Notes on Fourier Analysis (VIII);
Local properties of Fourier series.

By

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The object of this paper is to prove three localization theorems of Fourier series concerning absolute Riesz logarithmic summability. These are the analogue of theorems concerning absolute Cesàro summability.

Theorem 1. *If* $0<\alpha<\beta<2\pi$, *then there is an integrable function which is zero outside the interval* (α,β) *and whose fourier series is not summable* $|R, \log n, 1|$ *at* $t=0$. *That is,* $|R, \log n, 1|$ *summability is not the local property (in the ordinary sense).*

Proof. Let $s_v(x)$ be the $(v+1)$-th partial sum of the Fourier series of $f(x)$ and let

$$R_n \equiv \frac{1}{\log n}\sum_{v=1}^{n}\frac{s_v(0)}{v}.$$

Now

$$R_n - R_{n+1} = \frac{1}{\log n}\sum_{v=1}^{n}\frac{s_v(0)}{v} - \frac{1}{\log(n+1)}\sum_{v=1}^{n+1}\frac{s_v(0)}{v}$$

$$= \int_0^{2\pi} f(t)\left[\frac{1}{\log n}\sum_{v=1}^{n}\frac{D_v(t)}{v} - \frac{1}{\log(n+1)}\sum_{v=1}^{n+1}\frac{D_v(t)}{v}\right]dt,$$

where $D_v(t)$ dnotes the Dirichlet kernel. We have to prove the existence of a function $f(t)$ which is equal to zero outside the interval (α,β) and such that $\sum_{n=1}^{\infty}|R_n - R_{n+1}| = \infty$. This follows from the divergence of

$$\sum_{n=1}^{\infty}\left|\int_\alpha^\beta f(t)\left[\frac{1}{\log n}\sum_{v=1}^{n}\frac{\sin vt}{vt} - \frac{1}{\log(n+1)}\sum_{v=1}^{n+1}\frac{\sin vt}{vt}\right]dt\right|^{2)}.$$

We need the following

Lemma[1]. *Let* $\{g_n(t)\}$ *be a sequence of bounded measurable functions in the interval* (α,β). *Then a necessary and sufficient condition that, for any* *integrable function* $f(t)$,

$$\sum_{n=1}^{\infty} \left| \int_{\alpha}^{\beta} f(t)\, g_n(t)\, dt \right| < \infty$$

is that $\sum_{n=1}^{\infty} |g_n(t)|$ *s essentially bounded in the interval* (α,β).

If we put

$$\Delta_n(t) \equiv \frac{1}{\log n} \sum_{v=1}^{n} \frac{\sin vt}{v} - \frac{1}{\log(n+1)} \sum_{v=1}^{n+1} \frac{\sin vt}{v},$$

then, by Lemma, it is sufficint to prove that the series

$$J \equiv \sum_{n=2}^{\infty} |\Delta_n(t)|$$

is not essentially bounded.

$$J = \sum_{n=2}^{\infty} \left| \frac{1}{n \log^2 n} \sum_{v=1}^{n} \frac{\sin vt}{v} - \frac{1}{\log(n+1)} \frac{\sin(n+1)t}{n+1} \right| + O(1)$$

$$\geqq \sum_{n=2}^{\infty} \frac{|\sin nt|}{n \log n} - \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \left| \sum_{v=1}^{n} \frac{\sin vt}{v} \right| - O(1).$$

If we put $\varphi(t) \equiv (\pi - t)/2$ in $(0,2\pi)$ and $\varphi(t) = \varphi(t + 2\pi)$, then

$$\varphi(t) = \sum_{v=1}^{\infty} \frac{\text{siu } vt}{v}$$

and its partial sum is uniformly bounded in the interval (α,β). Hence

$$J \geqq \sum_{n=1}^{\infty} \frac{|\sin nt|}{n \log n} - O(1).$$

The right hand side series is divergent for $t \neq 0$ by the Fatou theorem. Thus the theorem is proved.

Theorem 2. *If* $a_n = o(1/\log^2 n)$, $b_n = o(1/\log^2 n)$, *then the* $|R, \log n, 1|$ *summability has local property.*

Proof. Let $0 < \delta < \pi$. It is sufficient to prove that

$$\sum_{n=2}^{\infty} \left| \int_{\delta}^{\pi} f(t) \left[\frac{1}{n \log^2 n} \sum_{v=1}^{n} \frac{\sin(v+1/2)t}{2v\sin t/2} - \frac{1}{\lg(n+1)} \frac{\sin(n+3/2)t}{2(n+1)\sin t/2} \right] dt \right|$$

is bounded for every even function $f(t)$ with Fourier coefficients satisfying the condition in the theorem since

$$\sum_{v=1}^{n} \frac{\sin vt}{v} = \frac{\pi - t}{2} + O\left(\frac{1}{nt}\right) = O(1)$$

in (δ, π), we have

$$\sum_{n=2}^{\infty} \frac{1}{n \log^2 n} \left| \int_{\delta}^{\pi} f(t) \sum_{v=1}^{n} \frac{\sin(v + 1/2)t}{2v\sin t/2}\, dt \right| = O\left(\sum_{n=2}^{\infty} \frac{1}{n \log^2 n}\right) = O(1).$$

Hence it is sufficient to prove that

$$\sum_{n=2}^{\infty} \frac{1}{n \log n} \left| \int_{\delta}^{\pi} f(t) \sin nt\, dt \right| = O(1).$$

For, the required one is quite similarly estimated. By

$$f(t) \sim \sum_{n=1}^{\infty} a_n \cos nt,$$

we have

$$\int_{\delta}^{\pi} f(t) \sin nt\, dt = a_n \int_{\delta}^{\pi} \cos nt \sin nt\, dt + \sum_{v=1}^{\infty} a_v \int_{\delta}^{\pi} \cos vt \sin nt\, dt$$

$$= \frac{a_n}{2n} \sin^2 n\delta + \sum_{\substack{u=1 \\ (n \neq v)}}^{\infty} a_v \left(\frac{\sin(n + w)\delta}{n + v} - \frac{\sin(n - v)\delta}{n - v} \right).$$

Firstly

$$\sum_{n-2}^{\infty} \frac{1}{n \log n} \left| \frac{a_n}{2n} \sin^2 n\delta \right| \leqq \sum_{n=2}^{\infty} \frac{1}{2n^2 \log^3 n} < \infty.$$

Secondly

$$\sum_{v=2}^{\infty} |a_v| \frac{|\sin(n + v)\delta|}{n + v} \leqq \sum_{v=2}^{\infty} \frac{1}{(n+v) \log^2 v}$$

$$= \sum_{v=2}^{n} + \sum_{v=n+1}^{\infty} = I_{1,n} + I_{2,n},$$

say. Hence

$$\sum_{n=2}^{\infty} \frac{I_{1,n}}{n \log n} = \sum_{v=2}^{\infty} \sum_{n=v}^{\infty} \frac{1}{\log^2 v \ (n+v) n \log n}$$

$$\leqq 4 \sum_{v=2}^{\infty} \sum_{n=v}^{\infty} \frac{1}{\log^2 v \ (n+v)^2 \log(n+v)} \leqq 4 \sum_{v=2}^{\infty} \frac{1}{v \log^3 v} < \infty,$$

$$\sum_{n=2}^{\infty} \frac{I_{2,n}}{n \log n} \leqq \sum_{n=2}^{\infty} \sum_{n=v}^{\infty} \frac{1}{v \log^2 v} \ \frac{1}{n \log^2 n}$$

$$\leqq C \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < \infty.$$

Finally

$$\sum_{\substack{v=2 \\ (v \neq n)}}^{\infty} |a_v| \frac{|\sin(n-v)\delta|}{n-v} \leqq \sum_{\substack{v=n \\ (v \neq n)}}^{\infty} \frac{1}{|n-v| \log^2 v}$$

$$= \sum_{v=2}^{n-1} + \sum_{v=n+1}^{2n} + \sum_{v=3n+1}^{\infty} \equiv J_{1,n} + J_{3,n} + J_{3,n},$$

say. We have

$$\sum_{n=2}^{\infty} \frac{J_{1,n}}{n \log n} = \sum_{v=2}^{\infty} \sum_{n=v+1}^{\infty} \frac{1}{\log^2 v \ (n-v) n \log n}$$

$$= \sum_{v=2}^{\infty} \Big(\sum_{n=n+1}^{2v} + \sum_{n=2v+1}^{\infty} \Big)$$

$$\leqq \sum_{v=2}^{\infty} \frac{1}{v \log^3 v} \sum_{n=v+1}^{2v} \left(\frac{1}{n-v} - \frac{1}{n} \right) + \sum_{v=2}^{\infty} \frac{2}{\log^2 v} \sum_{v=n}^{\infty} \frac{1}{n^2 \log n}$$

$$\leqq \sum_{v=2}^{\infty} \frac{1}{v \log^2 v} + 2 \sum_{v=3}^{\infty} \frac{1}{v \log^2 v} < \infty,$$

$$\sum_{n=2}^{\infty} \frac{J_{2,n}}{n \log n} \leqq 2 \sum_{n-2}^{\infty} \frac{1}{n \log n} \cdot \frac{1}{\log^2 n} \sum_{v=n+1}^{2n} \frac{1}{v-n}$$

$$\leqq 2 \sum_{n=2}^{\infty} \frac{1}{n \log^2 n} < \infty,$$

and

$$\sum_{n=2}^{\infty} \frac{J_{3,n}}{n \log n} \leqq \sum_{n=3}^{\infty} \frac{2}{n \log^2 n} < \infty.$$

Thus the theorem is proved.

Theorom 3.[3)] *If $f(x)$ is an integrable function such that for any y in the closed interval $(-\pi, \pi)$ there are function $f_y(x)$ and a $\delta > 0$ such as*

$$f_y(x) = f(x) \text{ for } |x-y| < \delta$$

and the Fourier series of $f_y(x)$ *is* $|R, \log n, 1|$ *summable, then the Fourier series of* $f(x)$ *is* $|R, \log n, 1|$ *summable. That is,* $|R, \log n, 1|$ *summability is the local propcrty in the Wiener sense.*

Proof. By the Borel's covering theorem, there are a finite number of overlapping intervals (d_i, d_i') covering $(-\pi, \pi)$ and functions $f_i(x)$ which coincide with $f(x)$ on (d_i, d_i') and is equal to zero outside the interval (d_i, d_i') and whose Fourier series is $|R, \log n, 1|$ summable. We can suppose that

$$d_i < d_{i-1}' < d_{i+1} < d_i'.$$

Let $g_i(x)$ be defined such that

$$g_i(x) = 1 \qquad (d_{i-1}' \leqq x \leqq d_{i+1}),$$

$$= 0 \qquad (x < d_i^*,\ d_i' < x),$$

and $0 \leqq g_i(x) \leqq 1$, $g_i(x) = 1$, and that the fcurth derivative $g''''(x)$ exists everywhere.

The n-th Fourier coefficient $c_n(g_i)$ of $g_i(x)$ is

$$c_n(g_i) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} g_i''''(x)\,\frac{e^{inx}}{x^4}\,dx = O\left(\frac{1}{n^3}\right).$$

Since $\sum g_i(x) = 1$, we have

$$f(x) = \sum f_i(x) g_i(x)$$

where

$$c_n(f_i g_i) = \sum_{m=-\infty}^{\infty} c_m(f_i)\ c_{n-m}(g_i),$$

the right hand side series evidently being convergent. We use the abbreviation

$$c_n(f_i g_i) \equiv c_n,\ c_m(f_i) \equiv b_m,\ c_m(g_i) \equiv a_m.$$

For the proof of the theorem, it is sufficient to prove that the Fourier series of $f_i(x)g_i(x)$ is $|R, \log n, 1|$ summable. Let $s_v(x)$ be the $(v+1)$-th partial sum of the Fourier series of $f_i(x)g_i(x)$, and let

$$R_n(x) \equiv \frac{1}{\log n}\sum_{v=1}^{n} \frac{s_v(x)}{v}.$$

Now

$$R_n(x) - R_{n+1}(x) = \frac{1}{\log n}\sum_{v=1}^{n}\frac{s_v(x)}{v} - \frac{1}{\log(n+1)}\sum_{v=1}^{n+1}\frac{s_v(x)}{v}$$

$$= \frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{s_v(x)}{v} - \frac{s_{n+1}(x)}{(n+1)\log(n+1)} + \varepsilon_n$$

$$= \frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{1}{v}\sum_{\mu=-v}^{v} c_\mu e^{i\mu x} + \frac{1}{(n+1)\log(n+1)}\sum_{\mu=-n-1}^{n+1} c_\mu e^{i\mu x} + \varepsilon_n,$$

where $\sum |\varepsilon_n| < \infty$. Since $c_\mu = \sum_{\lambda=-\infty}^{\infty} a_\lambda b_{\mu-\lambda}$, we have

$$R_n(x) - R_{n+1}(x) = \frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{1}{v}\sum_{\mu=-v}^{v} e^{i\mu x}\sum_{\lambda=-\infty}^{\infty} a_\lambda b_{\mu-\lambda}$$

$$- \frac{1}{(n+1)\log(n+1)}\sum_{\mu=-n-1}^{n+1} e^{i\mu x}\sum_{\lambda=-\infty}^{\infty} a_\lambda b_{\mu-\lambda}$$

$$= \frac{1}{n\log^2 n}\sum_{\lambda=-\infty}^{\infty} a_n e^{i\mu x}\sum_{v=1}^{n}\frac{1}{v}\sum_{\mu=-v}^{v} b_{\mu-\lambda} e^{i(\mu-\lambda)x}$$

$$- \frac{1}{(n+1)\log(n+1)}\sum_{\lambda=-\infty}^{\infty} a_\lambda e^{i\lambda x}\sum_{\mu=-n-1}^{n+1} b_{\mu-\lambda} e^{i(\mu-\lambda)x}.$$

Hence

$$\sum_{n=2}^{\infty} |R_n(x) - R_{n+1}(x)|$$

$$\leqq \sum_{\lambda=-\infty}^{\infty} |a_\lambda| \sum_{n=1}^{\infty}\left| \frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{1}{v}\sum_{\mu=-v}^{v} b_{\mu-\lambda} e^{i(\mu-\lambda)x} - \frac{1}{(n+1)\log(n+1)}\sum_{\mu=-n-1}^{n+1} b_{\mu-\lambda} e^{i(\mu-\lambda)x}\right|$$

$$\leqq \sum_{\lambda=-\infty}^{\infty} |a_\lambda| \sum_{n=2}^{\infty}\left| \frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{1}{v}\sum_{k=-v-\lambda}^{v-\lambda} b_k e^{ikx} - \frac{1}{(n+1)\log(n+1)}\sum_{k=-n-1-\lambda}^{n+1-\lambda} b_k e^{ikx}\right|$$

$$\leqq \sum_{\lambda=-\infty}^{\infty} |a_\lambda| \sum_{n=1}^{\infty}\left| \frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{1}{v}\sum_{k=-v}^{v} b_k e^{ikx} - \frac{1}{(n+1)\log(n+1)}\sum_{k=-n-1}^{n+1} b_k e^{ikx}\right|$$

$$+\sum_{\lambda=-\infty}^{\infty}|a_\lambda|\sum_{n=2}^{\infty}\left|\frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{1}{v}\left[\sum_{k=v-\lambda}^{v}b_k e^{ikx}-\sum_{k=-x-\lambda}^{-y}b_k e^{ikx}\right]\right.$$

$$\left.+\frac{1}{(n+1)\log(n+1)}\left[\sum_{k=-n-1-\lambda}^{-n-1}b_k e^{ikx}-\sum_{k=n+1-\lambda}^{n+1}b_k e^{ikx}\right]\right|.$$

The first term of the right hand side is $O\left(\sum_{\lambda=-\infty}^{\infty}|a_\lambda|\right)=O(1)$ by the hypothesis. Cencerning the second term, the general term of the inner sum is

$$\frac{1}{\pi}\int_0^{2\pi}f_i(x+u)\left[\frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{1}{v}\left(\sum_{k=v-\lambda}^{v}e^{iku}-\sum_{k=-v-\lambda}^{-v}e^{iku}\right)\right.$$

$$\left.+\frac{1}{(n+1)\log(n+1)}\left(\sum_{k=-n-1-\lambda}^{-n-1}e^{iku}-\sum_{k=n+1-\lambda}^{n+1}e^{iku}\right)\right]du$$

$$=\frac{2i}{\pi}\int_0^{2\pi}f_j(x+u)\left[\frac{1}{n\log^2 n}\sum_{v=1}^{u}\frac{\sin vu}{v}\right.$$

$$\left.-\frac{\sin(n+1)u}{(n+1)\log(n+1)}\right]\left(\sum_{k=1}^{\lambda}e^{-kui}\right)du$$

$$=\frac{2i}{\pi}\sum_{k=1}^{\lambda}\int_0^{2\pi}f_i(x+u)\left[\frac{1}{n\log^2 n}\sum_{v=1}^{n}\frac{\sin vu}{v}\right.$$

$$\left.-\frac{\sin(n+1)u}{(n+1)\log(n+1)}\right]e^{ixn}\,du,$$

which we denote by J_n. Then $\sum|J_n|=O(|\lambda|)$. For, if (s_n) is $|R,\log n,1|$ summable, then (a_n) is also.[4]

Thus the theorem is proved.[5]

Foot Notes

*) Received June 30th, 1949.

1) Randels, Bull. Am. Math. Soc., 1940. Bosanquet, Proc. London Math. Soc., 41 (1936), Bosanquet-Kestleman, ibidem, 45 (1938).

2) We can easily verify that the difference of this term and that replaced $\frac{\sin vt}{t}$ by $\frac{\sin(v+1/2)t}{2\sin t/2}$ converges absolutely. But the following arguement holds taking $\frac{\sin(v+1/2)t2}{\sin t/2}$ instead of $\frac{\sin vt}{t}$.

3) Mr. N. Matsuyama proved also this theorem after Randel's idea. See N. Matsuyama, Monthly of Real Analysis, 3 (1949) (in Japanese).

4) As an alternative proof, it is sufficient to prove that $\sum|a_n|/n\log n<\infty$. Now $s_n=n[\log n\cdot R_r-\log(n-1)\cdot R_{n-1}]$, whence

$$\sum_{n=2}^{\infty}\frac{a_n}{n\log n}=\sum_{n=2}^{\infty}\frac{|s_n-s_{n-1}|}{n\log n}$$

$$\leqq 2\sum_{n=2}^{\infty}|R_n-R_{n-1}|+3\sum_{n=2}^{\infty}\frac{|R_n-R_{n-1}|}{n}+\sum_{n=2}^{\infty}\frac{|R_n|}{n^2}<\infty.$$

This lemma is due to Mr. N. Matsuyama.

5) The author expresses his hearty thanks to Prof. A. Zygmund who gve me valuable remarks. Especially the original proof of Theorem 3 was incomplete.