]\rangle_{\boldsymbol{H_c}} - \langle \boldsymbol{v}_0, \rho[T_1]\rangle_{\boldsymbol{H_c}} + \langle \rho\boldsymbol{v}_1, \rho[T_1]\rangle_{\boldsymbol{H_c}} \\ &= \langle \boldsymbol{v}_0, [T_0]\rangle_{\boldsymbol{G_c}} - \langle \boldsymbol{v}_1, \rho[T_0]\rangle_{\boldsymbol{G_c}} - \langle \boldsymbol{v}_0, \rho[T_1]\rangle_{\boldsymbol{G_c}} + \langle \boldsymbol{v}_1, [T_1]\rangle_{\boldsymbol{G_c}}\,,\end{aligned}$$

since $\langle\ ,\ \rangle_{\boldsymbol{G_c}}|_{\boldsymbol{H_c}\times\boldsymbol{H_c}} = \langle\ ,\ \rangle_{\boldsymbol{H_c}}$ and $\langle\ ,\ \rangle_{\boldsymbol{G_c}}$ is $G_{\boldsymbol{c}}$-equivariant. This completes the proof of Lemma 6.1.

We will see the relevance of this lemma for the study of the structure of the ring of Hodge cycles on abelian varieties with complex multiplication by $K_{\boldsymbol{c}}$. First we recall some facts on Hodge cycles on abelian varieties of CM-type (see [5] for details). Let $T \subset G_{\boldsymbol{c}}$ be a CM-type and let A_T denote the abelian variety associated to T. One knows that the first cohomology group $H^1(A_T, \boldsymbol{C})$ can be identified with $\boldsymbol{C}^{G_c}$, and the complexification of the Hodge ring ($\subset \Lambda(\boldsymbol{C}^{G_c})$) admits as basis the set of basis vector of $\Lambda(\boldsymbol{C}^{G_c})$ corresponding to subsets P of $G_{\boldsymbol{c}}$ with the property that

$$\#(P \cap gT) = (\#P)/2 \quad \text{for any } \ g \in G_{\boldsymbol{c}}\,.$$

The above condition can be reformulated in terms of the group algebra $\boldsymbol{G_c}$ as

$$[P] \in ([gT] - \rho[gT])^{\perp}_{\boldsymbol{G_c}} \quad \text{for any } \ g \in G_{\boldsymbol{c}}\,.$$

For, we have the following series of equivalences:

$$\begin{aligned}&\#(P \cap gT) = (\#P)/2 \\ &\Leftrightarrow \#(P \cap gT) = \#(P \cap \rho gT) \\ &\Leftrightarrow \langle [P], [gT] - \rho[gT]\rangle_{\boldsymbol{G_c}} = 0\,.\end{aligned}$$

We can generalize the above consideration to deal with the Hodge ring of $A_T^N = A_T \times \cdots \times A_T$ (N times), by using the isomorphism $H^1(A_T^N, \boldsymbol{C}) \cong (\boldsymbol{C}^{G_c})^{\oplus N}$. For any $i \in [1, N]$, let e_g^i, $g \in G_{\boldsymbol{c}}$, denote the standard basis of the i-th direct summand of $(\boldsymbol{C}^{G_c})^{\oplus N}$. For any $\boldsymbol{v} = \sum_{g\in G_c} c_g.g \in \boldsymbol{G}_{\boldsymbol{c}}^{\geq 0}$ with $c_g \leq N$, we denote by $\langle \boldsymbol{v}\rangle$ the basis element of $\Lambda((\boldsymbol{C}^{G_c})^{\oplus N})$ defined by

$$\langle \boldsymbol{v}\rangle = \bigwedge_{g\in G_{\boldsymbol{c}}} \Bigg(\bigwedge_{1\leq i_g\leq c_g} e_g^{i_g} \Bigg)\,.$$

We have seen in [5] that $\langle \boldsymbol{v} \rangle$ is a Hodge cycle on A_T^N if and only if $\langle \boldsymbol{v}, [gT] - \rho[gT] \rangle_{\boldsymbol{G_c}} = 0$ for any $g \in \boldsymbol{G_c}$. Furthermore, when A_T is simple, one knows that $\langle \boldsymbol{v} \rangle$ is nondivisorial if and only if $c_g c_{\rho g} = 0$ holds for any $g \in \boldsymbol{G_c}$. In view of this, we put

$$\boldsymbol{G_c}(T)_{\text{Hodge}} = \{\boldsymbol{v} \in G_{\boldsymbol{c}}^{\geq 0};\ \langle \boldsymbol{v}, [gT] - \rho[gT] \rangle_{\boldsymbol{G_c}} = 0 \ \text{ for any } g \in G_{\boldsymbol{c}}\},$$
$$\boldsymbol{G_c}(T)_{\text{Hodge, nondiv}} = \boldsymbol{G_c}(T)_{\text{Hodge}} \cap (\boldsymbol{G_c^{\geq 0}})_{\text{nondiv}}.$$

Note that, since $[\rho g T] - \rho[\rho g T] = -([gT] - \rho[gT])$, we have

$$\boldsymbol{G_c}(T)_{\text{Hodge}} = \{\boldsymbol{v} \in \boldsymbol{G_c^{\geq 0}};\ \langle \boldsymbol{v}, [hT] - \rho[hT] \rangle_{\boldsymbol{G_c}} = 0 \ \text{ for any } h \in H_{\boldsymbol{c}}\}.$$

Furthermore, by Lemma 6.1 we can rewrite this as

$$\boldsymbol{G_c}(T)_{\text{Hodge}} = \{\boldsymbol{v} \in \boldsymbol{G_c^{\geq 0}};\ \langle \varphi(\boldsymbol{v}), \varphi([hT]) \rangle_{\boldsymbol{H_c}} = 0 \ \text{ for any } h \in H_{\boldsymbol{c}}\}. \tag{6.2}$$

A CM-type $T \subset \boldsymbol{G_c}$ is said to be *primitive* if the corresponding abelian variety A_T is simple. By [10], T is primitive if and only if there exists no $g \in G_c - \{\boldsymbol{0}\}$ such that $g.T = T$. We summarize the above argument in the following form:

PROPOSITION 6.2. *For any* $\boldsymbol{v} = \sum_{g \in G_{\boldsymbol{c}}} c_g.g \in \boldsymbol{G_c^{\geq 0}}$, *the following hold.*

(i) $\langle \boldsymbol{v} \rangle$ *is a Hodge cycle on some self-product of* A_T *if and only if* $\boldsymbol{v} \in G_{\boldsymbol{c}}(T)_{\text{Hodge}}$.

(ii) *When* T *is primitive*, $\langle \boldsymbol{v} \rangle$ *is a nondivisorial Hodge cycle on some self-product of* A_T *if and only if* $\boldsymbol{v} \in \boldsymbol{G_c}(T)_{\text{Hodge, nondiv}}$.

We will see below that the sets $\boldsymbol{G_c}(T)_{\text{Hodge}}$ and $\boldsymbol{G_c}(T)_{\text{Hodge, nondiv}}$ are related with a certain set of arrays investigated in the previous sections. For any element $\boldsymbol{w} = \sum_{h \in H_{\boldsymbol{c}}} d_h.h$ of $\boldsymbol{H_c}$, we define a window $\boldsymbol{t}^{\boldsymbol{w}} = (\boldsymbol{t}_{\boldsymbol{i}}^{\boldsymbol{w}})_{\boldsymbol{i} \in \boldsymbol{Z}^n}$ by the rule

$$\boldsymbol{t}_{\boldsymbol{i}}^{\boldsymbol{w}} = \begin{cases} d_{\pi_{\boldsymbol{c}}(\boldsymbol{i})}, & \boldsymbol{i} \in [\boldsymbol{0}, \boldsymbol{c} - \boldsymbol{1}], \\ 0, & \text{otherwise}, \end{cases}$$

where $\pi_{\boldsymbol{c}};\ \boldsymbol{Z}^n \to H_{\boldsymbol{c}}$ denotes the natural projection. We will see in the following theorem that the study of the structure of the Hodge ring of A_T^N, $N \geq 1$, is reduced to that of $\boldsymbol{A}^0_{\boldsymbol{t}^T}$. We denote by $\boldsymbol{A}(\boldsymbol{Z})$ the set of $\boldsymbol{Z}$-valued arrays, and let $\boldsymbol{A}_{\boldsymbol{t}}^0(\boldsymbol{Z}) = \boldsymbol{A}_{\boldsymbol{t}}^0 \cap \boldsymbol{A}(\boldsymbol{Z})$. In this notation, $\pi_{\boldsymbol{c}}$ induces an injective homomorphism $\pi_{\boldsymbol{c}}^* : \boldsymbol{H_c}(= (\boldsymbol{Z})^{H_{\boldsymbol{c}}}) \to \boldsymbol{A}(\boldsymbol{Z})(= (\boldsymbol{Z})^{\boldsymbol{Z}^n})$, whose image coincides with

$$\boldsymbol{A}(\boldsymbol{Z})^{\boldsymbol{c}} = \{(\boldsymbol{a_i})_{\boldsymbol{i} \in \boldsymbol{Z}^n} \in A(\boldsymbol{Z});\ \ \boldsymbol{a}_{\boldsymbol{i}+\boldsymbol{c}} = \boldsymbol{a_i} \ \text{ for any } \boldsymbol{i} \in \boldsymbol{Z}^n\},$$

the set of n-ply periodic arrays with period $\boldsymbol{c}$.

THEOREM 6.3. *Let* $T \subset G_{\boldsymbol{c}}$ *be a CM-type. For an element* $\boldsymbol{v} \in \boldsymbol{G_c^{\geq 0}}$ *to belong to* $\boldsymbol{G_c}(T)_{\text{Hodge}}$, *it is necessary and sufficient that* $\pi_{\boldsymbol{c}}^*(\varphi(\boldsymbol{v})) \in \boldsymbol{A}^0_{\boldsymbol{t}^{\varphi([T])}}(\boldsymbol{Z})^{\boldsymbol{c}}$. *Moreover, for any* $\boldsymbol{w} \in \boldsymbol{H_c}$, *we have* $\psi(\boldsymbol{w}) \in \boldsymbol{G_c}(T)_{\text{Hodge, nondiv}}$ *if and only if* $\pi_{\boldsymbol{c}}^*(\boldsymbol{w}) \in \boldsymbol{A}^0_{\boldsymbol{t}^{\varphi([T])}}(\boldsymbol{Z})^{\boldsymbol{c}}$.

PROOF. We can compute the degree of $\pi_{\boldsymbol{c}}^*(\varphi(\boldsymbol{v}))$ with respect to the translated window $\boldsymbol{t}^{\varphi([T])}+\boldsymbol{p}, \boldsymbol{p}\in \boldsymbol{Z}^n$, as follows:

$$\begin{aligned} d_{\boldsymbol{t}^{\varphi([T])}+\boldsymbol{p}}(\pi_{\boldsymbol{c}}^*(\varphi(\boldsymbol{v}))) &= \sum_{\boldsymbol{i}\in \boldsymbol{Z}^n} \boldsymbol{t}_{\boldsymbol{i}}^{\varphi([T])}\pi_{\boldsymbol{c}}^*(\varphi(\boldsymbol{v}))_{\boldsymbol{i}+\boldsymbol{p}} = \sum_{\boldsymbol{i}\in[\boldsymbol{0},\boldsymbol{c}-1]} \boldsymbol{t}_{\boldsymbol{i}}^{\varphi([T])}\varphi(\boldsymbol{v})_{\pi_{\boldsymbol{c}}(\boldsymbol{i}+\boldsymbol{p})} \\ &= \sum_{\boldsymbol{i}\in[\boldsymbol{0},\boldsymbol{c}-1]} \varphi([T])_{\pi_{\boldsymbol{c}}(\boldsymbol{i})}\varphi(\boldsymbol{v})_{\pi_{\boldsymbol{c}}(\boldsymbol{i}+\boldsymbol{p})} = \sum_{\boldsymbol{i}\in[\boldsymbol{0},\boldsymbol{c}-1]} \varphi([T])_{\pi_{\boldsymbol{c}}(\boldsymbol{i})}\varphi(\pi_{\boldsymbol{c}}(-\boldsymbol{p}).\boldsymbol{v})_{\pi_{\boldsymbol{c}}(\boldsymbol{i})} \\ &= \langle\varphi(\pi_{\boldsymbol{c}}(-\boldsymbol{p}).\boldsymbol{v}), \varphi([T])\rangle_{H_{\boldsymbol{c}}} = \langle\varphi(\boldsymbol{v}), \varphi([\pi_{\boldsymbol{c}}(\boldsymbol{p})T])\rangle_{H_{\boldsymbol{c}}}\,. \end{aligned}$$

Therefore, we see by (6.2) that $\pi_{\boldsymbol{c}}^*(\varphi(\boldsymbol{v})) \in \boldsymbol{A}^0_{\boldsymbol{t}^{\varphi([T])}}(\boldsymbol{Z})^{\boldsymbol{c}}$ if and only if $\boldsymbol{v}\in \boldsymbol{G}_{\boldsymbol{c}}(T)_{\mathrm{Hodge}}$. For the second assertion we have only to recall that $\varphi\circ\psi = id_{H_{\boldsymbol{c}}}$ and $\mathrm{Im}(\psi) = (\boldsymbol{G}_{\boldsymbol{c}}^{\geq 0})_{\mathrm{nondiv}}$. This completes the proof of Theorem 6.3.

In view of Proposition 6.2, Theorem 6.3 enables us to relate the study of Hodge cycles with that of discrete tomography in the following form:

THEOREM 6.4. *Let $T\subset G_{\boldsymbol{c}}$ be a CM-type and let $\boldsymbol{v}\in \boldsymbol{G}_{\boldsymbol{c}}^{\geq 0}$. Then $\langle\boldsymbol{v}\rangle$ is a Hodge cycle on some self-product of the abelian variety A_T if and only if $\pi_{\boldsymbol{c}}^*(\varphi(\boldsymbol{v}))\in \boldsymbol{A}^0_{\boldsymbol{t}^{\varphi[T]}}(\boldsymbol{Z})^{\boldsymbol{c}}$.*

Next we will study Hodge rings of an infinite family of abelian varieties constructed from a fixed finite subset of $\boldsymbol{Z}^n_{\geq 0}$. For any subset $S\subset H_{\boldsymbol{c}}$, let S' denote its complement in $H_{\boldsymbol{c}}$. Furthermore, for any subset $T\subset G_{\boldsymbol{c}}$, let $T_0 = T\cap H_{\boldsymbol{c}}$. Then we have $\varphi([T]) = [T_0]-[T_0']$ in this notation. For any window $\boldsymbol{t}$, let $(\boldsymbol{A}_{\boldsymbol{t}}^0)^{\boldsymbol{c}}$ denote the set of arrays in $\boldsymbol{A}_{\boldsymbol{t}}^0$ with period $\boldsymbol{c}$.

PROPOSITION 6.5. *For any subset $S\subset H_{\boldsymbol{c}}$ with $\#S\neq c_1\cdots c_n/2$, $(\boldsymbol{A}^0_{\boldsymbol{t}^{[S]-[S']}})^{\boldsymbol{c}} \supsetneqq \{\boldsymbol{0}\}$ if and only if $(\boldsymbol{A}^0_{\boldsymbol{t}^{[S]}})^{\boldsymbol{c}}\supsetneqq\{\boldsymbol{0}\}$.*

PROOF. Let $r : H_{\boldsymbol{c}}\to[\boldsymbol{0},\boldsymbol{c}-\boldsymbol{1}]\subset\boldsymbol{Z}^n$ be the product of n maps $\boldsymbol{Z}/c_j\boldsymbol{Z}\to[0,c_j-1]$, $1\leq j\leq n$, each of which chooses the minimal nonnegative representatives of the congruence classes. Then, by definition, the characteristic polynomial $m_{\boldsymbol{t}^{[S]}}$ is given by

$$m_{\boldsymbol{t}^{[s]}} = \sum_{\boldsymbol{i}\in r(S)} z^{\boldsymbol{i}}\,. \tag{6.3}$$

For the window $\boldsymbol{t}^{[S]-[S']}$ we have

$$\begin{aligned} m_{\boldsymbol{t}^{[S]-[S']}}(z) &= \sum_{\boldsymbol{i}\in r(S)} z^{\boldsymbol{i}} - \sum_{\boldsymbol{i}\in r(S')} z^{\boldsymbol{i}} = 2\sum_{\boldsymbol{i}\in r(S)} z^{\boldsymbol{i}} - \sum_{\boldsymbol{i}\in r(H_{\boldsymbol{c}})} z^{\boldsymbol{i}} \\ &= 2m_{\boldsymbol{t}^{[S]}}(z) - m_{\boldsymbol{t}^{[H_{\boldsymbol{c}}]}}(z)\,. \end{aligned} \tag{6.4}$$

On the other hand, we recall from Theorem 4.1 that for any window $\boldsymbol{t}$, we have $(\boldsymbol{A}_{\boldsymbol{t}}^0)^{\boldsymbol{c}}\supsetneqq\{\boldsymbol{0}\}$ if and only if $V_{\mu_{\boldsymbol{c}}}(m_{\boldsymbol{t}})\neq\emptyset$. Since $m_{\boldsymbol{t}^{[H_{\boldsymbol{c}}]}}(z) = \prod_{1\leq j\leq n}\sum_{0\leq i\leq c_j-1} z_j^i$, we see that $m_{\boldsymbol{t}^{[H_{\boldsymbol{c}}]}}$ vanishes identically on $\boldsymbol{\mu}_{\boldsymbol{c}}$ except for $z=\boldsymbol{1}$. Thus the equalities (6.3) and (6.4) assure the equivalence in the statement. This completes the proof of Proposition 6.5.

We describe how this proposition enables us to study the Hodge rings of a certain infinite family of abelian varieties. For any finite subset $S\subset\boldsymbol{Z}^n_{\geq 0}$, let $\mathrm{Rec}(S) = \{\boldsymbol{c}\in\boldsymbol{Z}^n_{\geq 2};\ [\boldsymbol{0},\boldsymbol{c}-1]\supset$

$S\}$, where $\boldsymbol{Z}_{\geq 2}$ denotes the set of integers greater than or equal to two. When $\boldsymbol{c} \in \mathrm{Rec}(S)$, we regard S as a subset of $H_{\boldsymbol{c}}$ through the natural projection. Let

$$\mathrm{Rec}(S)_{\mathrm{nonprim}} = \{\boldsymbol{c} \in \mathrm{Rec}(S)\,;\, h.S = S \text{ or } S' \text{ for some } h \in \boldsymbol{H_c} - \{(0, \dots, 0)\}\}\,,$$
$$\mathrm{Rec}(S)_{\mathrm{prim}} = \mathrm{Rec}(S) - \mathrm{Rec}(S)_{\mathrm{nonprim}}\,.$$

We denote by $\mathrm{AV}(S)$ the set of abelian varieties $A_{T_{\boldsymbol{c},S}}$, $\boldsymbol{c} \in \mathrm{Rec}(S)$, with complex multiplication by $K_{\boldsymbol{c}}$ such that its CM-type $T_{\boldsymbol{c},S}$ is given by

$$T_{\boldsymbol{c},S} = (\{0\} \times S) \cup (\{1\} \times (S')) \subset \boldsymbol{G_c}\,. \tag{6.5}$$

It follows from [10] that $A_{T_{\boldsymbol{c},S}}$ is simple if and only if $\boldsymbol{c} \in \mathrm{Rec}(S)_{\mathrm{prim}}$. The following theorem determines completely which abelian varieties in $\mathrm{AV}(S)$ are simple and stably nondegenerate.

THEOREM 6.6. *Notation being as above, let*

$$\mathrm{Period}(S) = \{\boldsymbol{c} \in \mathrm{Rec}(S);\ V_{\mu_{\boldsymbol{c}}}(m_{\boldsymbol{t}[S]}) \neq \emptyset \ \text{ and } \ c_1 \cdots c_n = 2\#S\}\,.$$

Then we have

$$\{\boldsymbol{c} \in \mathrm{Rec}(S);\ A_{T_{\boldsymbol{c},S}} \textit{ is simple and stably nondegenerate}\}$$
$$= \mathrm{Rec}(S)_{\mathrm{prim}} - \mathrm{Period}(S)\,.$$

PROOF. This is a consequence of the following series of equivalences, where we use the same symbol S to denote the image of $S \subset \boldsymbol{Z}^n_{\geq 0}$ under the projection $\pi_{\boldsymbol{c}}$ and put $T = T_{\boldsymbol{c},S}$ for simplicity:

$V_{\mu_{\boldsymbol{c}}}(m_{\boldsymbol{t}[S]}) \neq \emptyset$

$\Leftrightarrow V_{\mu_{\boldsymbol{c}}}(m_{\boldsymbol{t}[S]-[S']}) \neq \emptyset$ (by Proposition 6.5)

$\Leftrightarrow V_{\mu_{\boldsymbol{c}}}(m_{\boldsymbol{t}\varphi([T])}) \neq \emptyset$ (by (6.5))

$\Leftrightarrow (\boldsymbol{A}^0_{\boldsymbol{t}\varphi([T])}(\boldsymbol{Z}))^{\boldsymbol{c}} \supsetneqq \{\boldsymbol{0}\}$ (by Proposition 4.2)

$\Leftrightarrow$ there exists a $\boldsymbol{w} \in \boldsymbol{H_c}$ such that $\psi(\boldsymbol{w}) \in \boldsymbol{G_c}(T)_{\mathrm{Hodge,\,nondiv}}$ (by Theorem 6.3)

$\Leftrightarrow A_T$ is stably degenerate. (by Proposition 6.2)

This completes the proof of Theorem 6.6.

We will examine how this theorem contributes to the study of Hodge cycles through several examples. First we deal with the window $\boldsymbol{t}_{\mathrm{stairs}}(N)$ treated in Example 5.2.

EXAMPLE 6.7. Let $S = \boldsymbol{t}_{\mathrm{stairs}}(N)$, $N \geq 1$. In this case we have

$$\mathrm{Rec}(\boldsymbol{t}_{\mathrm{stairs}}(N))_{\mathrm{prim}} = \mathrm{Rec}(\boldsymbol{t}_{\mathrm{stairs}}(N)) = Z^2_{\geq N+1}\,,$$

where $\boldsymbol{Z}_{\geq n}$ denotes the set of integers $\geq n$ for any n. Furthermore Proposition 5.2 tells us that

$$\mathrm{Period}(\boldsymbol{t}_{\mathrm{stairs}}(N)) = \boldsymbol{Z}^2_{\geq N+1} \cap (\mathrm{Pair}_{N+1} \cup \mathrm{Pair}_{N+2}) \cup \{(N, N+1), (N+1, N)\}\,,$$

where we put $\mathrm{Pair}_n = \{(a, b) \in \boldsymbol{Z}^2_{\geq 0};\ (a, n), (b, n) > 1\} - \{(a, b) \in \boldsymbol{Z}^2_{\geq 0};\ (a, n) = (b, n) = 2\}$ for any n. (Note that $\mathrm{Pair}_2 = \emptyset$ by definition.) Thus it follows from Theorem 6.6 that for any positive N and for any $\boldsymbol{c} \in \boldsymbol{Z}^2_{\geq N+1} - (\mathrm{Pair}_{N+1} \cup \mathrm{Pair}_{N+2}) - \{(N, N+1), (N+1, N)\}$, the abelian variety $A_{T_{\boldsymbol{c},\boldsymbol{t}_{\mathrm{stairs}}(N)}}$ is simple and stably nondegenerate. In particular, we see that *there exist infinitely many nondegenerate abelian varieties in* $\mathrm{AV}(\boldsymbol{t}_{\text{stairs}}(N))$, *and hence Hodge conjecture holds for infinitely many abelian varieties in* $\mathrm{AV}(\boldsymbol{t}_{\text{stairs}}(N))$. Moreover the same theorem implies also that *there exist infinitely many degenerate abelian varieties in* $\mathrm{AV}(\boldsymbol{t}_{\text{stairs}}(N))$.

The following two examples examines the simplest and the second simplest n-dimensional windows.

PROPOSITION 6.8. *Let* $\boldsymbol{O} = \{(0, \dots, 0)\} \subset \boldsymbol{Z}^n$, $n \geq 2$. *Then any abelian varieties in* $\mathrm{AV}(\boldsymbol{O})$ *are simple and stably nondegenerate. In particular, the Hodge conjecture holds for every abelian variety in* $\mathrm{AV}(\boldsymbol{O})$.

PROOF. One can check easily that $\mathrm{Rec}(\boldsymbol{O})_{\text{prim}} = \mathrm{Rec}(\boldsymbol{O}) = \boldsymbol{Z}^n_{\geq 2}$. Since $m_{\boldsymbol{t}}\boldsymbol{o} \equiv 1$, it is evident that $V_{\mu_{\boldsymbol{c}}}(m_{\boldsymbol{t}}\boldsymbol{o}) = \emptyset$ for any $\boldsymbol{c} \in \mathrm{Rec}(\boldsymbol{O})$, and hence $\mathrm{Period}(\boldsymbol{O}) = \emptyset$. Therefore it follows from Theorem 6.6 that every abelian variety in $\mathrm{AV}(\boldsymbol{O})$ is simple and stably nondegenerate. This finishes the proof of Proposition 6.8.

In contrast to the simplicity of the proof, this proposition has an amusing consequence:

COROLLARY 6.8.1. *Let K be an arbitrary abelian CM-field which contains an imaginary quadratic subfield. Then there exists at least one CM-type for K such that the corresponding abelian variety satisfies the Hodge conjecture.*

REMARK. Actually, anyone with a little experience of computing Hodge cycles on abelian varieties could prove Proposition 6.8 directly without any knowledge about discrete tomography. The point is, however, that discrete tomography leads us naturally to the simplest window, which gives rise, a posteriori, to infinitely many stably nondegenerate abelian varieties as above.

The next example deals with the second simplest window. The result is, however, rather different. For any integer n, let $\mathbf{Z}_{\text{even}, \geq n}$ (resp. $\mathbf{Z}_{\text{odd}, \geq n}$) denote the set of even (resp. odd) integers $\geq n$.

PROPOSITION 6.9. *Let* **Domino** *denote the subset* $\{\mathbf{0}, \boldsymbol{e}_1\} \subset \boldsymbol{Z}^n$, *where* $\boldsymbol{e}_1 = \{1, 0, \dots, 0\}$. *Then we have*

$$\mathrm{Rec}(\mathbf{Domino})_{\text{nonprim}} = \{2\} \times \boldsymbol{Z}^{n-1}_{\geq 2}\,, \tag{6.6}$$

$$\mathrm{Rec}(\mathbf{Domino})_{\text{prim}} = \boldsymbol{Z}_{\geq 3} \times \boldsymbol{Z}^{n-1}_{\geq 2}\,. \tag{6.7}$$

Furthermore

$$\textit{every abelian variety } A_{T_{\boldsymbol{c}, \mathbf{Domino}}} \textit{ with } \boldsymbol{c} \in \mathbf{Z}_{\text{odd}, \geq 3} \times \boldsymbol{Z}^{n-1}_{\geq 2} \textit{ is stably nondegenerate}\,, \tag{6.8}$$

$$\textit{every abelian variety } A_{T_{\boldsymbol{c}, \mathbf{Domino}}} \textit{ with } \boldsymbol{c} \in \mathbf{Z}_{\text{even}, \geq 4} \times \boldsymbol{Z}^{n-1}_{\geq 2} \textit{ is stably degenerate}\,. \tag{6.9}$$

PROOF. It is easy to see that $\mathrm{Rec}(\mathbf{Domino}) = \boldsymbol{Z}^n_{\geq 2}$. Furthermore one can check that the CM-type $T_{\boldsymbol{c}, \mathbf{Domino}}$ is non-primitive if and only if $c_1 = 2$. Therefore we obtain the equalities (6.6) and (6.7). Furthermore, the characteristic polynomial of **Domino** is given by $m_{\boldsymbol{t}\,\mathbf{Domino}} = 1 + z_1$, and hence $V_{\mu_\infty^n}(m_{\boldsymbol{t}\,\mathbf{Domino}}) = \{-1\} \times \mu_\infty^{n-1}$ for any $\boldsymbol{c} \in \boldsymbol{Z}^n_{\geq 2}$. It follows that Period(**Domino**) $= \boldsymbol{Z}_{\text{even},\,\geq 2} \times \boldsymbol{Z}^{n-1}_{\geq 2}$. Thus the assertions (6.8) and (6.9) follows from Theorem 6.6. This completes the proof of Proposition 6.9.

REFERENCES

[1] R. E. EDWARDS, Fourier Series, Vol. 2, a Modern Introduction, Second edition, Grad. Texts in Math. 85, Springer-Verlag, New York, 1982.
[2] F. HAZAMA, Discrete tomography by hook-shapes (submitted).
[3] F. HAZAMA, Algebraic cycles on certain abelian varieties and powers of special surfaces, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 31 (1985), 487–520.
[4] F. HAZAMA, Hodge cycles on the Jacobian variety of the Catalan curve, Compositio Math. 107 (1997), 339–353.
[5] F. HAZAMA, Hodge cycles on abelian varieties with complex multiplication by cyclic CM-fields, J. Math. Soc. Univ. Tokyo 10 (2003), 581–598.
[6] T. KUBOTA, On the field extension by complex multiplication, Trans. Amer. Math. Soc. 118 (1965), 113–122.
[7] J. D. LEWIS, A survey of the hodge conjecture, Second edition, CRM Monograph Series, Vol. 10, American Mathematical Society, Providence, R.I., 1999.
[8] M. NIVAT, Sous-ensembles homogenes de $\boldsymbol{Z}^2$ et pavage du plan, C. R. Math. Acad. Sci. Paris 335 (2002), 83–86.
[9] M. NIVAT, On a tomographic equivalence between (0,1)-matrices, in Theory is forever, 216–234, Lecture Notes in Comput. Sci. 3113, Springer, Berlin, 2004.
[10] K. RIBET, Division fields of abelian varieties with complex multiplication, Mém. Soc. Math. France (2e serie) 2 (1980/81), 75–94.
[11] I. J. SCHOENBERG, A note on the cyclotomic polynomial, Mathematika 11 (1964), 131–136.
[12] T. SHIODA, Algebraic cycles on abelian varieties of Fermat type, Math. Ann. 258 (1981/82), 65–80.

DEPARTMENT OF NATURAL SCIENCES
COLLEGE OF SCIENCE AND ENGINEERING
TOKYO DENKI UNIVERSITY
HATOYAMA, SAITAMA 350–0394
JAPAN