ON A GENERALIZATION OF THE PRINCIPAL IDEAL THEOREM*)

By

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Preface (BY TADAO TANNAKA)

It is well known, for the case of cyclic class field, the principal ideal theorem of Furtwängler can be generalized in the form, "*All ambigous ideals in the class field are principal.*" I conjectured several years ago, that this allows the following generalization, "*If K/k is the arbitrary abelian absolute class field, then all the ambigous ideals in the cyclic subfield of K are principal, when considered as ideals in K.*"

I found this fact to be true by several p-groups of low degrees, but could not prove in general. Recently Mr. Terada took up this problem again, and he and I tried to solve it with great effort. After a considerably complicated calculation, he succeeded at last to master the problem. The following paper is the result of his hard work. I also succeeded to generalize the arithmetic part of Iyanaga's general principal ideal theorm in this form. This will be seen in my paper in this same volume†).

§ 1.

1. Introduction. Let k be a finite algebraic extension field of the rational field, and let K and $\bar{K}$ be the first and the second absolute class field of k, respectively. The field $\bar{K}$ is a normal extension field of k, and the Galois group $\mathfrak{G}$ of $\bar{K}/k$ is the so-called "*zweistufig metabelsche Gruppe*" (Hasse [1], p. 172); that is, the second commutator subgroup $\mathfrak{G}''$ of $\mathfrak{G}$ is an unit group. The subgroup of $\mathfrak{G}$ corresponding to K is, as is well known, the first commutator subgroup $\mathfrak{G}'$ of $\mathfrak{G}$.

Moreover, let Ω be a cyclic extension field of k, contained in K, and let $\mathfrak{H}$ be the corresponding subgroup of $\mathfrak{G}$. The ambigous class C of Ω is an absolute class of Ω such that $C^s=C$, where s is a generator of the cyclic group $\mathfrak{G}/\mathfrak{H}$. It is clear that this definition does not depend on the choice of s. An ideal, contained in one of the ambigous classes, will be called

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†) Meanwhile Tannaka has obtained a new treatment, which needs only few a pages.

an *ambigous* ideal in this paper. Any ideal in k is contained in one of the ambigous classes of Ω, but the converse is not true in general.

The principal ideal theorem of Furtwängler is that "*Any ideal in k is a principal ideal in K*," (Furtwängler [1], Hasse [1]). As stated in the preface, if K/k is the cyclic extention field, then this allows the following generalization "*Any ambigous ideal in K is principal*." Prof. Tannaka had conjectured the more generalized theorem, "*Any ambigous ideal in Ω is principal in K*." I shall prove this generalized principal ideal theorem in this paper.

2. LEMMA 1. *The subfield $\bar{\Omega}$ of $\bar{K}$, corresponding to the commutator subgroup $\mathfrak{H}'$ of $\mathfrak{H}$ is the absolute class field of Ω.*

PROOF. As is well known, the first absolute class field Ω' of Ω is the greatest, non-ramified, abelian extension field of Ω. Therefore it is easy to see that $\bar{\Omega}\subset\Omega'$. Since K is a non-ramified abelian extension field of Ω, K is contained in Ω'. Then Ω' is a non-ramified abelian extension field of K, and therefore is contained in $\bar{K}$, which is the first absolute class field of K; and therefore, is contained in $\bar{\Omega}$, for $\bar{\Omega}$ is the greatest abelian extension field of Ω contained in $\bar{K}$.

Following notations will be used in this paper:

s:	a fixed representative of the generator of the cyclic group $\mathfrak{G}/\mathfrak{H}$ in $\mathfrak{G}/\mathfrak{G}'$ $(s\epsilon\mathfrak{G}/\mathfrak{G}')$,
e:	the order of s with respect to $\mathfrak{H}$,
$\xi,\ \eta,\dots$:	general elements of $\mathfrak{H}/\mathfrak{G}'$,
$u(x)$:	a fixed representative of an element $x\epsilon\mathfrak{G}/\mathfrak{G}'$,
$[x,\ y]$:	an element $u(x)u(y)u(x)^{-1}u(y)^{-1}$ of $\mathfrak{G}'$, where $x,\ y\epsilon\mathfrak{G}/\mathfrak{G}'$,
g^x:	an element $u(x)gu(x)^{-1}$ of $\mathfrak{G}$, where $x\epsilon\mathfrak{G}/\mathfrak{G}'$ and $g\epsilon\mathfrak{G}'$,
$g^{\Sigma a_ix_i}$:	an element $\prod_i (g^{x_i})^{a_i}$ of $\mathfrak{G}'$, where $x_i\epsilon\mathfrak{G}/\mathfrak{G}'$ and a_i is an integer.

By the last definition, the group-ring $[\mathfrak{G}/\mathfrak{G}']$ of $\mathfrak{G}/\mathfrak{G}'$ may be considered as an operator-ring of the group $\mathfrak{G}'$, (Hasse [1]).

By the Artin's law of reciprocity, any ideal $\mathfrak{A}$ in Ω corresponds to an element $A=\left(\frac{\bar{\Omega}/\Omega}{\mathfrak{A}}\right)$ of $\mathfrak{H}$ mod. $\mathfrak{H}'$, and also to an element $V_{\mathfrak{H}\to\mathfrak{G}'}(A)=\left(\frac{\bar{K}/K}{\mathfrak{A}}\right)$ of $\mathfrak{G}'$, when considered as an ideal in K. If $\mathfrak{A}$ is an ambigous ideal, then $\mathfrak{A}^{s-1}$ is a principal ideal in Ω, so that we have by the reciprocity law and lemma 1

$$(1)\qquad A^{s-1}\equiv\left(\frac{\bar{\Omega}/\Omega}{\mathfrak{A}}\right)^{s-1}\equiv\left(\frac{\bar{\Omega}/\Omega}{\mathfrak{A}^{s-1}}\right)\equiv 1\qquad(\text{mod. }\mathfrak{H}').$$

Moreover, it is well known that

$$(2)\qquad V_{\mathfrak{H}\to\mathfrak{G}'}(A)=\prod v(\tau)A^f v(\tau)^{-1},$$

where the product is extended over all elements $\tau,\ldots$ of $\mathfrak{H}/\{\mathfrak{G}', A\}$, and $v(\tau)$ is the representative of τ mod. $\{\mathfrak{G}', A\}$, and f is the order of A with respect to $\mathfrak{G}'$ (Hasse [1] p. 171).

The right side of (2) is equal to

$$\prod_{\tau} v(\tau)A(v(\tau)A)^{-1}\cdot(v(\tau)A)\cdot A(v(\tau)A^2)^{-1}\ldots$$
$$(v(\tau)A^{f-2})A(v(\tau)A^{f-1})^{-1}(v(\tau)A^{f-1})Av(\tau)^{-1},$$

and the elements $v(\tau)$, $v(\tau)A$, $\ldots$, $v(\tau)A^{f-1}$, $(\tau\epsilon\mathfrak{H}/\{\mathfrak{G}', A\})$ form a complete system of representatives of $\mathfrak{H}$ mod. $\mathfrak{G}'$. Therefore, from (2) we have

$$V_{\mathfrak{H}\to\mathfrak{G}'}(A) = \prod u'(\xi)Au'(\xi\sigma)^{-1} \tag{3}$$

where σ is an element of $\mathfrak{H}/\mathfrak{G}'$ which contains A. Moreover let $u'(\xi)=g_\xi u(\xi)$ then

$$u'(\xi)Au'(\xi\sigma)^{-1} = g_\xi g_{\xi\sigma}^{-1}u(\xi)Au(\xi\sigma)^{-1},$$
$$\prod_{\xi} u'(\xi)Au'(\xi\sigma)^{-1} = \prod_{\xi} u(\xi)Au(\xi\sigma)^{-1},$$

and we have from (3)

$$V_{\mathfrak{H}\to\mathfrak{G}'}(A) = \prod u(\xi)Au(\xi\sigma)^{-1}. \tag{4}$$

The necessary and sufficient condition for $\mathfrak{A}$ to be a principal ideal in K is that

$$V_{\mathfrak{H}\to\mathfrak{G}'}(A) = \left(\frac{\bar{K}/K}{\mathfrak{A}}\right) = 1.$$

Therefore our theorem is translated to the

REDUCTION THEOREM 1. *For any element A of $\mathfrak{H}$ such that $A^{s-1}\epsilon\mathfrak{H}'$ we have*

$$V_{\mathfrak{H}\to\mathfrak{G}'}(A) = \prod_{\xi} u(\xi)Au(\xi\sigma)^{-1} = 1,$$

where this product is extended over all $\xi\epsilon\mathfrak{H}/\mathfrak{G}'$ and σ is an element of $\mathfrak{H}/\mathfrak{G}'$ which contains A.

3. Let $\mathfrak{H}^{s-1}$ be the subgroup of $\mathfrak{G}'$ generated by all the elements $H^{s-1}=u(s)Hu(s)^{-1}H^{-1}$, $H\epsilon\mathfrak{H}$. Then we have

LEMMA 2. $$\mathfrak{G}' = \mathfrak{H}^{s-1}\mathfrak{H}'.$$

PROOF. 1. It is easy to see that $\mathfrak{H}^{s-1}\mathfrak{H}'\subset\mathfrak{G}'$.

2. A system of generators of $\mathfrak{G}'$ consists of all the elements of the form $G_1G_2G_1^{-1}G_2^{-1}$, $G_i\epsilon\mathfrak{G}$. If

$$G_1 = u(s)^i u(\xi)g_1 \qquad g_1\epsilon\mathfrak{G}'$$
$$G_2 = u(s)^j u(\eta)g_2 \qquad g_2\epsilon\mathfrak{G}',$$

then

$$G_1G_2G_1^{-1}G_2^{-1}$$
$$= (u(s)^i u(\xi))(u(s)^j u(\eta))(u(s)^i u(\xi))^{-1}(u(s)^j u(\eta))^{-1}$$
$$\cdot(g^{1-s^j\eta})^{s^i\xi}(g_2^{1-s^i\xi})^{s^{i+j}\xi\eta}.$$

On the other hand, we get

$$\begin{aligned}[s^i\xi,\ s^j\eta] &= (u(s)^i u(\xi))(u(s)^j u(\eta))(u(s)^i u(\xi))^{-1}(u(s)^j u(\eta))^{-1} \\ &\equiv 1 \qquad (\mathrm{mod.}\ \mathfrak{H}^{1-s}\mathfrak{H}'),\end{aligned}$$

and

$$g^{1-s^j\eta} \equiv g^{1-s^i\xi} \equiv 1 \qquad (\mathrm{mod.}\ \mathfrak{H}^{1-s}\mathfrak{H}').$$

We can prove them by induction with respect to the exponents, for

$$[s^i\xi,\ s^j\eta] = [s^{i-1}\xi,\ s^j\eta]^s[s,\ \eta]^{s^j},$$

$$g^{1-s^j\eta} = g^{1-s}(g^s)^{1-s^{j-1}\eta}.$$

And we may conclude our lemma 2.

Let the abelian group $\mathfrak{H}/\mathfrak{G}'$ be expressed as a direct product of cyclic groups $\mathfrak{C}_1, \ldots, \mathfrak{C}_n$ with generators $\xi_1, \ldots, \xi_n$, respectively, and let e_i be the order of ξ_i. Then, each element of $\mathfrak{H}$ has the form $\prod u(\xi_i)^{\alpha_i}g$, and then, more precisely than lemma 2, we get

LEMMA 3.

$$\mathfrak{G}' = \{g^{s-1},\ g^{\xi_i-1},\ [\xi_i,\ s],\ [\xi_i,\ \xi_j]\} \qquad i, j = 1, \ldots, n, \tag{5}$$

$$\mathfrak{H}' = \{h^{s-1},\ h^{\xi_i-1},\ [\xi_i,\ s]^{s_j-1},\ [\xi_i,\ \xi_j]\}, \tag{6}$$

where g runs over all the elements of $\mathfrak{G}'$ and so is h of $\mathfrak{H}'$. The sign $\{\quad\}$ means that, $\mathfrak{G}'$, or $\mathfrak{H}'$, has these elements as its generators.

PROOF. By an analogous calculation as in the lemma 2, it is easy to see that

$$\mathfrak{H}' = \{g^{\xi_i-1},\ [\xi_i,\ \xi_j]\,;\ i, j = 1, \ldots, n\}. \tag{6'}$$

Starting from lemma 2, we get, by the same method as above, that

$$\mathfrak{G}' = \{\mathfrak{H}',\ \mathfrak{G}^{s-1},\ [\xi_i,\ s]\,;\ i = 1, \ldots, n\}. \tag{5'}$$

Now, putting (6) into (5)′, we may conclude (5), and from (5) and (6)′, we get the expression (6).

If $\mathfrak{G}$ is a p-group, by a successive application of lemma 3, as was carried out by Furtwängler (Furtwängler [1], Hasse [1], p. 185), we get

LEMMA 4. *If $\mathfrak{G}$ is a p-group, then*

$$\mathfrak{G}' = \{[\xi_i,\ s],\ [\xi_i,\ \xi_j]\,;\ i, j = 1, \ldots, n\}, \tag{7$_1$}$$

$$\mathfrak{H}' = \{[\xi_i,\ s]^{\xi_j-1},\ [\xi_i,\ \xi_j]\,;\ i, j = 1, \ldots, n\}, \tag{7$_2$}$$

where the sign $\{\quad\}$ means that, $\mathfrak{G}'$, or $\mathfrak{H}'$, has these elements as its generators permitting the symbolic power.

4. As is easily verified from (4),

$$V_{\mathfrak{H}\to\mathfrak{G}'}(AB) = V_{\mathfrak{H}\to\mathfrak{G}'}(A)\,V_{\mathfrak{H}\to\mathfrak{G}'}(B), \quad A,\ B\in\mathfrak{H} \tag{8}$$

(Hasse [1], p. 162), and if ξ is an element of $\mathfrak{H}/\mathfrak{G}'$ then

$$V_{\mathfrak{H}\to\mathfrak{G}'}(A)^{\xi} = V_{\mathfrak{H}\to\mathfrak{G}'}(A), \tag{9}$$

for $V_{\mathfrak{H}\to\mathfrak{G}'}(A)=\left(\dfrac{\overline{K}/K}{\mathfrak{A}}\right)$ and $\mathfrak{A}^{\xi}=\mathfrak{A}$. Furthermore, as was shown by deriving

(4) from (3),

(10) $V_{\mathfrak{H}\to\mathfrak{G}'}(A) = \prod u(\xi)Au(\xi\sigma)^{-1}$ does not depend on the choice of representatives $u(\xi)$ mod. $\mathfrak{G}'$.

To avoid the complication, we use following notations throughout this paper.

$$M_i^{(\gamma_i)} = 1 + \xi_i + \ldots + \xi_i^{\gamma_i-1} \qquad (0 < \gamma_i \leqslant e_i),$$
$$M_i = M_i^{(e_i)},$$
$$\theta = \prod_{i=1}^{n} M_i, \ \theta_i = \prod_{\substack{j=1 \\ j\neq i}}^{n} M_j,$$
$$\Gamma_i = M_i^{(\gamma_i)}\xi_1^{\gamma_1}\ldots\xi_{i-1}^{\gamma_{i-1}} \qquad (0 < \gamma_i \leqslant e_i).$$

From (10) we may change the representatives as follows:

$$u(\xi_i) = arbitrary,$$
$$u(\xi) = \prod_{i=1}^{n} u(\xi_i)^{\alpha_i} \qquad if \qquad \xi = \prod_{i=1}^{n} \xi_i^{\alpha_i}.$$

Then, as is well known,

(11) $$V_{\mathfrak{H}\to\mathfrak{G}'}(u(\xi_i)) = \prod_{\substack{\beta_j=0 \\ \beta_i=0}}^{e_j-1} u(\xi_1)^{\beta_1}\ldots u(\xi_n)^{\beta_n}u(\xi_i)^{e_i}u(\xi_n)^{-\beta_n}\ldots u(\xi_1)^{-\beta_1}$$
$$= u(\xi_i)^{e_i\theta_i},$$

((2), Hasse [1], Schumann [1]). Moreover, if $\sigma = \prod_{i=1}^{n} \xi_i^{\gamma_i}$, $(0 \leqslant \gamma_i < e_i)$ then

$$u(\sigma) = \prod_{i=1}^{n} u(\xi_i)^{\gamma_i},$$

and then, from (8), we have

(12) $$V_{\mathfrak{H}\to\mathfrak{G}'}(u(\sigma)) = \prod_{i=1}^{n} V_{\mathfrak{H}\to\mathfrak{G}'}(u(\xi_i))^{\gamma_i}.$$

Putting all $\xi_i = 1$ in Γ_i, we get γ_i, and therefore from (9), (12), we have

$$V_{\mathfrak{H}\to\mathfrak{G}'}(u(\sigma)) = \prod_{i=1}^{n} V_{\mathfrak{H}\to\mathfrak{G}'}(u(\xi_i))^{\Gamma_i},$$

from (11)

(13) $$= \prod_{i=1}^{n} u(\xi_i)^{e_i\Gamma_i\theta_i}.$$

On the other hand

$$u(\sigma)^{s-1} = [s, \ u(\sigma)] = [s, \ \prod u(\xi_i)^{\gamma_i}].$$

By using the relation

$$[a, \ bc] = [a, \ b][a, \ c]^b,$$

concerning the commutators, we get by a simple calculation

(14) $$u(\sigma)^{s-1} = \prod_{i=1}^{n} [s, \ \xi_i]^{\Gamma_i}.$$

Now let $A = u(\sigma)g$, $g \in \mathfrak{G}'$, then

$$A^{s-1} \equiv u(\sigma)^{s-1}g^{s-1} \qquad (\text{mod. } \mathfrak{H}'),$$

and therefore the assumption in Reduction theorem 1 is equivalent to the condition

$$(15) \qquad \prod_{i=1}^{n}[s,\ \xi_i]^{\Gamma_i} g^{s-1} \equiv 1 \qquad (\text{mod. } \mathfrak{H}').$$

From (8) and (13),

$$(16) \qquad V_{\mathfrak{H}\to\mathfrak{G}'}(A) = \prod_{i=1}^{n} u(\xi_i)^{e_i\Gamma_i\theta_i} g^{\theta}.$$

Therefore, the reduction theorem 1 is translated to the

REDUCTION THEOREM 2. *If the element* $\prod_{i=1}^{n}[s,\ \xi_i]^{\Gamma_i} g^{s-1}$ *of* $\mathfrak{G}'$ *is contained in* $\mathfrak{H}'$, *then we have*

$$\prod_{i=1}^{n} u(\xi_i)^{e_i\Gamma_i\theta_i}\cdot g^{\theta} = 1.$$

5. LEMMA 5. *It is sufficient to prove the theorem for a p-group.*[1]

PROOF. It is easy to see that, $V_{\mathfrak{H}\to\mathfrak{G}'}(h) = 1$, where h is an element of $\mathfrak{H}'$, and the ambigous elements of $\mathfrak{H}$ forms a subgroup $\mathfrak{A}$ of $\mathfrak{H}$. Then, from (8), it is sufficient to prove the theorem for a generator A_i of $\mathfrak{A}/\mathfrak{H}'$, and then the order of $V_{\mathfrak{H}\to\mathfrak{G}'}(A_i)$ is a power of prime number p.

Now let us assume that we have proved the theorem for a p-group. Let $\bar{\mathfrak{H}}/\mathfrak{H}$, $\mathfrak{U}/\mathfrak{G}'$ and $\mathfrak{B}/\mathfrak{U}'$ be the greatest subgroup with order prime to p of $\mathfrak{G}/\mathfrak{H}$, $\bar{\mathfrak{H}}/\mathfrak{G}'$ and $\mathfrak{U}/\mathfrak{U}'$ respectively. Then it is easy to see that $\mathfrak{U}/\mathfrak{B}$ is the abelian commutator subgroup of $\mathfrak{G}/\mathfrak{B}$. Our elements A of $\mathfrak{H}$ is ambigous with respect to $\mathfrak{H}$ and is also ambigous with respect to $\bar{\mathfrak{H}}$ mod $\mathfrak{B}$. Therefore applying the assumption for a p-group to the p-groups $\mathfrak{G}/\mathfrak{B}$ and $\bar{\mathfrak{H}}/\mathfrak{B}$ and to the elements A, we get $V_{\bar{\mathfrak{H}}\to\mathfrak{U}}(A) \equiv 1$ mod. $\mathfrak{B}$; and $V_{\bar{\mathfrak{H}}\to\mathfrak{U}}(A)^q$ is contained in $\mathfrak{U}'$, where q is the order of $\mathfrak{B}/\mathfrak{U}'$, which is prime to p. As is easily verified, for any element B of $\mathfrak{U}'$, we have $V_{\mathfrak{U}\to\mathfrak{G}'}(B) = 1$ (Hasse [1] p. 178). So that, we get

$$(17) \qquad V_{\bar{\mathfrak{H}}\to\mathfrak{G}'}(A)^q = V_{\mathfrak{U}\to\mathfrak{G}'}(V_{\bar{\mathfrak{H}}\to\mathfrak{U}}(A)^q) = 1.$$

On the other hand, A is commutative with a generator of the cyclic group $\bar{\mathfrak{H}}/\mathfrak{H}$ mod $\mathfrak{H}'$, as it is ambigous with respect to $\mathfrak{H}$. Then we have

$$V_{\bar{\mathfrak{H}}\to\mathfrak{H}}(A) \equiv A^{q'} (\text{mod. } \mathfrak{H}'),$$

where q' is the order of $\bar{\mathfrak{H}}/\mathfrak{H}$ so that it is prime to p. Therefore, calling the property (17) in mind, we get

$$1 = V_{\bar{\mathfrak{H}}\to\mathfrak{G}'}(A)^q = V_{\mathfrak{H}\to\mathfrak{G}'}(V_{\bar{\mathfrak{H}}\to\mathfrak{H}}(A))^q = V_{\mathfrak{H}\to\mathfrak{G}'}(A)^{qq'}$$

where qq' is prime to p. Thus we conclude our lemma.

§ 2.

6. From now on, we assume that $\mathfrak{G}$ is a p-group. Then, by lemma 4, $\mathfrak{G}'$ and $\mathfrak{H}'$ has as its symbolical generators the elements

$$\mathfrak{G}' = \{[\xi_i, S], [\xi_i, \xi_j]; i < j; i, j = 1, \ldots, n\},$$
$$\mathfrak{H}' = \{[\xi_i, S]^{\xi_j - 1}, [\xi_i, \xi_k]; i < k; i, j, k = 1, \ldots, n\}.$$

For any element $g \in \mathfrak{G}'$ such that

$$g = \prod [\xi_i, S]^{c_i^{(g)}} \prod [\xi_i, \xi_j]^{c_{i,j}^{(g)}},$$

we consider the exponential system

$$\{c_1^{(g)}, \ldots, c_n^{(g)}; c_{12}^{(g)}, \ldots, c_{rs}^{(g)}, \ldots, c_{n-1,n}^{(g)}\}, \; r < s.$$

Let

the exponential system $\{B_1^{(i)}, \ldots, B_n^{(i)}; A_{12}^{(i)}, \ldots, A_{n-1,n}^{(i)}\}$ of $u(\xi_i)^{e_i}$ be α_i,
the exponential system $\{B_1^{(0)}, \ldots, B_n^{(0)}; A_{12}^{(0)}, \ldots, A_{n-1,n}^{(0)}\}$ of g be α_0,
the exponential system $\{0, \ldots, 1, \ldots, 0; 0, \ldots, 0\}$ of $[\xi_i, s]$ be δ_i,
and the exponential system $\{0, \ldots, 0; 0, \ldots, 1, \ldots, 0\}$ of $[\xi_i, \xi_j]$ be ε_{ij},
where g is the element which occurs in Reduction theorem 2. As ε_{ij} was defined for $i<j$, we may define for $i \geqq j$ as follow,

$$\varepsilon_{ij} = - \varepsilon_{ji} \; (i > j), \; \varepsilon_{ii} = 0. \tag{18}$$

As is easily seen,

$$\Delta_i M_i = 0, \quad \text{where} \quad \Delta_i = \xi_i - 1, \tag{19}$$
$$\Delta_l \varepsilon_{ik} + \Delta_i \varepsilon_{kl} + \Delta_k \varepsilon_{li} = 0, \tag{20}$$
$$\Delta \varepsilon_{ik} + \Delta_i \delta_k - \Delta_k \delta_i = 0, \quad \textit{where} \quad \Delta = s - 1, \tag{21}$$
$$\Delta_k \alpha_i = M_i \varepsilon_{ik} \qquad (i \gtreqless k), \tag{22}$$
$$\Delta \alpha_i = M_i \delta_i, \tag{23}$$
$$\alpha_i = \sum_{r<s} A_{rs}^{(i)} \varepsilon_{rs} + \sum_t B_t^{(i)} \delta_t, \tag{24}$$
$$\alpha_0 = \sum_{r<s} A_{rs}^{(0)} \varepsilon_{rs} + \sum_t B_t^{(0)} \delta_t.$$

α_0 was the exponential system of the element g in the Reduction theorem 2. Without loss of generality, we may assume that

$$\alpha_0 = \sum_t B_t^{(0)} \delta_t.$$

For the element $\prod_{i<j} [\xi_i, \xi_j]^{A_{i,j}^{(0)}}$ is in $\mathfrak{H}'$, and then $V_{\mathfrak{H} \to \mathfrak{G}'}(\prod_{i<j} [\xi_i, \xi_j]^{A_{i,j}^{(0)}}) = 1$, and therefore does not affect our assumption and conclusion.

Using the exponential system, the Reduction theorem 2 is translated to the

REDUCTION THEOREM 3. *If there exists a relation*

$$\sum_{i=1}^n (\Gamma_i + \Delta B_i^{(0)}) \delta_i = \sum_{k,j} a_{k,j} \Delta_k \delta_j + \sum_{r<s} b_{rs} \varepsilon_{rs},$$

we have

$$\sum_{i=1}^n (\Gamma_i + \Delta B_i^{(0)}) \theta_i \alpha_i = 0.$$

Furthermore, by (19) and (22),

$$\Delta_k \theta_i \alpha_i = 0 \qquad \text{for all } i, k,$$

then, writing simply Γ_i instead of $\Gamma_i+\Delta B_i^{(0)}+\sum_k a_{ki}\Delta_k$,
we have

REDUCTION THEOREM 3′. *Let P, $(=[\mathfrak{G}/\mathfrak{G}'])$, be a commutaitve ring, and $\mathfrak{M}$ be a P-modul with generators $\varepsilon_{ij}(i\geqq j)$, δ_i, $(i, j=1, \ldots, n)$, which satisfy the condition* (18). *Furthermore we assume that there exist elements Δ, Δ_i, M_i, Γ_i, which satisfy the conditions* (19), (20), (21), *and*

$$\sum_{i=1}^{n}\Gamma_i\delta_i=\sum_{r<s}b_{rs}\varepsilon_{rs}, \tag{25}$$

then for any n elements α_i of $\mathfrak{M}$ which satisfies the conditions (22), (23) *we have*

$$\Xi=\sum\Gamma_i\theta_i\alpha_i=0.^{(2)} \tag{26}$$

7. As we defined θ and θ_i, we shall define in genearal $\theta_{i(1)\ldots i(r)}=\prod_{j\neq i(1)\ldots i(r)}M_j$.[3] Then we have

$$\theta_{i(1)\ldots i(r)}\varepsilon_{jk}=0 \qquad \text{if } j,\ k\neq i(1),\ \ldots,\ i(r). \tag{27}$$

For, as is easily seen, $M_jM_k\varepsilon_{jk}=0$ $\quad(j,\ k=1,\ 2,\ \ldots,\ n)$.
As A_{rs} was defined for $r<s$, we may define for $r\geqq s$

$$A_{rs}=-A_{sr}\ (r>s),\ A_{rr}=0. \tag{28}$$

With this notations we have

$$\begin{aligned}\theta_{i(1)}\alpha_{i(1)}=&-\Delta_{i(1)}\sum_{i(2)\neq i(1)}\theta_{i(1)i(2)}A_{i(1)i(2)}^{(i(1))}\alpha_{i(2)}\\&+\Delta\sum_{i(2)\neq i(1)}\theta_{i(1)i(2)}B_{i(2)}^{(i(1))}\alpha_{i(2)}+\theta_{i(1)}B_{i(1)}^{(i(1))}\delta_{i(1)}.\end{aligned} \tag{29}$$

For, from (24) and (27), (Hasse [1], p. 189)

$$\begin{aligned}\theta_{i(1)}\alpha_{i(1)}=&\ \theta_{i(1)}\sum_{i(2)<i(1)}A_{i(2)i(1)}^{(i(1))}\varepsilon_{i(2)i(1)}+\theta_{i(1)}\sum_{i(2)>i(1)}A_{i(1)i(2)}^{(i(1))}\varepsilon_{i(1)i(2)}\\&+\theta_{i(1)}\sum_{i(2)}B_{i(2)}^{(i(1))}\delta_{i(2)},\end{aligned}$$

and from (28)

$$\begin{aligned}=&\ \theta_{i(1)}\sum_{i(2)\neq i(1)}A_{i(1)i(2)}^{(i(1))}\varepsilon_{i(1)i(2)}+\theta_{i(1)}\sum_{i(2)\neq i(1)}B_{i(2)}^{(i(1))}\delta_{i(2)}\\&+\theta_{i(1)}B_{i(1)}^{(i(1))}\delta_{i(1)},\end{aligned}$$

and then, from (22), (23), we conclude (29).

Putting (29) into (26), we have

$$\begin{aligned}\Xi=&-\sum_{i(1)}\sum_{i(2)\neq i(1)}\Gamma_{i(1)}\Delta_{i(1)}\theta_{i(1)i(2)}A_{i(1)i(2)}^{(i(1))}\alpha_{i(2)}\\&+\sum_{i(1)}\sum_{i(2)\neq i(1)}\Gamma_{i(1)}\Delta\theta_{i(1)i(2)}B_{i(2)}^{(i(1))}\alpha_{i(2)}+\sum_{i(1)}\Gamma_{i(1)}\theta_{i(1)}B_{i(1)}^{(i(1))}\delta_{i(1)}.\end{aligned} \tag{30}$$

On the other hand, from (22), we have

$$0=\sum_{i(1)}\sum_{i(2)\neq i(1)}\Gamma_{i(1)}\Delta_{i(1)}\theta_{i(1)i(2)}A_{i(1)i(2)}^{(i(2))}\alpha_{i(1)}. \tag{31}$$

We write the left side of (25), $\sum\Gamma_i\delta_i$, briefly Γ, then by (22),

$$
(32)\qquad \begin{aligned}
\sum_{i(2)}\theta_{i(2)}B_{i(2)}^{(i(2))}\cdot\Gamma &= \sum_{i(2)}\sum_{i(1)}\theta_{i(2)}B_{i(2)}^{(i(2))}\Gamma_{i(1)}\delta_{i(1)} \\
&= \sum_{i(1)}\sum_{i(2)\neq i(1)}\Gamma_{i(1)}\Delta\theta_{i(1)i(2)}B_{i(2)}^{(i(2))}\alpha_{i(1)} + \sum_{i(1)}\Gamma_{i(1)}\theta_{i(1)}B_{i(1)}^{(i(1))}\delta_{i(1)}.
\end{aligned}
$$

Subtracting (31) and (32) from (30), we get

$$
\begin{aligned}
\Xi = \sum_{i(1)}\sum_{i(2)\neq i(1)}\Gamma_{i(1)}\theta_{i(1)i(2)}\Bigg[\Delta_{i(1)}&\begin{vmatrix}\alpha_{i(1)} & \alpha_{i(2)} \\ A_{i(1)i(2)}^{(i(1))} & A_{i(1)i(2)}^{(i(2))}\end{vmatrix} \\
&- \Delta\begin{vmatrix}\alpha_{i(1)} & \alpha_{i(2)} \\ B_{i(2)}^{(i(1))} & B_{i(2)}^{(i(2))}\end{vmatrix}\Bigg] + \sum_{i(1)}\theta_{i(1)}B_{i(1)}^{(i(1))}\cdot\Gamma.
\end{aligned}
$$

Putting (24) into the above expression, we get the following formula.

THE FIRST MODIFICATION THEOREM.

$$
(33)\qquad \begin{aligned}
\Xi = \sum_{\substack{i(1),\, i(2) \\ i(1)\neq i(2)}}\Gamma_{i(1)}\theta_{i(1)i(2)}\Bigg[&\Delta_{i(1)}\sum_{r<s}\begin{vmatrix}A_{rs}^{(i(1))} & A_{rs}^{(i(2))} \\ A_{i(1)i(2)}^{(i(1))} & A_{i(1)i(2)}^{(i(2))}\end{vmatrix}\varepsilon_{rs} \\
&+ \Delta_{i(1)}\sum_{t}\begin{vmatrix}B_{t}^{(i(1))} & B_{t}^{(i(2))} \\ A_{i(1)i(2)}^{(i(1))} & A_{i(1)i(2)}^{(i(2))}\end{vmatrix}\delta_t - \Delta\sum_{r<s}\begin{vmatrix}A_{rs}^{(i(1))} & A_{rs}^{(i(2))} \\ B_{i(2)}^{(i(1))} & B_{i(2)}^{(i(2))}\end{vmatrix}\varepsilon_{rs} \\
&- \Delta\sum_{t}\begin{vmatrix}B_{t}^{(i(1))} & B_{t}^{(i(2))} \\ B_{i(2)}^{(i(1))} & B_{i(2)}^{(i(2))}\end{vmatrix}\delta_t\Bigg] + \sum\theta_{i(1)}B_{i(1)}^{(i(1))}\cdot\Gamma.
\end{aligned}
$$

8. In case of $n=2$, we get our main theorem from the first modification theorem, and it will be shown in § 3. We now proceed to the case of $n\geqq 3$.

From now on, we write a determinant

$$
\begin{vmatrix}
B_{i(1)}^{(i(1))} & B_{i(1)}^{(i(2))} & \cdots & B_{i(1)}^{(i(r))} & B_{i(1)}^{(k(1))} & \cdots & B_{i(1)}^{(k(s))} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
B_{i(r)}^{(i(1))} & B_{i(r)}^{(i(2))} & \cdots & B_{i(r)}^{(i(r))} & B_{i(r)}^{(k(1))} & \cdots & B_{i(r)}^{(k(s))} \\
A_{k(1)l(1)}^{(i(1))} & A_{k(1)l(1)}^{(i(2))} & \cdots & A_{k(1)l(1)}^{(i(r))} & A_{k(1)l(1)}^{(k(1))} & \cdots & A_{k(1)l(1)}^{(k(s))} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots \\
A_{k(s)l(s)}^{(i(1))} & A_{k(s)l(s)}^{(i(2))} & \cdots & A_{k(s)l(s)}^{(i(r))} & A_{k(s)l(s)}^{(k(1))} & \cdots & A_{k(s)l(s)}^{(k(s))}
\end{vmatrix}
$$

simply

$$
\begin{pmatrix} i(1), & i(2), & \dots, & i(r), & k(1), & \dots, & k(s) \\ i(1), & i(2), & \dots, & i(r), & k(1)l(1), & \dots, & k(s)l(s) \end{pmatrix},
$$

and a determinant with α in its r-th row

$$
\begin{vmatrix}
B_{i(1)}^{(i(1))} & B_{i(1)}^{(i(2))} & \cdots & B_{i(1)}^{(i(r))} & \cdots & B_{i(1)}^{(i(m))} \\
\vdots & \vdots & & \vdots & & \vdots \\
\alpha_{i(1)} & \alpha_{i(2)} & \cdots & \alpha_{i(r)} & \cdots & \alpha_{i(m)} \\
\vdots & \vdots & & \vdots & & \vdots \\
A_{kl}^{(i(1))} & A_{kl}^{(i(2))} & \cdots & A_{kl}^{(i(r))} & \cdots & A_{kl}^{(i(m))}
\end{vmatrix}
$$

simply

$$
\begin{pmatrix} i(1), & i(2), & \dots, & i(r), & \dots, & i(m) \\ i(1), & i(2), & \dots, & \alpha, & \dots, & kl \end{pmatrix}.
$$

By these notations (33) has the following description,

$$
\Xi = \sum_{\substack{i(1),\, i(2) \\ i(1)\neq i(2)}}\Gamma_{i(1)}\theta_{i(1)i(2)}\Bigg[\Delta_{i(1)}\sum_{r<s}\begin{pmatrix} i(1), & i(2) \\ rs, & i(1)i(2)\end{pmatrix}\varepsilon_{rs}
$$

$$+ \Delta_{i(1)}\sum_t\begin{pmatrix} i(1), & i(2) \\ t, & i(1)i(2)\end{pmatrix}\delta_t + \Delta\sum_{r<s}\begin{pmatrix} i(1), & i(2) \\ i(2), & rs\end{pmatrix}\varepsilon_{rs}$$
$$+ \Delta\sum_t\begin{pmatrix} i(1), & i(2) \\ i(2), & t\end{pmatrix}\delta_t\Bigg] + \sum_{i(1)}\theta_{i(1)}B_{i(1)}^{(i(1))}\cdot\Gamma.$$

Let

$$\Xi_1 = \sum_{\substack{i(1),\, i(2) \\ i(1)\neq i(2)}}\Gamma_{i(1)}\theta_{i(1)i(2)}\Delta_{i(1)}\sum_{r<s}\begin{pmatrix} i(1), & i(2) \\ rs, & i(1)i(2)\end{pmatrix}\varepsilon_{rs},$$
$$\Xi_2 = \sum_{\substack{i(1),\, i(2) \\ i(1)\neq i(2)}}\Gamma_{i(1)}\theta_{i(1)i(2)}\Bigg[\Delta_{i(1)}\sum_t\begin{pmatrix} i(1), & i(2) \\ t, & i(1)i(2)\end{pmatrix}\delta_t + \Delta\sum_{r<s}\begin{pmatrix} i(1), & i(2) \\ i(2), & rs\end{pmatrix}\varepsilon_{rs}\Bigg],$$
$$\Xi_3 = \sum_{\substack{i(1),\, i(2) \\ i(1)\neq i(2)}}\Gamma_{i(1)}\theta_{i(1)i(2)}\Delta\sum_t\begin{pmatrix} i(1), & i(2) \\ i(2), & t\end{pmatrix}\delta_t.$$

[I] The modification of Ξ_1. We can restrict the summation $\sum\limits_{r<s}$ to

$$\sum(r<s;\ r\neq i(2),\ s=i(1))+\sum(r<s;\ r=i(1),\ s\neq i(2))$$
$$+\sum(r<s;\ r\neq i(1),\ s=i(2))+\sum(r<s;\ r=i(2),\ s\neq i(1)),$$

for if $r\neq i(1),\ i(2)$; and $s\neq i(1),\ i(2)$, then these terms vanish by (27), and also the terms $r=i(1)$, $s=i(2)$ and $r=i(2)$, $s=i(1)$ vanish, since these determinants$=0$. We write $i(3)$ instead of r or s, we get[3]

$$\Xi_1 = \sum_{\substack{i(1),\, i(2) \\ i(1)\neq i(2)}}\Delta_{i(1)}\Gamma_{i(1)}\theta_{i(1)i(2)}\Bigg[\sum_{\substack{i(3)<i(1) \\ i(3)\neq i(2)}}\begin{pmatrix} i(1), & i(2) \\ i(3)i(1), & i(1)i(2)\end{pmatrix}\varepsilon_{i(3)i(1)}$$
$$+\sum_{\substack{i(3)>i(1) \\ i(3)\neq i(2)}}\begin{pmatrix} i(1), & i(2) \\ i(1)i(3), & i(1)i(2)\end{pmatrix}\varepsilon_{i(1)i(3)} + \sum_{\substack{i(3)<i(2) \\ i(3)\neq i(1)}}\begin{pmatrix} i(1), & i(2) \\ i(3)i(2), & i(1)i(2)\end{pmatrix}\varepsilon_{i(3)i(2)}$$
$$+\sum_{\substack{i(3)>i(2) \\ i(3)\neq i(1)}}\begin{pmatrix} i(1), & i(2) \\ i(2)i(3), & i(1)i(2)\end{pmatrix}\varepsilon_{i(2)i(3)}\Bigg].$$

By (18) and (28)

$$\Xi_1 = \sum_{\substack{i(1),\, i(2),\, i(3) \\ i(1)\neq i(2)\neq i(3)}}\Delta_{i(1)}\Gamma_{i(1)}\theta_{i(1)i(2)}\Bigg[\begin{pmatrix} i(1), & i(2) \\ i(3)i(1), & i(1)i(2)\end{pmatrix}\varepsilon_{i(3)i(1)}$$
$$+\begin{pmatrix} i(1), & i(2) \\ i(3)i(2), & i(1)i(2)\end{pmatrix}\varepsilon_{i(3)i(2)}\Bigg].$$

By (22)

$$\Xi_1 = \sum_{\substack{i(1),\, i(2),\, i(3) \\ i(1)\neq i(2)\neq i(3)}}\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta_{i(1)}\Bigg[\begin{pmatrix} i(1), & i(2) \\ i(3)i(1), & i(1)i(2)\end{pmatrix}\Delta_{i(1)}\alpha_{i(3)}$$
$$+\begin{pmatrix} i(1), & i(2) \\ i(3)i(2), & i(1)i(2)\end{pmatrix}\Delta_{i(2)}\alpha_{i(3)}\Bigg].$$

We divide the summation $\sum(i(1),i(2),i(3);\,i(1)\neq i(2)\neq i(3))$ into two parts

$$\sum(i(1),i(2),i(3);\ i(1)\neq i(2),i(3),\,i(2)<i(3))$$
$$+\sum(i(1),i(2),i(3);\ i(1)\neq i(2),i(3),\,i(2)>i(3)).$$

In the second term, we exchange the letters $i(2)$ and $i(3)$ one another. Then

$$\Xi_1=\sum_{i(1)\neq i(2),i(3)}\ \sum_{\substack{i(2),\,i(3)\\ i(2)<i(3)}}\Delta_{i(1)}\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Big[\begin{pmatrix} i(1), & i(2)\\ i(3)i(1), & i(1)i(2)\end{pmatrix}\Delta_{i(1)}\alpha_{i(3)}$$
$$+\begin{pmatrix} i(1), & i(2)\\ i(3)i(2), & i(1)i(2)\end{pmatrix}\Delta_{i(2)}\alpha_{i(3)}+\begin{pmatrix} i(1), & i(3)\\ i(2)i(1), & i(1)i(3)\end{pmatrix}\Delta_{i(1)}\alpha_{i(2)}$$
$$+\begin{pmatrix} i(1), & i(3)\\ i(2)i(3), & i(1)i(3)\end{pmatrix}\Delta_{i(3)}\alpha_{i(2)}\Big].$$

Calling the equality (22) in mind, we may write

$$\Xi_1=\sum_{i(1)\neq i(2),i(3)}\ \sum_{\substack{i(2),\,i(3)\\ i(2)<i(3)}}\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Big[\Delta^2_{i(1)}\begin{pmatrix} i(1), & i(2), & i(3)\\ \alpha, & i(1)i(2), & i(1)i(3)\end{pmatrix}$$
$$+\Delta_{i(1)}\Delta_{i(2)}\begin{pmatrix} i(1), & i(2), & i(3)\\ \alpha, & i(1)i(2), & i(2)i(3)\end{pmatrix}$$
$$+\Delta_{i(1)}\Delta_{i(3)}\begin{pmatrix} i(1), & i(2), & i(3)\\ \alpha, & i(3)i(2), & i(1)i(3)\end{pmatrix}\Big],$$

and this may be written as follows.

$$\Xi_1=\sum_{i(1)\neq i(2),i(3)}\ \sum_{\substack{i(2),i(3)\\ i(2)<i(3)}}\ \sum_{\substack{a(1)+a(2)+a(3)=2\\ a(1),\,a(2),\,a(3)\geqq 0}}\ \sum_{j(2),\,j(3)}\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta^{a(1)}_{i(1)}\Delta^{a(2)}_{i(2)}\Delta^{a(3)}_{i(3)}$$
$$\cdot\begin{pmatrix} i(1), & i(2), & i(3)\\ \alpha, & j(2)i(2), & j(3)i(3)\end{pmatrix}.$$

Putting (24) into the above expression, we get

$$\Xi_1=\sum\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta^{a(1)}_{i(1)}\Delta^{a(2)}_{i(2)}\Delta^{a(3)}_{i(3)}\cdot\Big[\sum_{r<s}\begin{pmatrix} i(1), & i(2), & i(3)\\ rs, & j(2)i(2), & j(3)i(3)\end{pmatrix}\varepsilon_{rs}$$
$$(34)\qquad +\sum_t\begin{pmatrix} i(1), & i(2), & i(3)\\ t, & j(2)i(2), & j(3)i(3)\end{pmatrix}\delta_t\Big],$$

where the summation is extended over $i(1)$, $i(2)$, $i(3)$, $a(1)$, $a(2)$, $a(3)$, $j(2), j(3)$, and

(35)
- (i) $i(1)\neq i(2), i(3)$; $i(2)<i(3)$,
- (ii) $a(1), a(2), a(3)$ with $a(1)+a(2)+a(3)=2$,
- (iii) $j(2)$, $j(3)$ take the value $i(1)$ in $a(1)$ times, $i(2)$ in $a(2)$ times and $i(3)$ in $a(3)$ times.

[II] The modification of Ξ_2. In the first term of Ξ_2, we divide the summation $\sum_t$ into $\sum(t,\ t\neq i(1),\ i(2))$ and $\sum(t,\ t=i(1),\ i(2))$, and in the second term of Ξ_2, as was in [I], we divide the summation $\sum_{r<s}$ into $\sum(r\neq i(1), i(2)\ ;\ s=i(1), i(2))$ and $\sum(r=i(1), s=(2))$.

And then, writing $i(3)$ instead of t and r, we get

$$\Xi_2=\sum_{\substack{i(1),\,i(2),\,i(3)\\ i(1)\neq i(2)\neq i(3)}}\Gamma_{i(1)}\theta_{i(1)i(2)}\Big[\Delta_{i(1)}\begin{pmatrix} i(1), & i(2)\\ i(3), & i(1)i(2)\end{pmatrix}\delta_{i(3)}$$
$$+\Delta\begin{pmatrix} i(1), & i(2)\\ i(2), & i(3)i(1)\end{pmatrix}\varepsilon_{i(3)i(1)}+\Delta\begin{pmatrix} i(1), & i(2)\\ i(2), & i(3)i(2)\end{pmatrix}\varepsilon_{i(3)i(2)}\Big]$$
$$+\sum_{\substack{i(1),\,i(2)\\ i(1)\neq i(2)}}\Gamma_{i(1)}\theta_{i(1)i(2)}\Big[\Delta_{i(1)}\begin{pmatrix} i(1), & i(2)\\ i(1), & i(1)i(2)\end{pmatrix}\delta_{i(1)}$$

$$+ \Delta_{i(1)}\begin{pmatrix} i(1), & i(2) \\ i(1), & i(1)i(2) \end{pmatrix}\delta_{i(2)} + \Delta\begin{pmatrix} i(1), & i(2) \\ i(2), & i(1)i(2) \end{pmatrix}\varepsilon_{i(1)i(2)}\Big].$$

Exchanging the letters $i(2)$ and $i(3)$ one another in the second and the sixth term, and in addition applying (22) and (23) to the first three terms, and also, applying (21) to the sixth term, we get

$$\begin{aligned}(36)\qquad \Xi_2 = &\sum_{\substack{i(1),\, i(2),\, i(3) \\ i(1)\neq i(2)\neq i(3)}} \Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta\Big[\Delta_{i(1)}\begin{pmatrix} i(1), & i(2) \\ i(3), & i(1)i(2) \end{pmatrix}\alpha_{i(3)} \\ &+ \Delta_{i(1)}\begin{pmatrix} i(1), & i(3) \\ i(3), & i(2)i(1) \end{pmatrix}\alpha_{i(2)} + \Delta_{i(2)}\begin{pmatrix} i(1), & i(2) \\ i(2), & i(3)i(2) \end{pmatrix}\alpha_{i(3)}\Big] \\ &+ \sum_{i(1),\, i(2)} \Gamma_{i(1)}\theta_{i(1)i(2)}\Big[\Delta_{i(1)}\begin{pmatrix} i(1), & i(2) \\ i(1), & i(1)i(2) \end{pmatrix} \\ &+ \Delta_{i(2)}\begin{pmatrix} i(1), & i(2) \\ i(2), & i(1)i(2) \end{pmatrix}\Big]\delta_{i(1)}.\end{aligned}$$

Adding the first and the second term, and, exchanging the letters $i(2)$, $i(3)$ one another, we get

$$(37)\qquad \sum_{\substack{i(1),\, i(2),\, i(3) \\ i(1)\neq i(2)\neq i(3)}} \Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta\Delta_{i(1)}\begin{pmatrix} i(1), & i(2), & i(3) \\ i(2), & \alpha, & i(1)i(3) \end{pmatrix},$$

for, from (22), $\Delta_{i(1)}\alpha_{i(1)}=0$. By the same reason, we get from the third term of (36)

$$(38)\qquad \sum_{\substack{i(1),\, i(2),\, i(3) \\ i(1)\neq i(2)\neq i(3)}} \Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta\Delta_{i(2)}\begin{pmatrix} i(1), & i(2), & i(3) \\ i(2), & \alpha, & i(2)i(3) \end{pmatrix}$$

$$(39)\qquad +\sum_{\substack{i(1),\, i(2),\, i(3) \\ i(1)\neq i(2)\neq i(3)}} \Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta\Delta_{i(2)}\begin{pmatrix} i(2), & i(3) \\ i(2), & i(2)i(3) \end{pmatrix}\alpha_{i(1)}.$$

From now on, we say "*change the letters* $i(1)$, $i(2)$, ..., $i(r)$ *as* $i(1) \rightarrow i(2) \rightarrow \ldots\ldots \rightarrow i(r) \rightarrow i(1)$" in the following sense, that is, *we write* $i(1)$ *instead of* $i(r)$, $i(2)$ *instead of* $i(1)$, ..., and $i(r)$ *instead of* $i(r-1)$.

Changing the letters $i(1)$, $i(2)$, $i(3)$ as $i(1)\rightarrow i(3)\rightarrow i(2)\rightarrow i(1)$ in (39), and applying (23), we get

$$(39)'\qquad \sum_{\substack{i(1),\, i(2) \\ i(1)\neq i(2)}} \theta_{i(1)i(2)}\Delta_{i(1)}\begin{pmatrix} i(1), & i(2) \\ i(1), & i(1)i(2) \end{pmatrix}\cdot\sum_{i(3)\neq i(1),\, i(2)} \Gamma_{i(3)}\delta_{i(3)}.$$

Changing the letters $i(1), i(2)$ as $i(1)\rightarrow i(2)\rightarrow i(1)$ in the fifth term of (36), and, adding the forth term of (36), we get

$$(40)\qquad \sum_{\substack{i(1),\, i(2) \\ i(1)\neq i(2)}} \theta_{i(1)i(2)}\Delta_{i(1)}\begin{pmatrix} i(1), & i(2) \\ i(1), & i(1)i(2) \end{pmatrix}\Big(\Gamma_{i(1)}\delta_{i(1)} + \Gamma_{i(2)}\delta_{i(2)}\Big).$$

Therefore,

$$\begin{aligned}\Xi_2 &= (37) + (38) + (39') + (40) \\ &= \sum_{\substack{i(1),\, i(2),\, i(3) \\ i(1)\neq i(2)\neq i(3)}} \Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta\Big[\Delta_{i(1)}\begin{pmatrix} i(1), & i(2), & i(3) \\ i(2), & \alpha, & i(1)i(3) \end{pmatrix} \\ &\quad + \Delta_{i(2)}\begin{pmatrix} i(1), & i(2), & i(3) \\ i(2), & \alpha, & i(2)i(3) \end{pmatrix}\Big]\end{aligned}$$

$$+\sum_{\substack{i(1),\,i(2)\\ i(1)\neq i(2)}}\theta_{i(1)i(2)}\Delta_{i(1)}\begin{pmatrix} i(1), & i(2)\\ i(1), & i(1)i(2)\end{pmatrix}\cdot\Gamma$$

$$=\sum\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta\Delta_{i(1)}^{a(1)}\Delta_{i(2)}^{a(2)}\Delta_{i(3)}^{a(3)}\begin{pmatrix} i(1), & i(2), & i(3)\\ i(2), & \alpha, & j(3)i(3)\end{pmatrix}$$

$$+\sum_{\substack{i(1),\,i(2)\\ i(1)\neq i(2)}}\theta_{i(1)i(2)}\Delta_{i(1)}\begin{pmatrix} i(1), & i(2)\\ i(1), & i(1)i(2)\end{pmatrix}\cdot\Gamma,$$

where the summation in the first term has the following sense,

$$(41)\qquad \begin{cases} i(1), i(2), i(3) \quad \text{with} \quad i(1)\neq i(2)\neq i(3)\neq i(1),\\ a(1), a(2), a(3) \quad \text{with} \quad 1+a(1)+a(2)+a(3)=2 \quad \text{and}\\ \qquad\qquad\qquad\qquad a(1), a(2), a(3)\geqq 0,\\ j(3) \;\text{ takes the value } i(1) \text{ in } a(1) \text{ times, } i(2) \text{ in } a(2) \text{ times, and}\\ i(3) \text{ in } a(3) \text{ times.}\end{cases}$$

Putting (24) into the above expression

$$\Xi_2=\sum\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta\Delta_{i(1)}^{a(1)}\Delta_{i(2)}^{a(2)}\Delta_{i(3)}^{a(3)}\Bigg[\sum_{r<s}\begin{pmatrix} i(1), & i(2), & i(3)\\ i(2), & rs, & j(3)i(3)\end{pmatrix}^{rs}$$

$$(42)\qquad +\sum_{t}\begin{pmatrix} i(1), & i(2), & i(3)\\ i(2), & t, & j(3)i(3)\end{pmatrix}^{\delta_t}\Bigg]$$

$$+\sum_{\substack{i(1),\,i(2)\\ i(1)\neq i(2)}}\theta_{i(1),\,i(2)}\Delta_{i(1)}\begin{pmatrix} i(1), & i(2)\\ i(1), & i(1)i(2)\end{pmatrix}\cdot\Gamma.$$

[III] The modification of Ξ_3. First of all, we divide the summation $\sum_t$ into $\sum(t,\ t\neq i(1),\ i(2))$ and $\sum(t,\ t=i(1),\ i(2))$ and describe $i(2)$ instead of t in the first term, then by (23)

$$\Xi_3=\sum_{\substack{i(1),\,i(2),\,i(3)\\ i(1)\neq i(2)\neq i(3)}}\Delta^2\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\begin{pmatrix} i(1), & i(2)\\ i(2), & i(3)\end{pmatrix}^{\alpha_{i(3)}}$$

$$+\sum_{\substack{i(1),\,i(2)\\ i(1)\neq i(2)}}\Delta\Gamma_{i(1)}\theta_{i(1)i(2)}\begin{pmatrix} i(1), & i(2)\\ i(2), & i(1)\end{pmatrix}^{\delta_{i(1)}}.$$

In the first term, we divide the $\sum\limits_{i(2)\neq i(3)}$ into $\sum\limits_{i(2)<i(3)}$ and $\sum\limits_{i(2)>i(3)}$ and, in the latter, exchange the letters $i(2)$ and $i(3)$ one-another. In the second term, we also divide the $\sum\limits_{i(1)\neq i(2)}$ into $\sum\limits_{i(1)<i(2)}$ and $\sum\limits_{i(1)>i(2)}$ and, in the latter, exchange the letters $i(1)$ and $i(2)$ one another. Then, arranging the rows and the columns in order, we get

$$\Xi_3=\sum_{i(1)\neq i(2),i(3)}\ \sum_{\substack{i(2),\,i(3)\\ i(2)<i(3)}}\Delta^2\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\left[\begin{pmatrix} i(1), & i(2)\\ i(2), & i(3)\end{pmatrix}^{\alpha_{i(3)}}-\begin{pmatrix} i(1), & i(3)\\ i(2), & i(3)\end{pmatrix}^{\alpha_{i(2)}}\right]$$

$$-\sum_{\substack{i(1),\,i(2)\\ i(1)<i(2)}}\Delta\theta_{i(1)i(2)}\begin{pmatrix} i(1), & i(2)\\ i(1), & i(2)\end{pmatrix}^{(\Gamma_{i(1)}\delta_{i(1)}+\Gamma_{i(2)}\delta_{i(2)})}$$

$$(43)\qquad =\sum_{i(1)\neq i(2),i(3)}\ \sum_{\substack{i(2),\,i(3)\\ i(2)<i(3)}}\Delta^2\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\begin{pmatrix} i(1), & i(2), & i(3)\\ i(2), & i(3), & \alpha\end{pmatrix}$$

$$-\sum_{i(1)\neq i(2),i(3)}\sum_{\substack{i(2),\, i(3)\\ i(2)<i(3)}}\Delta^2\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\binom{i(2),\, i(3)}{i(2),\, i(3)}\alpha_{i(1)}$$
$$-\sum_{\substack{i(1),\, i(2)\\ i(1)<i(2)}}\Delta\theta_{i(1)i(2)}\binom{i(1),\, i(2)}{i(1),\, i(2)}\left(\Gamma_{i(1)}\delta_{i(1)}+\Gamma_{i(2)}\delta_{i(2)}\right).$$

Changing the letters $i(1)$, $i(2)$, $i(3)$ as $i(1)\rightarrow i(3)\rightarrow i(2)\rightarrow i(1)$ in the second term, and applying (23), we have

$$-\sum_{\substack{i(1),\, i(2)\\ i(1)<i(2)}}\Delta\theta_{i(1)i(2)}\binom{i(1),\, i(2)}{i(1),\, i(2)}\sum_{i(3)\neq i(1),\, i(2)}\Gamma_{i(3)}\delta_{i(3)}.$$

Adding this and the third term of (43), and putting (24) into the first term of (43), we get

$$\Xi_3=\sum_{i(1)\neq i(2),i(3)}\sum_{\substack{i(2),\, i(3)\\ i(2)<i(3)}}\Delta^2\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Bigg[\sum_{r<s}\binom{i(1),\, i(2),\, i(3)}{i(2),\, i(3),\ \ rs}\varepsilon_{rs}$$
$$+\sum_t\binom{i(1),\, i(2),\, i(3)}{i(2),\, i(3),\ \ t}\delta_t\Bigg]-\sum_{\substack{i(1),\, i(2)\\ i(1)<i(2)}}\Delta\theta_{i(1)i(2)}\binom{i(1),\, i(2)}{i(1),\, i(2)}\cdot\Gamma. \tag{44}$$

Adding (34), (42), (44), and the last term of (33), we get

THE SECOND MODIFICATION THEOREM. *In case of $n\geqq 3$, we get*

$$\Xi=\sum\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta_{i(2)}^{a(2)}\Delta_{i(1)}^{a(1)}\Delta_{i(3)}^{a(3)}\Bigg[\sum_{r<s}\binom{i(1),\quad i(2),\qquad i(3)}{rs,\ \ j(2)i(2),j(3)i(3)}\varepsilon_{rs}$$
$$+\sum_t\binom{i(1),\quad i(2),\qquad i(3)}{rs,\ \ j(2)i(2),j(3)i(3)}\delta_t\Bigg]$$
$$+\sum\Gamma_{i(1)}\theta_{i(1)i(2)i(3)}\Delta\Delta_{i(1)}^{a(1)}\Delta_{i(2)}^{a(1)}\Delta_{i(3)}^{a(3)}\Bigg[\sum_{r<s}\binom{i(1),\, i(2),\quad i(3)}{i(2),\ \ rs,\ \ j(3)i(3)}\varepsilon_{rs}$$
$$+\sum_t\binom{i(1),\, i(2),\quad i(3)}{i(2),\ \ t,\ \ j(3)i(3)}\delta_t\Bigg]$$
$$+\sum\Gamma_{i(1)}\theta_{i(1)j(2)i(3)}\Delta^2\Bigg[\sum_{r<s}\binom{i(1),\, i(2),\, i(3)}{i(2),\, i(3),\ \ rs}\varepsilon_{rs}$$
$$+\sum_t\binom{i(1),\, i(2),\, i(3)}{i(2),\, i(3),\ \ t}\delta_t\Bigg]$$
$$+\Bigg[\sum_{\substack{i(1),\, i(2)\\ i(1)\neq i(2)}}\theta_{i(1),\, i(2)}\Delta_{i(1)}\binom{i(1),\qquad i(2)}{i(1),\ i(1)i(2)}-\sum_{\substack{i(1),\, i(2)\\ i(1)<i(2)}}\theta_{i(1)i(2)}\Delta\binom{i(1),\, i(2)}{i(1),\, i(2)}$$
$$+\sum_{i(1)}B_{i(1)}^{(i(1))}\theta_{i(1)}\Bigg]\cdot\Gamma, \tag{45}$$

where the summation is extended over $i(k)$, $a(l)$, $j(m)$ as was described in (35), (41), (44).

9. In general, we get the following

THE m-TH MODIFICATION THEOREM. *In case of $n\geqq m+1$, we get*

$$\Xi=\sum\Gamma_{i(1)}\theta_{i(1)i(2)\ldots i(m+1)}\Delta^u\Delta_{i(1)}^{a(1)}\Delta_{i(1)}^{a(2)}\cdots\Delta_{i(m+1)}^{a(m+1)}$$
$$\cdot\Bigg[\sum_{r<s}\binom{i(1),\ldots,\ \ i(u),\ \ i(u+1),\quad i(u+2),\qquad\ldots,\qquad i(m+1)}{i(2),\ldots,i(u+1),\quad rs,\quad j(u+2)i(u+2),\ldots,j(m+1)i(m+1)}\varepsilon_{rs}$$
$$+\sum_t\binom{i(1),\ldots,\ \ i(u),\ \ i(u+1),\quad i(u+2),\qquad\ldots,\qquad i(m+1)}{i(2),\ldots,i(u+1),\quad t,\quad j(u+2)i(u+2),\ldots,j(m+1)i(m+1)}\delta_t\Bigg] \tag{46}$$

$$+\Gamma(\Xi_m+\Xi_{m-1}+\dots+\Xi_1),$$

where the summation is extended over

(47)
(i) $u=0,1,2,\dots,m,$
(ii) $i(1),i(2),\dots,i(m+1)$, *with* $i(1)\neq i(2)\neq\dots\neq i(m+1)$, *and* $i(2)<i(3)<\dots<i(u+1),\ i(u+2)<i(u+3)<\dots<i(m+1),$
(iii) $a(1),a(2),\dots,a(m+1)$ *with* $a(1),a(2),\dots,a(m+1)\geqq 0$ *and* $u+a(1)+a(2)+\dots+a(m+1)=m,$
(iv) $j(u+2),j(u+3),\dots,j(m+1)$, *taking values* $i(1)$ *in* $a(1)$ *times,* $i(2)$ *in* $a(2)$ *times,* $\dots,i(m+1)$ *in* $a(m+1)$ *times,*

and

$$\Xi_k=\sum(-1)^u\theta_{i(1)\dots i(k)}\Delta^u\Delta_{i(1)}^{a(1)}\dots\Delta_{i(k)}^{a(k)}\cdot\begin{pmatrix} i(1),\dots,i(u+1), & i(u+2), & \dots, & i(k)\\ i(1),\dots,i(u+1), & j(u+2)i(u+2), & \dots, & j(k)i(k)\end{pmatrix}, \tag{48}$$

where the summation is extened over

(49)
(i) $u=0,1,\dots,k-1,$
(ii) $i(1),\dots,i(k)$ *with* $i(1)\neq i(2)\neq\dots\neq i(k)$ *and* $i(1)<\dots<i(u+1),\ i(u+2)<\dots<i(k),$
(iii) $a(1),\dots,a(k)$ *with* $a(1),a(2),\dots,a(k)\geqslant 0$ *and* $u+a(1)+\dots+a(k)=k-1,$
(iv) $j(u+2),\dots,j(k)$, *taking values* $i(1)$ *in* $a(1)$ *times,* $i(2)$ *in* $a(2)$ *times,* $\dots,i(k)$ *in* $a(k)$ *times.*

PROOF. We shall prove this theorem by induction. Assume that we have already gotten $m-1$-th modification theorem in this form. We shall prove that, modifying the expression

$$\begin{aligned}&\sum\Gamma_{i(1)}\theta_{i(1)i(2)\dots i(m)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\Delta_{i(2)}^{a(2)}\dots\Delta_{i(m)}^{a(m)}\\&\qquad\cdot\sum_t\begin{pmatrix} i(1),\dots,i(u-1), & i(u), & i(u+1), & \dots, & i(m)\\ i(2),\dots, & i(u), & t, & j(u+1)i(u+1),\dots, & j(m)i(m)\end{pmatrix}\delta_t\\&+\sum\Gamma_{i(1)}\theta_{i(1)i(2)\dots i(m)}\Delta^u\Delta_{i(1)}^{a(1)}\Delta_{i(2)}^{a(2)}\dots\Delta_{i(m)}^{a(m)}\\&\qquad\cdot\sum_{r<s}\begin{pmatrix} i(1),\dots & i(u), & i(u+1), & i(u+2), & \dots, & i(m)\\ i(2),\dots & i(u+1), & rs, & j(u+2)i(u+2), & \dots, & j(m)i(m)\end{pmatrix}\varepsilon_{rs},\end{aligned}\tag{50}$$

for $0\leqslant u\leqslant m$, where the summations are extended over all i,a,j analogously as was described in (ii),(iii),(iv) of (47), we have

$$\begin{aligned}&\sum\Gamma_{i(1)}\theta_{i(1)i(2)\dots i(m+1)}\Delta^u\Delta_{i(1)}^{a(1)}\Delta_{i(2)}^{a(2)}\dots\Delta_{i(m+1)}^{a(m+1)}\\&\qquad\cdot\begin{pmatrix} i(1),\dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m+1)\\ i(2),\dots, & i(u+1), & \alpha, & j(u+2)i(u+2), & \dots, & j(m+1)i(m+1)\end{pmatrix}\\&+\sum(-1)^{u-1}\theta_{i(1)\dots i(m)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\dots\Delta_{i(m)}^{a(m)}\\&\qquad\cdot\begin{pmatrix} i(1),\dots i(u), & i(u+1), & \dots, & i(m)\\ i(1),\dots i(u), & j(u+1)i(u+1), & \dots, & j(m)i(m)\end{pmatrix}\cdot\Gamma\end{aligned}\tag{51}$$

where the summations are extended over all i, j, a as was described in (ii), (iii), (iv), of (47) and (49), respectively. For $u=0$, or $u=m$, the expression (50) contains only the second, or the first term, and yet we may prove in these cases as well as in the case of $0<u<m$. This can be seen easily in the following proof of the case of $0<u<m$. We write the first and the second term of (50) by (50. 1) and (50. 2) respectively. From now on, we write the summation with respect to a and j simply by $\sum\limits_{a,\,j}$.

In (50. 1), we divide the summation $\sum\limits_{t}$ into

$$(50.1.1) \qquad \sum(t \neq i(1), \ldots, i(m)),$$

$$(50.1.2) \qquad \sum(t = i(1)),$$

$$(50.1.3) \qquad \sum(t = i(2), \ldots, i(m)) = \sum(t = i(u+1), \ldots, i, i(m)).$$

In (50. 1. 1), changing the letters $t, i(u+1), \ldots, i(m)$ as $t \to i(u+1) \to i(u+2) \to \ldots \to i(m) \to i(m+1)$, and in addition, taking (23) in consideration, we get

$$(50.1.1.) = \sum_{\substack{i(1)i(2)<\ldots<i(u)\\ i(u+2)<\ldots<i(m+1)}} \sum_{i(u+1)} \sum_{a,\,j} \Gamma_{i(1)} \theta_{i(1)i(2)\ldots i(m+1)} \Delta^u \Delta^{a(1)}_{i(1)} \ldots \Delta^{a(m+1)}_{i(m+1)}$$

$$\cdot \begin{pmatrix} i(1), \ldots, i(u-1), & i(u), & i(u+2), & \ldots, & i(m+1) \\ i(2), \ldots, \quad i(u), & i(u+1), & j(u+2)i(u+2), & \ldots, & j(m+1)i(m+1) \end{pmatrix} \alpha_{i(u+1)},$$

where $i(u+1) \neq i(1), \ldots, i(m+1)$. Although the term $\Delta^{a(u+1)}_{i(u+1)}$ is wanting in the above expression, we can assume that it exists, for $\Delta^{a(u+1)}_{i(u+1)} \cdot \alpha_{i(u+1)} = 0$.

We say the process in the following sense, *to put the letter* $i(u+1)$ *into an inequality* $i(2)<\ldots<i(u)$, that is, dividing the summation with respect to $i(2), \ldots, i(u+1)$ into the following cases,

(i) $i(u+1) < i(2) < i(3) < \ldots < i(u)$,

(ii) $i(2) < i(u+1) < i(3) < \ldots < i(u)$,

......

(iii) $i(2) < i(3) < \ldots < i(k) < i(u+1) < i(k+1) < \ldots < i(u)$,

......

(iv) $i(2) < i(3) < \ldots < i(u) < i(u+1)$,

and then, we change the letters $i(2), i(3), \ldots, i(u+1)$ as follows,

(i) in case of (i) above, $i(u+1) \to i(2) \to i(3) \to \ldots \to i(u) \to i(u+1)$,

(ii) in case of (ii) above, $i(u+1) \to i(3) \to i(4) \to \ldots \to i(u) \to i(u+1)$,

......

(iii) in case of (iii) above, $i(u+1) \to i(k+1) \to \ldots \to i(u) \to i(u+1)$,

......

(iv) in case of (iv) above, all letters are fixed.

Now, putting $i(u+1)$ into the inequality $i(2)<\ldots<i(u)$ and, arranging the rows and the columns in order, we get

$$(50.1.1) = (-1)^{u-1} \sum_{i(1)} \sum_{\substack{i(2)<\ldots<i(u+1)\\ i(u+2)<\ldots<i(m+1)}} \sum_{a,\,j} \Gamma_{i(1)} \theta_{i(1)\ldots i(m+1)} \Delta^u \Delta^{a(1)}_{i(1)} \ldots \Delta^{a(m+1)}_{i(m+1)}$$

$$(52)\qquad \cdot\sum_{\lambda=2}^{u+1}(-1)^{\lambda}\begin{pmatrix} i(1), \ldots, i(\lambda-1), i(\lambda+1), \ldots \\ i(2), \ldots,\ \ i(\lambda),\ \ i(\lambda+1), \ldots \end{pmatrix.$$
$$\begin{pmatrix} \ldots, i(u+1), & i(u+2), & \ldots, & i(m+1) \\ \ldots, i(u+1), & j(u+2)i(u+2), & \ldots, & j(m+1)i(m+1) \end{pmatrix}\alpha_{i(\lambda)}.$$

In (50. 1. 2), after changing the letters $i(1), i(2), \ldots, i(u)$ as $i(1)\to i(u)\to i(u-1)\to\ldots\to i(2)\to i(1)$, we put $i(u)$ into the inequality $i(1)<\ldots<i(u-1)$. Then, arranging the rows and the columns in order, we get

$$(50.1.2) = (-1)^{u-1}\sum_{\substack{i(1)<\ldots<i(u) \\ i(u+1)<\ldots<i(m)}}\sum_{a,\,j}\theta_{i(1)\ldots i(m)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\Delta_{i(2)}^{a(2)}\cdots\Delta_{i(m)}^{a(m)}$$

$$(53)\qquad \cdot\begin{pmatrix} i(1), \ldots, i(u), & i(u+1), & \ldots, & i(m) \\ i(1), \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(m)i(m) \end{pmatrix}\left(\sum_{\lambda=1}^{u}\Gamma_{i(\lambda)}\delta_{i(\lambda)}\right).$$

In (50. 1. 3), we may assume that $t=i(u+1), \ldots, i(m)$, for otherwise the determinants are equal to zero. First of all, we change the letters as follows,

(54)

(i) if $t=i(u+1)$ all letters are fixed,

(ii) if $t=i(u+2)$, $i(u+2)\to i(u+1)\to i(u+2)$, all others are fixed,

……

(iii) if $t=i(u+k)$, $i(u+k)\to i(u+1)\to i(u+2)\to\ldots\to i(u+k-1)\to i(u+k)$, all others are fixed,

……

(iv) if $t=i(m)$, $i(m)\to i(u+1)\to i(u+2)\to\ldots\to i(m-1)\to i(m)$.

Then, after arranging the rows and columns in order, we get

$$\sum_{i(1)}\sum_{\substack{i(2)<\ldots<i(u) \\ i(u+2)<\ldots<i(m)}}\sum_{i(u+1)}\sum_{a,\,j}\Gamma_{i(1)}\theta_{i(1)i(2)\ldots i(m)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\Delta_{i(2)}^{a(2)}\cdots\Delta_{i(m)}^{a(m)}$$
$$\cdot\begin{pmatrix} i(1), \ldots, & i(u), & i(u+1), & \ldots, & i(m) \\ i(2), \ldots, & i(u+1), & j(u+1)i(u+1), & \ldots, & j(m)i(m) \end{pmatrix}\delta_{i(u+1)}.$$

Moreover, putting the letter $i(u+1)$ into the inequality $i(2)<\ldots<i(u)$ and also arranging the rows and columns in order, we have

$$(50.1.3) = \sum_{i(1)}\sum_{\substack{i(2)<\ldots<i(u+1) \\ i(u+2)<\ldots<i(m)}}\sum_{a,\,j}\Gamma_{i(1)}\theta_{i(1)\ldots i(m)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\cdots\Delta_{i(m)}^{a(m)}$$

$$(55)\qquad \cdot\sum_{k=2}^{u+1}\begin{pmatrix} i(1), \ldots, & i(u), & i(u+1), & i(u+2), & \ldots, & i(m) \\ i(2), \ldots, & i(u+1), & j(k)i(k), & j(u+2)i(u+2), & \ldots, & j(m)i(m) \end{pmatrix}\delta_{i(k)}.$$

In (50. 2), we first divide the summation $\sum(r<s)$ into

(i) $\sum(r<s;\ r\neq i(1), \ldots, i(m), s\neq i(1), \ldots, i(m))$,

(ii) $\sum(r<s;\ r=i(1), \ldots, i(m), s\neq i(1), \ldots, i(m))$,

(iii) $\sum(r<s;\ r\neq i(1), \ldots, i(m), s=i(1), \ldots, i(m))$,

(iv) $\sum(r<s;\ r=i(1), s=\text{all } i(k) \text{ such that } i(1)<i(k))$,

(v) $\sum(r<s;\ s=i(1), r=\text{all } i(k) \text{ such that } i(1)>i(k))$,

(vi) $\sum(r<s;\ r=i(2), s=\text{all } i(k) \text{ such that } i(2)<i(k)$ and

$i(k) \neq i(1))$,

(vii) $\sum(r < s;\ s = i(2), r = \text{all } i(k)$ such that $i(2) > i(k)$ and $i(k) \neq i(1))$,

......

(viii) $\sum(r < s;\ r = i(l),\ s = \text{all } i(k)$ such that $i(l) < i(k)$ and $i(k) \neq i(1), i(2), \ldots, i(l-1))$,

(ix) $\sum(r < s;\ s = i(l),\ r = \text{all } i(k)$ such that $i(l) > i(k)$ and $i(k) \neq i(1), \ldots, i(l-1))$,

......

(x) $\sum(r < s;\ r = i(m-1), s = i(m))$.

According to (27), each term of (i) is equal to zero. In the case of (ii) and (iii), writing $i(m+1)$ instead of r and s, respectively, and adding them, we get by virtue of the fact $A_{rs}\varepsilon_{rs} = A_{sr}\varepsilon_{sr}$,

$$(50.2.1) \equiv \sum_{i(1)} \sum_{\substack{i(2)<\ldots<i(u+1) \\ i(u+2)<\ldots<i(m)}} \sum_{i(m+1)} \sum_{a,\,j} \Gamma_{i(1)} \theta_{i(1)i(2)\ldots(m)} \Delta^u \Delta^{a(1)}_{i(1)} \ldots \Delta^{a(m)}_{i(m)}$$

$$(56) \qquad \cdot \sum_{k=1}^{m} \begin{pmatrix} i(1), \ldots, & i(u), & i(u+1), & i(u+2), & \ldots & \ldots, & i(m) \\ i(2), \ldots, & i(u+1), & i(m+1)i(k), & j(u+2)i(u+2), & \ldots & \ldots, & j(m)i(m) \end{pmatrix} \varepsilon_{i(m+1)i(k)}.$$

Analogously, adding (iv), and (v), we get

$$\sum_{i(1)} \sum_{\substack{i(2)<\ldots<i(u+1) \\ i(u+2)<\ldots<i(m)}} \sum_{a,\,j} \Gamma_{i(1)} \theta_{i(1)\ldots i(m)} \Delta^u \Delta^{a(1)}_{i(1)} \ldots \Delta^{a(m)}_{i(m)}$$

$$\cdot \sum_{k=2}^{m} \begin{pmatrix} i(1), \ldots, & i(u), & i(u+1), & i(u+2), & \ldots, & i(m) \\ i(2), \ldots, & i(u+1), & i(1)i(k), & j(u+2)i(u+2), & \ldots, & j(m)i(m) \end{pmatrix} \varepsilon_{i(1)i(k)}.$$

Also, adding (vi) and (vii),

$$\sum_{i(1)} \sum_{\substack{i(2)<\ldots<i(u+1) \\ i(u+2)<\ldots<i(m)}} \sum_{a,\,j} \Gamma_{i(1)} \theta_{i(1)\ldots i(m)} \Delta^u \Delta^{a(1)}_{i(1)} \ldots \Delta^{a(m)}_{i(m)}$$

$$\cdot \sum_{k=3}^{m} \begin{pmatrix} i(1), \ldots, & i(u), & i(u+1), & i(u+2), & \ldots, & i(m) \\ i(2), \ldots, & i(u+1), & i(2)i(k), & j(u+2)i(u+2), & \ldots, & j(m)i(m) \end{pmatrix} \varepsilon_{i(2)i(k)}.$$

In general, adding (viii), and (ix),

$$\sum_{i(1)} \sum_{\substack{i(2)<\ldots<i(u+1) \\ i(u+2)<\ldots<i(m)}} \sum_{a,\,j} \Gamma_{i(1)} \theta_{i(1)\ldots i(m)} \Delta^u \Delta^{a(1)}_{i(1)} \ldots \Delta^{a(m)}_{i(m)}$$

$$\cdot \sum_{k=l+1}^{m} \begin{pmatrix} i(1), \ldots, & i(u), & i(u+1), & i(u+2), & \ldots, & i(m) \\ i(2), \ldots, & i(u+1), & i(l)i(k), & j(u+2)i(u+2), & \ldots, & j(m)i(m) \end{pmatrix} \varepsilon_{i(l)i(k)}.$$

Finally, in case of (x), we get

$$\sum_{i(1)} \sum_{\substack{i(2)<\ldots<i(u+1) \\ i(u+2)<\ldots<i(m)}} \sum_{a,\,j} \Gamma_{i(1)} \theta_{i(1)\ldots i(m)} \Delta^u \Delta^{a(1)}_{i(1)} \ldots \Delta^{a(m)}_{i(m)}$$

$$\cdot \begin{pmatrix} i(1), \ldots, & i(u), & i(u+1), & i(u+2), & \ldots & \ldots, & i(m) \\ i(2), \ldots, & i(u+1), & i(m-1)i(m), & j(u+2)i(u+2), & \ldots & \ldots, & j(m)i(m) \end{pmatrix} \varepsilon_{i(m-1)i(m)}.$$

Therefore, adding (iv), (v), ..., (x), we get

$$(50.2.2) \equiv \sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m)}} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m)}\Delta^u \Delta_{i(1)}^{a(1)} \dots \Delta_{i(m)}^{a(m)} \sum_{l=1}^{m-1}\sum_{k=l+1}^{m}$$

$$(57) \qquad \begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & i(l)i(k), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix} \varepsilon_{i(l)i(k)}.$$

(50. 2) is equal to the sum of (50. 2. 1) and (50. 2. 2), and we shall modify these expressions as is in (50. 1).

First of all, in (56), using the property (22), we get

$$(50.2.1) = \sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m)}} \sum_{i(m+1)} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m+1)}\Delta^u \Delta_{i(1)}^{a(1)} \dots \Delta_{i(m+1)}^{a(m+1)}$$

$$(58) \qquad \cdot\sum_{k=1}^{m} \begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & i(m+1)i(k), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix} \Delta_{i(k)}\alpha_{i(m+1)},$$

where the term $\Delta_{i(m+1)}^{a(m+1)}$ does not exist at first, and yet we may write as above because of the property $\Delta_{i(m+1)}\alpha_{i(m+1)}=0$. Now, the expression

$$\sum_{k=1}^{m} \begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & i(k)i(m+1), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix} \Delta_{i(k)}\alpha_{i(m+1)}$$

may be written as

$$\sum_{j(m+1),\, a} \Delta_{i(1)}^{a(1)} \dots \Delta_{i(m+1)}^{a(m+1)} \begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & j(m+1)i(m+1), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix} \alpha_{i(m+1)},$$

where $a(1)+\dots+a(m+1)=1$ and $j(m+1)$ takes the value $i(k)$ in $a(k)$ times. And moreover, arranging the columns in order, it is equal to

$$(-1)^{m-u-1} \sum_{j(m+1),\, a} \Delta_{i(1)}^{a(1)} \dots \Delta_{i(m+1)}^{a(m+1)} \cdot \begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & j(u+2)i(u+2), & \dots, & j(m+1)i(m+1) \end{pmatrix} \alpha_{i(m+1)}.$$

Therefore

$$(50.2.1) = (-1)^{m-u} \sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m)}} \sum_{i(m+1)} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m+1)}\Delta^u \Delta_{i(1)}^{a(1)} \dots \Delta_{i(m+1)}^{a(m+1)}$$

$$(59) \qquad \cdot \begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & j(u+2)i(u+2), & \dots, & j(m+1)i(m+1) \end{pmatrix} \alpha_{i(m+1)}.$$

Now, putting the letter $i(m+1)$ into the inequality $i(u+2)<\dots<i(m)$, and, arranging the rows and columns in order, we get

$$(50.2.1) = (-1)^{u-1} \sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m+1)}} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m+1)}\Delta^u \Delta_{i(1)}^{a(1)} \dots \Delta_{i(m+1)}^{a(m+1)}$$

$$(60) \qquad \cdot\sum_{\lambda=u+2}^{m+1} (-1)^{\lambda} \begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & \dots \\ i(2), \dots, & i(u+1), & j(u+2)i(u+2), & \dots \end{pmatrix}$$

$$\begin{matrix} \dots, i(\lambda-1), & i(\lambda+1) & \dots, & i(m+1) \\ \dots, j(\lambda)i(\lambda), & j(\lambda+1)i(\lambda+1), & \dots, & j(m+1)i(m+1) \end{matrix}\Bigg)\alpha_{i(\lambda)}.$$

In (50. 2. 2), by means of the relation (21), we get from (57),

$$\sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m)}} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m)}\Delta^{u-1}\Delta^{a(1)}_{i(1)}\dots\Delta^{a(m)}_{i(m)}$$

$$\cdot\sum_{l<k}^{1,\dots,m}\Bigg[\begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & i(l)i(k), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix}\Delta_{i(k)}\delta_{i(l)}$$

$$-\begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & i(l)i(k), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix}\Delta_{i(l)}\delta_{i(k)}\Bigg].$$

Exchanging the letters k and l in this first term, and, adding to the second term, and then, calling the meaning of j in mind, we get

$$(50.\,2.\,2) = -\sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m)}} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m)}\Delta^{u-1}\Delta^{a(1)}_{i(1)}\dots\Delta^{a(m)}_{i(m)}$$

$$\cdot\sum_{k=1}^{m}\begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & j(k)i(k), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix}\delta_{i(k)}$$

$$= -\sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m)}} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m)}\Delta^{u-1}\Delta^{a(1)}_{i(1)}\dots\Delta^{a(m)}_{i(m)} \tag{61}$$

$$\cdot\begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & j(k)i(k), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix}\delta_{i(1)}$$

$$-\sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m)}} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m)}\Delta^{u-1}\Delta^{a(1)}_{i(1)}\dots\Delta^{a(m)}_{i(m)} \tag{62}$$

$$\cdot\sum_{k=2}^{m}\begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m) \\ i(2), \dots, & i(u+1), & j(k)i(k), & j(u+2)i(u+2), & \dots, & j(m)i(m) \end{pmatrix}\delta_{i(k)}.$$

In the first term, the expression (61), changing the letters $i(1)$, $i(2)$, $\dots$, $i(u+1)$ as $i(u+1)\to i(u)\to\dots\to i(2)\to i(1)\to i(u+1)$, and then, putting the letter $i(u+1)$ into the inequality $i(u+2)<\dots<i(m)$, we get

$$(-1)^{u-1}\sum_{\substack{i(1)<\dots<i(u) \\ i(u+1)<\dots<i(m)}} \sum_{a,\, j} \theta_{i(1)\dots i(m)}\Delta^{u-1}\Delta^{a(1)}_{i(1)}\dots\Delta^{a(m)}_{i(m)} \tag{63}$$

$$\cdot\begin{pmatrix} i(1), \dots, i(u), & i(u+1), & \dots, & i(m) \\ i(1), \dots, i(u), & j(u+1)i(u+1), & \dots, & j(m)i(m) \end{pmatrix}\sum_{\lambda=u+1}^{m}\Gamma_{i(\lambda)}\delta_{i(\lambda)}.$$

(50. 2) is equal to the sum of (60), (62), and (63).

We now proceed to the calculation of (50. 1) + (50. 2). Adding (52) and (60), we get

$$(50.\,1.\,1) + (50.\,2.\,1)$$

$$= \sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m+1)}} \sum_{a,\, j} \Gamma_{i(1)}\theta_{i(1)\dots i(m+1)}\Delta^{u}\Delta^{a(1)}_{i(1)}\dots\Delta^{a(m+1)}_{i(m+1)}$$

$$\cdot\begin{pmatrix} i(1), \dots, & i(u), & i(u+1), & i(u+2), & \dots, & i(m+1) \\ i(2), \dots, & i(u+1), & \alpha, & j(u+2)i(u+2), & \dots, & j(m+1)i(m+1) \end{pmatrix} \tag{64}$$

$$+ (-1)^{u-1}\sum_{i(1)} \sum_{\substack{i(2)<\dots<i(u+1) \\ i(u+2)<\dots<i(m+1)}} \sum_{a,\, i} \Gamma_{i(1)}\theta_{i(1)\dots i(m+1)}\Delta^{u}\Delta^{a(1)}_{i(1)}\dots\Delta^{a(m+1)}_{i(m+1)}$$

$$\cdot\begin{pmatrix} i(2), \ldots, i(u+1), & i(u+2), & \ldots, & i(m+1) \\ i(2), \ldots, i(u+1), & j(u+2)i(u+2), & \ldots, & j(m+1)i(m+1) \end{pmatrix}\alpha_{i(1)}.$$

In the second term, using the property (23), that is, $\Delta\alpha_{i(1)}=M_{i(1)}\delta_{i(1)}$, we get

$$(-1)^{u-1}\sum_{i(1)}\sum_{\substack{i(2)<\ldots<i(u+1)\\ i(u+2)<\ldots<i(m+1)}}\sum_{a,\,j}\theta_{i(2)\ldots i(m+1)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\ldots\Delta_{i(m+1)}^{a(m+1)}$$

$$\cdot\begin{pmatrix} i(2), \ldots, i(u+1), & i(u+2), & \ldots, & i(m+1) \\ i(2), \ldots, i(u+1), & j(u+2)i(u+2), & \ldots, & j(m+1)i(m+1) \end{pmatrix}\Gamma_{i(1)}\delta_{i(1)}.$$

Exchanging the letters $i(1), \ldots, i(m+1)$ as $i(1)\to i(m+1)\to i(m)\to\ldots\to i(2)\to i(1)$, and by means of the relation $\theta_{i(1)\ldots i(m)}\Delta_{i(m+1)}=0$, we get

$$(-1)^{u-1}\sum_{\substack{i(1)<\ldots<i(u)\\ i(u+1)<\ldots<i(m)}}\sum_{a,\,j}\theta_{i(1)\ldots i(m)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\ldots\Delta_{i(m)}^{a(m)}$$

$$\cdot\begin{pmatrix} i(1), \ldots, i(u), & i(u+1), & \ldots, & i(m) \\ i(1), \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(m)i(m) \end{pmatrix}\sum_{i(m+1)\neq i(1), \ldots, i(m)}\Gamma_{i(m+1)}\delta_{i(m+1)}.$$

Adding this to (53) and (63), we get

$$(65)\qquad (-1)^{u-1}\sum_{\substack{i(1)<\ldots<i(u)\\ i(u+1)<\ldots<i(m)}}\sum_{a,\,j}\theta_{i(1)\ldots i(m)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\ldots\Delta_{i(m)}^{a(m)}$$

$$\cdot\begin{pmatrix} i(1), \ldots, i(u), & i(u+1), & \ldots, & i(m) \\ i(1), \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(m)i(m) \end{pmatrix}\cdot\Gamma,$$

where $\Gamma=\sum_{i=1}^{n}\Gamma_i\delta_i$.

Finally adding (55) and (62), we get

$$-\sum_{i(1)}\sum_{\substack{i(2)<\ldots<i(u+1)\\ i(u+2)<\ldots<i(m)}}\sum_{a,\,j}\Gamma_{i(1)}\theta_{i(1)i(2)\ldots i(m)}\Delta^{u-1}\Delta_{i(1)}^{a(1)}\ldots\Delta_{i(m)}^{a(m)}$$

$$\cdot\sum_{k=u+2}^{m}\begin{pmatrix} i(1), \ldots, & i(u), & i(u+1), & i(u+2), & \ldots, & i(m) \\ i(2), \ldots, & i(u+1), & j(k)i(k), & j(u+2)i(u+2), & \ldots, & j(m)i(m) \end{pmatrix}\delta_{i(k)},$$

and this is equal to zero in accordance to the fact described below. For $k=u+s$, we consider $j(u+s)$, and if $j(k)=i(p), j(u+s)=i(q)$, $(p, q=1,\ldots, m)$ we pick up the cases when $j(k)=i(q)$ and $f(u+s)=i(p)$. These two terms have the opposite sign and cancel one another.

Therefore $(50)=(50.1)+(50.2)$ is (65) and the first term of (64), and is equal to (51) as was required.

In the case of $u=0$ the expression (50) is only the (50.2) and the summation $\sum(r<s)$ is divided into

(i) $\sum(r<s\,;\, r, s\neq i(1), \ldots, i(m))$,

(ii) $\sum\begin{pmatrix} r<s\,;\, r\neq i(1), \ldots, i(m), s=i(1), \ldots, i(m) \text{ and} \\ r=i(1), \ldots, i(m), s\neq i(1), \ldots, i(m) \end{pmatrix}$,

(iii) $\sum(r<s\,;\, r=i(k), s=i(l), k, l=1, \ldots, m)$.

In the case of (i), all the terms are equal to zero. In case of (iii), the proof of Furtwängler's theorem (Furtwängler [1]) is applicable to our case(4), and is equal to zero in all. There remains only the case of (ii), and we get (56) with $u=0$, and then without (50.1.1), we get (64) with $u=0$, that is,

$$\sum_{i(1)}\sum_{i(2)<\dots<i(m+1)}\sum_{\alpha,j}\Gamma_{i(1)}\theta_{i(1)\dots i(m+1)}\Delta_{i(1)}^{\alpha(1)}\dots\Delta_{i(m+1)}^{\alpha(m+1)}$$

$$(51')\qquad \cdot\left[\begin{pmatrix} i(1), & i(2), & \dots, & i(m+1) \\ \alpha, & j(2)i(2), & \dots, & j(m+1)i(m+1)\end{pmatrix}\right.$$
$$\left.-\begin{pmatrix} i(2), & \dots, & i(m+1) \\ j(2)i(2), & \dots, & j(m+1)i(m+1)\end{pmatrix}\alpha_{i(1)}\right],$$

and in the second term, the terms in which $j(k)\neq i(1)\,(k=2, \dots, m+1)$ are equal to zero in all, for an arbitrary non-zero determinant has one of opposite sign in this terms.[5] And the terms, such that one of the $j(k)$ is equal to $i(1)$ is also zero because of the property $\Delta_{i(1)}\alpha_{i(1)}=0$. Therefore, there remains only the first term of the above expression.

In the case of $u=m$, there is only the expression (50.1), the first term of (50), and we have only the (50.1.1) and (50.1.2), and by a same method as in (50.1), we have

$$(51'')\qquad \sum_{i(1)}\sum_{i(2)<\dots<i(m+1)}\Gamma_{i(1)}\theta_{i(1)\dots i(m+1)}\Delta^{m}\begin{pmatrix} i(1), & \dots, & i(m), & i(m+1) \\ i(2), & \dots, & i(m+1), & \alpha\end{pmatrix}$$
$$+(-1)^{m-1}\sum_{i(1)<\dots<i(m)}\theta_{i(1)\dots i(m)}\Delta^{m-1}\begin{pmatrix} i(1), & \dots, & i(m) \\ i(1), & \dots, & i(m)\end{pmatrix}\cdot\Gamma.$$

Putting the value of α_i into the expression (51), (51′), (51″), adding u from 0 to m, we get easily the expession (46) and thus complete the proof of m-th modification theorem.

10. If $n=m+1$, as is easily seen from the proof of the m-th modification theorem, we get

$$\Xi=\Gamma(\Xi_{m+1}+\Xi_m+\dots+\Xi_2+\Xi_1)$$

Therefore

REDUCTION THEOREM 4. *Let P be a commutative ring, and $\mathfrak{M}$ be a P-modul with generators $\varepsilon_{ij}(i<j)$, δ_i, $(i,j=1, \dots, n)$ which satify the condition* (18). *Furthermore, we assume that there exist elements α_i, Δ, Δ_i, M_i and Γ which satisfy the conditions* (19), (20), (21) *and*

$$(25)\qquad \Gamma=\sum_{r<s}a_{rs}\varepsilon_{rs},$$

then

$$(66)\qquad \Xi=\Gamma(\Xi_1+\Xi_2+\dots+\Xi_n)=0,$$

where Ξ_k is as in (48).

§ 3.

11. We now proceed to the proof of our theorem. As in the case of $n=1$, we get $\Gamma=0$, and it follows $\Xi=0$ immediately.

As in the case $n\geqq 2$, putting (25) into (66), we get

$$\Xi = (\sum_{r<s} a_{rs}\varepsilon_{rs})(\Xi_1 + \Xi_2 + \dots + \Xi_n),$$

and we shall prove that

$$\varepsilon_{rs}(\Xi_1 + \Xi_2 + \dots + \Xi_n) = 0,$$

for each $r<s$. Without loss of generality, we may assume that $r=1, s=2$. We shall namely prove that

(67) $$\varepsilon_{12}(\Xi_1 + \Xi_2 + \dots + \Xi_n) = 0.$$

First of all, we calculate

$$\varepsilon_{12}\Xi_1 = \varepsilon_{12}\cdot\sum_{i(1)} \theta_{i(1)} B_{i(1)}^{(i(1))}.$$

As $i(1) \neq 1, 2, \varepsilon_{12}\cdot\theta_{i(1)}=0$ by (27), we get

$$\varepsilon_{12}\cdot\Xi_1 = \varepsilon_{12}(\theta_1 B_1^{(1)} + \theta_2 B_2^{(2)}).$$

Using (22), we get

$$= \theta_{12}(\alpha_1\Delta_2 B_2^{(2)} - \alpha_2\Delta_1 B_1^{(1)}).$$

By means of the property $\alpha_i\Delta_i=0$, this may be written as

$$= -\theta_{12}\begin{vmatrix} B_1^{(1)} B_1^{(2)} \\ \alpha_1 \ \alpha_2 \end{vmatrix}\Delta_1 - \theta_{12}\begin{vmatrix} B_2^{(1)} B_2^{(2)} \\ \alpha_1 \ \alpha_2 \end{vmatrix}\Delta_2$$

$$= -\theta_{12}\left[\Delta_1\begin{pmatrix}1, 2 \\ 1, \alpha\end{pmatrix} + \Delta_2\begin{pmatrix}1, 2 \\ 2, \alpha\end{pmatrix}\right]$$

(68) $$= -\theta_{12}\sum_{j,\,a} \Delta_1^{a(1)}\Delta_2^{a(2)}\begin{pmatrix}1, 2 \\ j, \alpha\end{pmatrix},$$

where $a(1)+a(2)=1$ and j takes the values 1 and 2 in $a(1)$ and $a(2)$ times, respectively.

Secondly, we calculate

$$\varepsilon_{12}\Xi_2 = \varepsilon_{12}\left[\sum_{i(1),\,i(2)} \theta_{i(1)i(2)}\Delta_{i(1)}\begin{pmatrix}i(1), \ i(2) \\ i(1), \ i(1)i(2)\end{pmatrix} - \sum_{i(1)<i(2)} \theta_{i(1)i(2)}\Delta\begin{pmatrix}i(1), i(2) \\ i(1), i(2)\end{pmatrix}\right]$$

$$=\Phi_1+\Phi_2 \text{ (say)}.$$

By virtue of (27), we divide the summation $\sum_{i(1),\,i(2)}$ in Φ_1 into

(i) $i(1) = 1, \ i(2) > 2,$
(ii) $i(2) = 1, \ i(1) > 2,$
(iii) $i(1) = 2, \ i(2) > 2,$
(iv) $i(2) = 2, \ i(1) > 2,$
(v) $i(1) = 1, \ i(2) = 2$ and $i(1) = 2, \ i(2) = 1.$

(For $n=2$, we only have the case (v)). Now, changing the letter $i(2)$ to $i(1)$ in (i) and (iii), we get

$$\text{(i)} + \text{(ii)} + \text{(iii)} + \text{(iv)}$$

$$= \sum_{i(1)>2}\left[\theta_{1i(1)}\Delta_1\begin{pmatrix}1, \ i(1) \\ 1, \ 1i(1)\end{pmatrix} + \theta_{2i(1)}\Delta_2\begin{pmatrix}2, \ i(1) \\ 2, \ 2i(1)\end{pmatrix}\right.$$

$$\left. + \theta_{1i(1)}\Delta_{i(1)}\begin{pmatrix}i(1), \ \ 1 \\ i(1), \ i(1)1\end{pmatrix} + \theta_{2i(1)}\Delta_{i(1)}\begin{pmatrix}i(1), \ \ 2 \\ i(1), \ i(1)2\end{pmatrix}\right]\cdot\varepsilon_{12}.$$

Using the property (22), we get

$$=-\sum_{i(1)>2}\theta_{12i(1)}\Big[\Delta_1^2\binom{1,\ i(1)}{1,\ 1i(1)}\alpha_2-\Delta_2^2\binom{2,\ i(1)}{2,\ 2i(1)}\alpha_1$$
$$+\Delta_1\Delta_{i(1)}\binom{1,\quad i(1)}{i(1),\ 1i(1)}\alpha_2-\Delta_2\Delta_{i(1)}\binom{2,\quad i(1)}{i(1),\ 2i(1)}\alpha_1\Big].$$

By means of the relations $\Delta_i\alpha_i=0$

$$=-\sum_{i(1)>2}\sum_{j,\ a}\theta_{12i(1)}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\Big[\Delta_1\binom{1,\quad i(1)}{1,j(1)i(1)}\alpha_2$$
$$-\Delta_2\binom{2,\quad i(1)}{2,j(1)i(1)}\alpha_1+\Delta_{i(1)}\binom{1,\quad i(1)}{i(1),j(1)i(1)}\alpha_2-\Delta_{i(1)}\binom{2,\quad i(1)}{i(1),j(1)i(1)}\alpha_1\Big]$$

(69) $$=-\sum_{i(1)>2}\sum_{j,\ a}\theta_{12i(1)}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\Big[\binom{1,\quad i(1)}{j,j(1)i(1)}\alpha_2-\binom{2,\quad i(1)}{j,j(1)i(1)}\alpha_1\Big].$$

On the other hand

$$(\mathrm{v})=\theta_{12}\Big[\Delta_1\binom{1,\ 2}{1,\ 12}+\Delta_2\binom{1,\ 2}{2,\ 12}\Big]\varepsilon_{12}.$$

From (24), we have

(70) $$A_{12}^{(i)}\cdot\varepsilon_{12}=\alpha_i-\sum_{\substack{r<s\\(r,\ s)\neq(1,\ 2)}}A_{rs}^{(i)}\varepsilon_{rs}-\sum_t B_t^{(i)}\delta_t.$$

Putting this into the above expression, we get

(71) $$(\mathrm{v})=\theta_{12}\sum_{k=1,\ 2}\Delta_k\Big[\binom{1,2}{k,\alpha}-\sum_{\substack{r<s\\(r,\ s)\neq(1,\ 2)}}\binom{1,2}{k,rs}\varepsilon_{rs}-\sum_t\binom{1,2}{k,t}\delta_t\Big].$$

In the second term, there remain only the cases of $r=1$, $s>2$ and $r=2$, $s>2$. Changing the letters to $i(1)$ it is equal to

$$-\theta_{12}\sum_{k=1,\ 2}\Delta_k\sum_{i(1)>2}\Big[\binom{1,\quad 2}{k,\ 1i(1)}\varepsilon_{1i(1)}+\binom{1,\quad 2}{k,\ 2i(1)}\varepsilon_{2i(1)}\Big]$$
$$=\sum_{k=1,2}\sum_{i(1)>2}\theta_{12i(1)}\Delta_k\Big[\binom{1,\quad 2}{k,\ 1i(1)}\Delta_1\alpha_{i(1)}+\binom{1,\quad 2}{k,\ 2i(1)}\Delta_2\alpha_{i(1)}\Big]$$
$$=\sum_{i(1)>2}\sum_{a,\ j}\theta_{12i(1)}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,\quad 2}{j,j(1)i(1)}\alpha_{i(1)}.$$

The last summation of (71) can be divided into $\sum(t\neq1,2)$ and $\sum\ (t=1,2)$. Changing the letter t to $i(1)$ in the former, we have

$$-\theta_{12}\sum_{k=1,\ 2}\Delta_k\Big[\sum_{i(1)>2}\binom{1,\quad 2}{k,\ i(1)}\delta_{i(1)}-\sum_{t=1,\ 2}\binom{1,2}{k,t}\delta_t\Big],$$

by (23)

$$=-\sum_{i(1)>2}\sum_{k=1,\ 2}\theta_{12i(1)}\Delta_k\binom{1,\quad 2}{k,\ i(1)}\Delta\alpha_{i(1)}-\theta_{12}\binom{1,2}{1,2}(\Delta_1\delta_2-\Delta_2\delta_1);$$

using (21) in the latter, we get finally

$$=-\sum_{i(1)>2}\sum_{j,\ a}\theta_{12i(1)}\Delta\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,\quad 2}{j,\ i(1)}\alpha_{i(1)}+\theta_{12}\binom{1,2}{1,2}\Delta\varepsilon_{12}.$$

Therefore

$$(\mathrm{v})=\theta_{12}\sum_{j,\ a}\Delta_1^{a(1)}\Delta_2^{a(2)}\binom{1,2}{j,\alpha}+\theta_{12}\binom{1,2}{1,2}\Delta\varepsilon_{12}$$
$$+\sum_{i(1)>2}\sum_{j,\ a}\theta_{12i(1)}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,\quad 2}{j,j(1)i(1)}\alpha_{i(1)}$$

$$-\sum_{i(1)>2}\sum_{j,\,a}\theta_{12i(1)}\Delta\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,\ 2}{j,\ i(1)}\alpha_{i(1)}.$$

Adding this to (69), we get

$$\Phi_1=\theta_{12}\sum_{j,\,a}\Delta_1^{a(1)}\Delta_2^{a(2)}\binom{1,2}{j,\alpha}+\theta_{12}\binom{1,2}{1,2}\Delta\varepsilon_{12}$$
$$-\sum_{i(1)>2}\sum_{j,\,a}\theta_{12i(1)}\Delta\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,\ 2}{j,\ i(1)}\alpha_{i(1)}$$
$$-\sum_{i(1)>2}\sum_{j,\,a}\theta_{12i(1)}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,2,\ \ i(1)}{j,\alpha,j(1)i(1)}.$$

Now we proceed to the computation of the second term

$$\Phi_2=-\ \varepsilon_{12}\sum_{i(1)<i(2)}\theta_{i(1)i(2)}\Delta\binom{i(1),i(2)}{i(1),i(2)}.$$

According to (27), there remain only the cases where at least one of the $i(1), i(2)=1,\ 2$; and we get

$$\Phi_2=\ \ \varepsilon_{12}\sum_{i(2)>2}\left[\theta_{1i(2)}\Delta\binom{1,i(2)}{1,i(2)}-\theta_{2i(2)}\Delta\binom{2,i(2)}{2,i(2)}\right]-\varepsilon_{12}\theta_{12}\Delta\binom{1,2}{1,2}.$$

Changing the letter $i(2)$ to $i(1)$, and using (22), we get

$$\Phi_2=-\sum_{i(1)>2}\theta_{12i(1)}\Delta\left[-\Delta_1\binom{1,i(1)}{1,i(1)}\alpha_2+\Delta_2\binom{2,i(1)}{2,i(1)}\alpha_1\right]-\varepsilon_{12}\theta_{12}\Delta\binom{1,2}{1,2}.$$

And, as is easily calculated,

$$\Delta_1\binom{1,i(1)}{1,i(1)}\alpha_2=\sum_{j,\,a}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,i(1)}{j,\ i(1)}\alpha_2,$$
$$\Delta_2\binom{2,i(1)}{2,i(1)}\alpha_1=\sum_{j,\,a}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{2,i(1)}{j,\ i(1)}\alpha_1.$$

Therefore

$$\Phi_2=\sum_{i(1)>2}\sum_{j,\,a}\theta_{12i(1)}\Delta\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\left[\binom{1,i(1)}{j,\ i(1)}\alpha_2-\binom{2,i(1)}{j,\ i(1)}\alpha_1\right]$$
$$-\ \varepsilon_{12}\theta_{12}\Delta\binom{1,2}{1,2}.$$

Adding Φ_1 and Φ_2, we get

$$\Xi_2\cdot\varepsilon_{12}=-\sum_{i(1)>2}\sum_{j,a}\theta_{12i(1)}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,2,\ \ i(1)}{j,\alpha,j(1)i(1)}$$
$$(72)\qquad -\sum_{i(1)>2}\sum_{j,a}\theta_{12i(1)}\Delta\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\binom{1,\ \ 2,\ \ i(1)}{j,i(1),\ \ \alpha}$$
$$+\ \theta_{12}\sum_{j,a}\Delta_1^{a(1)}\Delta_2^{a(2)}\binom{1,2}{j,\alpha}.$$

12. *In general, we get*

$$\Xi_k\varepsilon_{12}=\sum\nolimits_{(k-1)}(-1)^{u+1}\theta_{12i(1)\dots i(k-1)}\Delta^u\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-1)}^{a(k+1)}$$
$$\cdot\binom{1,2,i(1),\dots,i(u),\qquad i(u+1),\qquad\dots,\qquad i(k-1)}{j,\alpha,i(1),\dots,i(u),j(u+1)i(u+1),\dots,j(k-1)i(k-1)}$$
$$(73)\qquad +\sum\nolimits_{(k-2)}(-1)^u\theta_{12i(1)\dots i(k-2)}\Delta^u\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$
$$\cdot\binom{1,2,i(1),\dots,i(u),\qquad i(u+1),\qquad\dots,\qquad i(k-2)}{j,\alpha,i(1),\dots,i(u),j(u+1)i(u+1),\dots,j(k-2)i(k-2)},$$

where the summations $\sum_{(k)}$ *are extended over all* u, $i(r)$, $a(r)$, $j(r)$ *as follows,*

$i(1), \ldots, i(k)$ such that

$$2 < i(1) < \ldots < i(u), 2 < i(u+1) < \ldots < i(k);$$

$u=0, \ldots, k$;

$a(1), \ldots, a(k+2)$ such that $u+a(1)+\ldots+a(k+2)=k$;

and $j(u+1), \ldots, j(k)$ which take the values

1 in $a(1)$ times,

2 in $a(2)$ times,

$i(1)$ in $a(3)$ times.

......

$i(k-1)$ in $a(k-1)$ times,

We now proceed to the proof of the above equality (73). Let

$$(\Xi_k\varepsilon_{12})_u=(-1)^{u-1}\sum_{\substack{i(1)<\ldots<i(u)\\ i(u+1)<\ldots<i(k)}}\sum_{a,j}\theta_{i(1)\ldots i(k)}\Delta^{u-1}\Delta^{a(1)}_{i(1)}\ldots\Delta^{a(k)}_{i(k)} \tag{74}$$

$$\cdot\begin{pmatrix} i(1), \ldots, i(u), & i(u+1), & \ldots, & i(k) \\ i(1), \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(k)i(k)\end{pmatrix}\cdot\varepsilon_{12},$$

and we shall prove the equality

$$(\Xi_k\varepsilon_{12})_u = E_{u-1} + F_{u-1} + G_{1,u-1} + G_{2,u} + H_{u-2} - H_{u-1}(u = 1, \ldots, k), \tag{75}$$

where

$$E_u = (-1)^u\sum_{\substack{2<i(1)<\ldots<i(u)\\ 2<i(u+1)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^u\Delta^{a(1)}_1\Delta^{a(2)}_2\Delta^{a(3)}_{i(1)}\ldots\Delta^{a(k)}_{i(k-2)}$$

$$\cdot\begin{pmatrix} 1, 2, i(1), \ldots, i(u), & i(u+1), & \ldots, & i(k-2) \\ j, \alpha, i(1), \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(k-2)i(k-2)\end{pmatrix}$$

$$(u = 0, \ldots, k-2),$$

$$E_{k-1} = 0,$$

$$F_u = (-1)^u\sum_{\substack{2<i(1)<\ldots<i(u)\\ 2<i(u+1)<\ldots<i(k-1)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-1)}\Delta^u\Delta^{a(1)}_1\Delta^{a(2)}_2\Delta^{a(3)}_{i(1)}\ldots\Delta^{a(k+1)}_{i(k-1)}$$

$$\cdot\left[\begin{pmatrix} 2, i(1), \ldots, i(u), & i(u+1), & \ldots, & i(k-1) \\ j, i(1), \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(k-1)i(k-1)\end{pmatrix}\alpha_1\right.$$

$$\left.-\begin{pmatrix} 1, i(1), \ldots, i(u), & i(u+1), & \ldots, & i(k-1) \\ j, i(1), \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(k-1)i(k-1)\end{pmatrix}\alpha_2\right]$$

$$(u = 0, \ldots, k-1),$$

$$G_{1,u} = (-1)^u\sum_{\substack{2<i(1)<\ldots<i(u)\\ 2<i(u+1)<\ldots<i(k-1)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-1)}\Delta^u\Delta^{a(1)}_1\Delta^{a(2)}_2\Delta^{a(3)}_{i(1)}\ldots\Delta^{a(k+1)}_{i(k-1)}$$

$$\cdot\sum_{l=u+1}^{k-1}(-1)^{l-1}\begin{pmatrix} 1, & 2, & i(1), \ldots\ldots, & i(u), & \ldots \\ j, & i(1), & \ldots\ldots, i(u), & j(u+1)i(u+1), & \ldots\end{pmatrix} \tag{76}$$

$$\begin{matrix} \ldots, i(l-1), & i(l+1), & \ldots, & i(k-1) \\ \ldots, j(l)i(l), & j(l+1)i(l+1), & \ldots, & j(k-1)i(k-1)\end{matrix}\Bigg)\alpha_{i(l)}$$

$$(u = 0, \ldots, k-1),$$

$$G_{2,u} = (-1)^u\sum_{\substack{2<i(1)<\ldots<i(u)\\ 2<i(u+1)<\ldots<i(k-1)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-1)}\Delta^u\Delta^{a(1)}_1\Delta^{a(2)}_2\Delta^{a(3)}_{i(1)}\ldots\Delta^{a(k+1)}_{i(k-1)}$$

$$\cdot\sum_{l=1}^{u}(-1)^{l-1}\begin{pmatrix} 1, & 2, & i(1), & \ldots\ldots\ldots, & i(l-1), & i(l+1), & \ldots \\ j, & i(1), & \ldots\ldots, & i(l-1), & i(l), & i(l+1), & \ldots\end{pmatrix} \tag{77}$$

$$\begin{pmatrix} \ldots, i(u), & i(u+1), & \ldots, & i(k-1) \\ \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(k-1)i(k-1) \end{pmatrix} \alpha_{i(l)} \qquad (u = 1, \ldots, k-1),$$

and especially

$$G_{2,0} = 0,$$
$$G_{2,k} = 0,$$
$$H_u = (-1)^u \sum_{\substack{2<i(1)<\ldots<i(u) \\ 2<i(u+1)<\ldots<i(k-2)}} \sum_{a,j} \theta_{12i(1)\ldots i(k-2)} \Delta^u \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \cdots \Delta_{i(k-2)}^{a(k)}$$
$$\cdot \sum_{\lambda} \begin{pmatrix} 1, 2, i(1), \ldots, i(u), & i(u+1), & \ldots, & i(k-2) \\ j, \lambda, i(1), \ldots, i(u), & j(u+1)i(u+1), & \ldots, & j(k-2)i(k-2) \end{pmatrix} \delta_\lambda \qquad (u = 0, \ldots k-2),$$

where λ takes the values $1, 2, i(1), \ldots, i(k-2)$, and especially

$$H_{-1} = 0,$$
$$H_{k-1} = 0.$$

It is easy to see that $\sum_{u=0}^{k-1}(F_u + G_{1,u} + G_{2,u})$ and $\sum_{u=0}^{k-2} E_u$ are equal to the first and the second term of the equality (73), and therefore, from

$$\begin{aligned} \Xi_k \mathcal{E}_{12} &= \sum_{u=1}^{k} (\Xi_k \mathcal{E}_{12})_u \\ &= \sum_{u=1}^{k} E_{u-1} + \sum_{u=1}^{k}(F_{u-1} + G_{1,u-1} + G_{2,u}) + \sum_{u=1}^{k}(H_{u-2} - H_{u-1}) \\ &= \sum_{u=0}^{k-2} E_u + \sum_{u=0}^{k-1}(F_u + G_{1,u} + G_{2,u}) + E_{k-1} + G_{2,k} + H_{-1} \\ &= \sum_{u=0}^{k-2} E_u + \sum_{u=0}^{k-1}(F_u + G_{1,u} + G_{2,u}), \end{aligned}$$

we have immediately our equality (73).

13. Next we shall prove the equality (75). Property (27) allows in (74) only the cases where one of the $i(s) = 1, 2, (s = 1, 2, \ldots, k)$, and we divide these cases into the following,

(i) $i(1) = 1, 2 < i(2) < \ldots < i(u), 2 < i(u+1) < \ldots < i(k)$,
(ii) $2 < i(1) < \ldots < i(u), i(u+1) = 1, 2 < i(u+2) < \ldots < i(k)$;
(iii) $i(1) = 2, 2 < i(2) < \ldots < i(u), 2 < i(u+1) < \ldots < i(k)$,
(iv) $2 < i(1) < \ldots < i(u), i(u+1) = 2, 2 < i(u+2) < \ldots < i(k)$,
(v) $i(1) = 1, 2 < i(2) < \ldots < i(u), i(u+1) = 2, 2 < i(u+2) < \ldots < i(k)$,
(vi) $i(1) = 2, 2 < i(2) < \ldots < i(u), i(u+1) = 1, 2 < i(u+2) < \ldots < i(k)$,
(vii) $2 < i(1) < \ldots < i(u), i(u+1) = 1, i(u+2) = 2, 2 < i(u+3) < \ldots < i(k)$,
(viii) $i(1) = 1, i(2) = 2, 2 < i(3) < \ldots < i(u), 2 < i(u+1) < \ldots < i(k)$,

and it is shown that

(78) $\quad$ (i) + (ii) + (iii) + (iv) $= F_{u-1}$,
(79) $\quad$ (v) + (vi) + (vii) + (viii) $= E_{u-1} + G_{1,u-1} + G_{2,u} + H_{u-2} - H_{u-1}$.

14. The proof of (78). In the case (i), changing the letters $i(2), \ldots, i(k)$ as $i(k) \to i(k-1) \to i(k-2) \to \ldots \to i(3) \to i(2) \to i(1)$, we get

$$(\mathrm{i}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{1i(1)\ldots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_{i(1)}^{a(2)} \cdots \Delta_{i(k-1)}^{a(k)}$$
$$\cdot\begin{pmatrix} 1, i(1), \ldots, i(u-1), & i(u), & \ldots, & i(k-1) \\ 1, i(1), \ldots, i(u-1), & j(u) i(u), & \ldots, & j(k-1) i(k-1) \end{pmatrix} \varepsilon_{12}.$$

Using (22), and calling the relations $\Delta_i \alpha_i = 0$ in mind, we get

$$(\mathrm{i}) = -(-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{12i(1)\ldots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \cdots \Delta_{i(k-1)}^{a(k+1)}$$
$$(80) \qquad \cdot\begin{pmatrix} 1, i(1), \ldots, i(u-1), & i(u), & \ldots, & i(k-1) \\ 1, i(1), \ldots, i(u-1), & j(u) i(u), & \ldots, & j(k-1) i(k-1) \end{pmatrix} \Delta_1 \alpha_2.$$

Similarly, we get

$$(\mathrm{iii}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{12i(1)\ldots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \cdots \Delta_{i(k-1)}^{a(k+1)}$$
$$(81) \qquad \cdot\begin{pmatrix} 2, i(1), \ldots, i(u-1), & i(u), & \ldots, & i(k-1) \\ 2, i(1), \ldots, i(u-1), & j(u) i(u), & \ldots, & j(k-1) i(k-1) \end{pmatrix} \Delta_2 \alpha_1.$$

In case of (ii), changing the letters $i(u+2), \ldots, i(k)$ as $i(k) \to i(k-1) \to \ldots \to i(u+4) \to i(u+3) \to i(u+2)$,

$$(\mathrm{ii}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u)\\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{1i(1)\ldots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_{i(1)}^{a(2)} \cdots \Delta_{i(k-1)}^{a(k)}$$
$$\cdot\begin{pmatrix} i(1), \ldots, i(u), & 1, & i(u+1), & \ldots, & i(k-1) \\ i(1), \ldots, i(u), & j1, & j(u+1) i(u+1), & \ldots, & j(k-1) i(k-1) \end{pmatrix} \varepsilon_{12}.$$

Calling the meaning of j in mind,

$$(\mathrm{ii}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u)\\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{1i(1)\ldots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_{i(1)}^{a(2)} \cdots \Delta_{i(k-1)}^{a(k)}$$
$$\cdot\sum_{\lambda=1}^{k-1} \begin{pmatrix} i(1), \ldots, i(u), & 1, & i(u+1), & \ldots, & i(k-1) \\ i(1), \ldots, i(u), & i(\lambda) 1, & j(u+1) i(u+1), & \ldots, & j(k-1) i(k-1) \end{pmatrix} \Delta_{i(\lambda)} \varepsilon_{12}.$$

From (20), (19), we get,

$$(82) \qquad \begin{aligned} \theta_{1i(1)\ldots i(k-1)} \Delta_p \varepsilon_{q2} &= \theta_{1i(1)\ldots i(k-1)} (\Delta_2 \varepsilon_{qp} + \Delta_q \varepsilon_{p2}) \\ &= \theta_{1i(1)\ldots i(k-1)} \Delta_q \varepsilon_{p2} \end{aligned}$$

for $p, q = 1, i(1), \ldots, i(k-1)$; whence

$$(\mathrm{ii}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u)\\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{1i(1)\ldots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_{i(1)}^{a(2)} \cdots \Delta_{i(k-1)}^{a(k)}$$
$$\cdot\sum_{\lambda=1}^{k-1} \begin{pmatrix} i(1), \ldots, i(u), & 1, & i(u+1), & \ldots, & i(k-1) \\ i(1), \ldots, i(u), & i(\lambda) 1, & j(u+1) i(u+1), & \ldots, & j(k-1) i(k-1) \end{pmatrix} \Delta_1 \varepsilon_{i(\lambda)2}.$$

On the other hand, we get

$$(83) \qquad 0 = (-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u)\\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{1i(1)\ldots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_{i(1)}^{a(2)} \cdots \Delta_{i(k-1)}^{a(k)}$$
$$\cdot\sum_{\lambda=1}^{k-1} \sum_{\mu=1}^{k-1} \begin{pmatrix} i(1), \ldots, i(u), & 1, & i(u+1), & \ldots, & i(k-1) \\ i(1), \ldots, i(u), & i(\lambda) i(\mu), & j(u+1) i(n+1), & \ldots, & j(k-1) i(k-1) \end{pmatrix} \Delta_{i(\mu)} \varepsilon_{i(\lambda)2}.$$

For, from (82),

$$\theta_{1i(1)\ldots i(k-1)} \sum_{\lambda<\mu}^{1,\, \ldots,\, k-1} \begin{pmatrix} i(1), \ldots, i(u), & 1, & i(u+1), & \ldots \\ i(1), \ldots, i(u), & i(\lambda) i(\mu), & j(u+1) i(u+1), & \ldots \end{pmatrix}$$

$$\begin{pmatrix} \ldots, & i(k-1) \\ \ldots, & j(k-1)i(k-1) \end{pmatrix} \Delta_{i(\mu)}\varepsilon_{i(\lambda)2}$$

$$=\theta_{1i(1)\ldots i(k-1)} \sum_{\lambda<\mu}^{1,\ldots,k-1} \begin{pmatrix} i(1), \ldots, i(u), & 1, & i(u+1), & \ldots \\ i(1), \ldots, i(u), & i(\lambda)i(\mu), & j(u+1)i(u+1), & \ldots \end{pmatrix}$$
$$\begin{pmatrix} \ldots, & i(k-1) \\ \ldots, & j(k-1)i(k-1) \end{pmatrix} \Delta_{i(\lambda)}\varepsilon_{i(\mu)2},$$

and, exchanging the letter λ, μ in this right side, and transfering this to the left side, we get our equality (83). Adding (83) to (ii) and calling the meaning of j in mind, we get

$$\text{(ii)} = -(-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u) \\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{1i(1)\ldots i(k-1)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_{i(1)}^{a(2)} \cdots \Delta_{i(k-1)}^{a(k)}$$

(84) $$\cdot \sum_{\lambda=1}^{k-1} \begin{pmatrix} i(1), \ldots, i(u), & 1, & i(u+1), & \ldots \\ i(1), \ldots, i(u), & j(\lambda)i(\lambda), & j(u+1)i(u+1), & \ldots \end{pmatrix}$$
$$\begin{pmatrix} \ldots, & i(k-1) \\ \ldots, & j(k-1)i(k-1) \end{pmatrix} \varepsilon_{i(\lambda)2}.$$

The summation $\sum\limits_{\lambda=1}^{k-1}$ is, however, equal to $\sum\limits_{\lambda=1}^{u}$, for

$$-(-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u) \\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{a,\, j} \theta_{1i(1)\ldots i(k-1)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_{i(1)}^{a(2)} \cdots \Delta_{i(k-1)}^{a(k)}$$
$$\cdot \begin{pmatrix} i(1), \ldots i(u), & 1, & i(u+1), & \ldots, & i(k-1) \\ i(1), \ldots i(u), & j(\lambda)i(\lambda), & j(u+1)i(u+1), & \ldots, & j(k-1)i(k-1) \end{pmatrix} \varepsilon_{i(\lambda)2}$$
$$= 0 \qquad\qquad (\lambda = u+1, \ldots, k),$$

as is easily verified. Moreover, we change the letters $i(1), \ldots, i(u)$, as follows,

if $\lambda = 1$, $\quad i(1) \to i(u) \to i(u-1) \to \ldots \to i(2) \to i(1)$,

if $\lambda = 2$, $\quad i(2) \to i(u) \to i(u-1) \to \ldots \to i(3) \to i(2)$,

......

if $\lambda = r$, $\quad i(1), \ldots, i(r-1)$ are fixed
$\quad i(r) \to i(u) \to i(u-1) \to \ldots \to i(r+1) \to i(r)$,

......

if $\lambda = u$, $\quad$ all letters are fixed.

Then, we get

$$\sum_{\substack{2<i(1)<\ldots<i(u) \\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{\lambda=1}^{u} = \sum_{\substack{2<i(1)<\ldots<i(u-1) \\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{2<i(u)},$$

that is,

$$\text{(ii)} = (-1)^{u-1} \sum_{\substack{2<i(1)<\ldots<i(u-1) \\ 2<i(u+1)<\ldots<i(k-1)}} \sum_{2<i(u)} \sum_{a,\, j} \theta_{1i(1)\ldots i(k-1)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_{i(1)}^{a(2)} \cdots \Delta_{i(k-1)}^{a(k)}$$
$$\cdot \begin{pmatrix} i(1), \ldots, i(u-1), & 1, & i(u), & \ldots, & i(k-1) \\ i(1), \ldots, i(u-1), & i(u), & j(u)i(u), & \ldots, & j(k-1)i(k-1) \end{pmatrix} \varepsilon_{i(u)2}.$$

Moreover, change the letters $i(u), i(u+1), \ldots, i(k-1)$ as $i(u) \to i(k-1) \to i(k-2) \to \ldots \to i(u+2) \to i(u+1) \to i(u)$, and then, putting the letter $i(k-1)$ into the inequality $2<i(u)<\ldots<i(k-2)$, we get at last

$$(\mathrm{ii}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-1)}} \sum_{a,\,j} \theta_{1i(1)\dots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_{i(1)}^{a(2)} \dots \Delta_{i(k-1)}^{a(k)}$$
$$\cdot \sum_{\lambda=u}^{k-1} \begin{pmatrix} 1, & i(1), \dots, i(u-1), & i(u), & \dots, & i(k-1) \\ i(\lambda), & i(1), \dots, i(u-1), & j(u)i(u), & \dots, & j(k-1)i(k-1) \end{pmatrix} \varepsilon_{i(\lambda)2}.$$

As is easily seen, $\sum_{\lambda=u}^{k-1} = \sum_{\lambda=1}^{k-1}$ in the above equality, and, as was done in the calculation of (80), by virtue of the relations (22),

$$(\mathrm{ii}) = -(-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-1)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \dots \Delta_{i(k-1)}^{a(k+1)} \tag{85}$$
$$\cdot \sum_{\lambda=1}^{k-1} \begin{pmatrix} 1, & i(1), \dots, i(u-1), & i(u), & \dots, & i(k-1) \\ i(\lambda), & i(1), \dots, i(u-1), & j(u)i(u), & \dots, & j(k-1)i(k-1) \end{pmatrix} \Delta_{i(\lambda)} \alpha_2.$$

Similarly, we get

$$(\mathrm{iv}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-1)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \dots \Delta_{i(k-1)}^{a(k+1)} \tag{86}$$
$$\cdot \sum_{\lambda=1}^{k-1} \begin{pmatrix} 2, & i(1), \dots, i(u-1), & i(u), & \dots, & i(k-1) \\ i(\lambda), & i(1), \dots, i(u-1), & j(u)i(u), & \dots, & j(k-1)i(k-1) \end{pmatrix} \Delta_{i(\lambda)} \alpha_1.$$

Now, adding (80) and (85), and, calling the meaning of j in mind we get

$$(\mathrm{i}) + (\mathrm{ii}) = -(-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-1)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \dots \Delta_{i(k-1)}^{a(k+1)}$$
$$\cdot \begin{pmatrix} 1, i(1), \dots, i(u-1), & i(u), & \dots, & i(k-1) \\ j, i(1), \dots, i(u-1), & j(u)i(u), & \dots, & j(k-1)i(k-1) \end{pmatrix} \alpha_2,$$

and similarly, from (81) and (86),

$$(\mathrm{iii}) + (\mathrm{iv}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-1)}} \sum_{a\,j} \theta_{12i(1)\dots i(k-1)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \dots \Delta_{i(k-1)}^{a(k+1)}$$
$$\cdot \begin{pmatrix} 2, i(1), \dots, i(u-1), & i(u), & \dots, & i(k-1) \\ j, i(1), \dots, i(u-1), & j(u)i(u), & \dots, & j(k-1)i(k-1) \end{pmatrix} \alpha_1.$$

Adding above two equalities, we get finally our equality (78) as was desired.

15. The proof of (79). In case of (v), changing the letters $i(2), \dots, i(u)$ as $i(u) \to i(u-1) \to i(u-2) \to \dots \to i(3) \to i(2) \to i(1)$, and also the letters $i(u+2), \dots, i(k)$ as $i(k) \to i(k-2), \dots, i(r) \to i(r-2), \dots, i(u+2) \to i(u)$, we get

$$(\mathrm{v}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-2)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \dots \Delta_{i(k-2)}^{a(k)}$$
$$\cdot \begin{pmatrix} 1, i(1), \dots, i(u-1), 2, & i(u), & \dots, & i(k-2) \\ 1, i(1), \dots, i(u-1), j2, & j(u)i(u), & \dots, & j(k-2)i(k-2) \end{pmatrix} \varepsilon_{12}$$
$$= (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-2)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1} \Delta_1^{a(1)} \Delta_2^{a(2)} \Delta_{i(1)}^{a(3)} \dots \Delta_{i(k-2)}^{a(k)} \tag{87}$$
$$\cdot \Bigg[\begin{pmatrix} 1, i(1), \dots, i(u-1), 2, & i(u), & \dots, & i(k-2) \\ 1, i(1), \dots, i(u-1), 12, & j(u)i(u), & \dots, & j(k-2)i(k-2) \end{pmatrix} \Delta_1 \varepsilon_{12}$$
$$+ \sum_{\lambda=1}^{k-2} \begin{pmatrix} 1, i(1), \dots, i(u-1), & 2 & i(u), & \dots, & i(k-2) \\ 1, i(1), \dots, i(u-1), & i(\lambda)2, & j(u)i(u), & \dots, & j(k-2)i(k-2) \end{pmatrix} \Delta_{i(\lambda)} \varepsilon_{12} \Bigg].$$

Similary, we get

$$(\mathrm{vi}) = (-1)^{u-1} \sum_{\substack{2<i(1)\dots<i(u-1)\\ 2<i(u)\dots<i(k-2)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\tag{88} \cdot\left[\begin{pmatrix} 1, i(1), \dots, i(u-1), 2, & i(u), & \dots, & i(k-2) \\ 2, i(1), \dots, i(u-1), 12, j(u)i(u), & & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_2\varepsilon_{12} + \sum_{\lambda=1}^{k-2}\begin{pmatrix} 1, i(1), \dots, i(u-1), & 2, & i(u), & \dots, & i(k-2) \\ 2, i(1), \dots, i(u-1), 1i(\lambda), j(u)i(u), & & \dots & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{12}\right].$$

Next, we modify the case (vii). First of all, we change the letters $i(k), \dots, i(u+3)$ as $i(k)\to i(k-2), \dots, i(r)\to i(r-2), \dots, i(u+3)\to i(u+1)$, then we get

$$(\mathrm{vii}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u)\\ 2<i(u+1)<\dots<i(k-2)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\begin{pmatrix} i(1), \dots, i(u), 1, 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots, i(u), j1, j2, j(u+1)i(u+1), & & \dots, & j(k-2)i(k-2) \end{pmatrix}\varepsilon_{12}.$$

Calling the meaning of j in mind, we divide this into three parts as was done when getting (87), (88); that is,

$$(\mathrm{vii}) = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u)\\ 2<i(u+1)<\dots<i(k-2)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\left[\sum_{\lambda=1}^{k-2}\begin{pmatrix} i(1), \dots, i(u), 1, & 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots, i(u), 21, i(\lambda)2, j(u+1)i(u+1), & & \dots & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_2\Delta_{i(\lambda)}\varepsilon_{12}\right.$$

$$+\sum_{\lambda=1}^{k-2}\begin{pmatrix} i(1), \dots i(u), & 1, & 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots i(u), i(\lambda)1, 12, j(u+1)i(u+1), & & & \dots & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_1\Delta_{i(\lambda)}\varepsilon_{12}$$

$$\left.+\sum_{\lambda,\,\mu}^{1,\dots,k-2}\begin{pmatrix} i(1), \dots, i(u), & 1, & 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots, i(u), i(\lambda)1, i(\mu)2, j(u+1)i(u+1), & & & \dots & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\lambda)}\Delta_{i(\mu)}\varepsilon_{12}\right].$$

The sum S of the first and the second term of this equality is equal to

$$(-1)^{u-1} \sum_{\substack{2<i(1)<\dots i(u-1)\\ 2<i(u)<\dots<i(k-2)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\tag{89} \cdot\sum_{\lambda=1}^{k-2}\begin{pmatrix} 1, & i(1), \dots, i(u-1), 2, & i(u), & \dots, & i(k-2) \\ i(\lambda), & i(1), \dots, i(u-1), 12, j(u)i(u), & & \dots & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{12},$$

for,

$$\tag{90} S = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u)\\ 2<i(u+1)<\dots<i(k-2)}} \sum_{a,\,j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda=1}^{k-2}\sum_{\xi=1,\,2}\begin{pmatrix} i(1), \dots, i(u), & 1, & 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots, i(u), & 12, & \xi i(\lambda), & j(u+1)i(u+1), & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{\xi}\Delta_{i(\lambda)}\varepsilon_{12}.$$

On the other hand, exchanging the letter λ, μ we get

$$\sum_{\lambda<\mu}^{1,\dots,k-2}\begin{pmatrix} i(1), \dots, i(u), & 1, & 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots, i(u), & 12, & i(\mu)i(\lambda), & j(u+1)i(u+1), & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\mu)}\Delta_{i(\lambda)}\varepsilon_{12}$$

$$= \sum_{\mu<\lambda}^{1,\dots,k-2}\begin{pmatrix} i(1), \dots, i(u), & 1, & 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots, i(u), & 12, & i(\lambda)i(\mu), & j(u+1)i(u+1), & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\mu)}\Delta_{i(\lambda)}\varepsilon_{12}.$$

Transfering the right side of this equality to the left one, we may have easily

$$0 = (-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u)\\ 2<i(u+1)<\dots<i(k-2)}}\sum_{a,\,j}\theta_{12i(1)\dots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda,\,\mu}^{1,\dots,k-2}\begin{pmatrix} i(1), \dots, i(u), & 1, & 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots, i(u), & 12, & i(\mu)i(\lambda), & j(u+1)i(u+1), & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\mu)}\Delta_{i(\lambda)}\varepsilon_{12}. \tag{91}$$

Adding (90) and (91), we get

$$S = (-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u)\\ 2<i(u+1)<\dots<i(k-2)}}\sum_{a,\,j}\theta_{12i(1)\dots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda=1}^{k-2}\begin{pmatrix} i(1), \dots, i(u), & 1, & 2, & i(u+1), & \dots, & i(k-2) \\ i(1), \dots, i(u), & 12, & j(\lambda)i(\lambda), & j(u+1)i(u+1), & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{12},$$

and then we get our equality (89) in the same manner as was done when we deduced (85) from (84).

Therefore

$$(\mathrm{vii}) = (-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-2)}}\sum_{a,\,j}\theta_{12i(1)\dots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda=1}^{k-2}\begin{pmatrix} 1, & i(1), \dots, i(u-1), & 2, & i(u), & \dots, & i(k-2) \\ i(\lambda), & i(1), \dots, i(u-1), & 12, & j(u)i(u), & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{12} \tag{92}$$

$$+ (-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u)\\ 2<i(u+1)<\dots<i(k-2)}}\sum_{a,\,j}\theta_{12i(1)\dots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda,\,\mu}^{1,\dots,k-2}\begin{pmatrix} i(1), \dots, i(u), & 1, & 2, & i(u+1), & \dots \\ i(1), \dots, i(u), & i(\lambda)1, & i(\mu)2, & j(u+1)i(u+1), & \dots \end{pmatrix}$$

$$\begin{pmatrix} \dots, & i(k-2) \\ \dots, j(k-2)i(k-2) \end{pmatrix} \Delta_{i(\mu)}\Delta_{i(\lambda)}\varepsilon_{12}.$$

In the last case (viii), we change the letters $i(k), \dots, i(3)$ as $i(r) \to i(r-2)\,(r=3, \dots, k)$, then

$$\text{(viii)} = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-2) \\ 2<i(u-1)<\dots<i(k-2)}} \sum_{a, j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$

(93)
$$\cdot\begin{pmatrix} 1, 2, i(1), \dots, i(u-2), & i(u-1), & \dots, & i(k-2) \\ 1, 2, i(1), \dots, i(u-2), & j(u-1)i(u-1), & \dots, & j(k-2)i(k-2) \end{pmatrix}\varepsilon_{12}.$$

We must notice the fact that this occurs only in the cases where $u \geqq 2$, and therefore, there exists always the element Δ.

Now, we calculate the sum of the equality (87), (88), (92) and (93). Adding the first terms of the equalities (87), (88) and (92), we get

$$(-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1) \\ 2<i(u)<\dots<i(k-2)}} \sum_{a, j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$
$$\cdot\begin{pmatrix} 1, i(1), \dots, i(u-1), 2, & i(u), & \dots, & i(k-2) \\ j, i(1), \dots, i(u-1), 12, & j(u)i(u), & \dots, & j(k-2)ik(-2) \end{pmatrix}\varepsilon_{12}.$$

Putting (70) into the above expression, we get

$$E_{u-1} - L_{u-1} - M_{u-1},$$

where

$$L_{u-1} = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1) \\ 2<i(u)<\dots<i(k-2)}} \sum_{a, j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$
$$\cdot\sum_{r<s}\begin{pmatrix} 1, i(1), \dots, i(u-1), 2, & i(u), & \dots, & i(k-2) \\ j, i(1), \dots, i(u-1), rs, & j(u)i(u), & \dots, & j(k-2)i(k-2) \end{pmatrix}\varepsilon_{rs},$$

$$M_{u-1} = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1) \\ 2<i(u)<\dots<i(k-2)}} \sum_{a, j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$
$$\cdot\sum_{t}\begin{pmatrix} 1, i(1), \dots, i(u-1), 2, & i(u), & \dots, & i(k-2) \\ j, i(1), \dots, i(u-1), t, & j(u)i(u), & \dots, & j(k-2)i(k-2) \end{pmatrix}\delta_t.$$

Now, adding (87), (88), (92), and (93), we get

$$\text{(v)} + \text{(vi)} + \text{(vii)} + \text{(viii)} = E_{u-1} - L_{u-1} - M_{u-1} + P + Q + R + S,$$

where L_{u-1} and M_{u-1} were already described above, and

$$P = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1) \\ 2<i(u)<\dots<i(k-2)}} \sum_{a, j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$
$$\cdot\sum_{\lambda=1}^{k-2}\begin{pmatrix} 1, i(1), \dots, i(u-1), & 2, & i(u), & \dots, & i(k-2) \\ 1, i(1), \dots, i(u-1), & 2i(\lambda)j(u), & i(u) & \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{12},$$

$$Q = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u-1) \\ 2<i(u)<\dots<i(k-2)}} \sum_{a, j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$
$$\cdot\sum_{\lambda=1}^{k-2}\begin{pmatrix} 1, i(1), \dots i(u-1), & 2, & i(u), & \dots \\ 2, i(1), \dots i(u-1), & 1i(\lambda), & j(u)i(u), & \dots \end{pmatrix}$$
$$\begin{pmatrix} \dots, & i(k-2) \\ \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{12},$$

$$R = (-1)^{u-1} \sum_{\substack{2<i(1)<\dots<i(u) \\ 2<i(u+1)<\dots<i(k-2)}} \sum_{a, j} \theta_{12i(1)\dots i(k-2)} \Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda,\mu=1}^{k-2}\begin{pmatrix} i(1), \dots i(u), & 1, & 2, & i(u+1), & \dots \\ i(1), \dots i(u), & i(\lambda)1, & i(\mu)2, & j(u+1)i(u+1), & \dots \end{pmatrix}$$

$$\begin{pmatrix} \dots, & i(k-2) \\ \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\mu)}\Delta_{i(\lambda)}\varepsilon_{12},$$

$$S = (-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u-2)\\ 2<i(u-1)<\dots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\dots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\begin{pmatrix} 1, 2, i(1), \dots, i(u-2), & i(u-1), & \dots, & i(k-2) \\ 1, 2, i(1), \dots, i(u-2), & j(u-1)i(u-1), & \dots, & j(k-2)i(k-2) \end{pmatrix}\varepsilon_{12}.$$

We shall prove in the next section that

$$L_{u-1} - S = -G_{1,u-1} - H_{u-2} + P + Q + R, \tag{94}$$

$$M_{u-1} = -G_{2,u} + H_{u-1}, \tag{95}$$

and then we have our equality (79) from these two equalities.

16. The proof of the equality (94). First of all, we calculate L_{u-1}. By virtue of the property (27), there remain only the following cases,

(a) $r = 1, 2, \quad s > 2, s \neq i(1), \dots, i(k-2)$;

(b) $r = i(\mu), \quad s > i(\mu), s \neq i(\lambda)$ such that $i(\lambda) > i(\mu)$.
$s = i(\mu), \quad 2 < r < i(\mu), r \neq i(\lambda)$ such that $i(\lambda) < i(\mu)$;
$(\mu = 1, \dots, k-2)$,

(c) $r = 1, 2, \quad s = i(\mu) \quad (\mu = 1, 2, \dots, k-2)$;

(d) $r = i(\mu), \quad s = i(\lambda)(\lambda, \mu = 1, \dots, k-2)$ such that $i(\mu) < i(\lambda)$.

In case (a), describing $i(k-1)$ instead of s, and then, using the property (22), we get easily,

$$(a) = -(-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-2)}}\sum_{2<i(k-1)}\sum_{a,j}\theta_{12i(1)\dots i(k-1)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-1)}^{a(k+1)}$$

$$\cdot\sum_{\lambda=1,2}\begin{pmatrix} 1, i(1), \dots, i(u-1), & 2, & i(u), & \dots \\ j, i(1), \dots, i(u-1), & \lambda i(k-1), & j(u)i(u), & \dots \end{pmatrix}$$

$$\begin{pmatrix} \dots, & i(k-2) \\ \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_\lambda\alpha_{i(k-1)}.$$

In case (b), the summation with respect to r and s may be considered identical with the sum in which

$$r = i(\lambda), s = i(\mu)\,;\ \lambda, \mu = 1, 2, \dots, k-2.$$

Therefore, writing $i(k-1)$ instead of s, and calling the property (22) in mind, we get

$$(b) = -(-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-2)}}\sum_{2<i(k-1)}\sum_{a,j}\theta_{12i(1)\dots i(k-1)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-1)}^{a(k+1)}$$

$$\cdot\sum_{\lambda=1}^{k-2}\begin{pmatrix} 1, i(1), \dots, i(u-1), & 2, & i(u), & \dots \\ j, i(1), \dots, i(u-1), & i(\lambda)i(k-1), & j(u)i(u), & \dots \end{pmatrix}$$

$$\begin{pmatrix} \dots, & i(k-2) \\ \dots, & j(k-2)i(k-2) \end{pmatrix}\Delta_{i(\lambda)}\alpha_{i(k-1)}.$$

Moreover

$$(c) = (-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u-1)\\ 2<i(u)<\dots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\dots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda=1}^{k-2}\sum_{\xi=1,2}\begin{pmatrix}1, i(1), \ldots, i(u-1), & 2, & i(u), & \ldots \\ j, i(1), \ldots, i(u-1), & \xi i(\lambda), & j(u)i(u), & \ldots\end{pmatrix.}$$

$$\begin{pmatrix.}\ldots, & i(k-2) \\ \ldots, & j(k-2)i(k-2)\end{pmatrix}\varepsilon_{\xi i(\lambda)},$$

$$(\mathrm{d}) = (-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\ldots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda<\mu}^{1,\ldots,k-2}\begin{pmatrix}1, i(1), \ldots, i(u-1), & 2, & i(u), & \ldots \\ j, i(1), \ldots, i(u-1), & i(\lambda)i(\mu), & j(u)i(u), & \ldots\end{pmatrix.}$$

$$\begin{pmatrix.}\ldots, & i(k-2) \\ \ldots, & j(k-2)i(k-2)\end{pmatrix}\varepsilon_{i(\lambda)i(\mu)}.$$

Now, adding (a) and (b), we get

$$(\mathrm{a}) + (\mathrm{b}) = -(-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-2)}}\sum_{2<i(k-1)}\sum_{a,j}\theta_{12i(1)\ldots i(k-1)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}$$

$$\ldots\Delta_{i(k-1)}^{a(k+1)}\begin{pmatrix}1, i(1), \ldots, i(u-1), & 2, & i(u), & \ldots \\ j, i(1), \ldots, i(u-1), & j(k-1)i(k-1), & j(u)i(u), & \ldots\end{pmatrix.}$$

$$\begin{pmatrix.}\ldots, & i(k-2) \\ \ldots, & j(k-2)i(k-2)\end{pmatrix}\alpha_{i(k-1)},$$

and then, putting the letter $i(k-1)$ into the inequality $2<i(u)<\ldots<i(k-2)$, and finally, arranging the rows and columns in order, we get $-G_{1,u-1}$. Therefore, we have only to prove that

$$(96)\qquad -S + (\mathrm{c}) + (\mathrm{d}) = -H_{u-2} + P + Q + R.$$

S is the expression (93), and, as there exists always the element Δ in this expression, we get by virtue of the property (21)

$$-S = (-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u-2)\\ 2<i(u-1)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-2}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\ldots\Delta_{i(k-2)}^{a(k)}$$

$$(97)\qquad \cdot\sum_{\eta,\xi=1,2}\begin{pmatrix}1, 2, i(1), \ldots, i(u-2), & i(u-1), & \ldots \\ \eta, \xi, i(1), \ldots, i(u-2), & j(u-1)i(u-1), & \ldots\end{pmatrix.}$$

$$\begin{pmatrix.}\ldots, & i(k-2) \\ \ldots, & j(k-2)i(k-2)\end{pmatrix}\Delta_\eta\delta_\xi.$$

The letter j in the first columns of (c) and (d) takes the values $1, 2, i(1), \ldots, i(k-2)$, and, if we write $1, 2, i(1), \ldots, i(k-2)$ instead of j, we must affix the element $\Delta_1, \Delta_2, \Delta_{i(1)}, \ldots, \Delta_{i(k-2)}$ to it, respectively. Therefore, calling the property (20) in mind, we get

$$(\mathrm{c}) = (-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\ldots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda=1}^{k-2}\sum_{\xi=1,2}\sum_{\eta=1,2,i(1),\ldots,i(k-2)}\begin{pmatrix}1, i(1), \ldots, i(u-1), & 2, & i(u), & \ldots \\ \eta, i(1), \ldots, i(u-1), & \xi i(\lambda), & j(u)i(u), & \ldots\end{pmatrix.}$$

$$\begin{pmatrix.}\ldots, & i(k-2) \\ \ldots, & j(k-2)i(k-2)\end{pmatrix}(\Delta_{i(\lambda)}\varepsilon_{\xi\eta} + \Delta_\xi\varepsilon_{\eta i(\lambda)}).$$

Similarly,

$$(\mathrm{d}) = (-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\ldots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda<\mu}^{1,\ldots,k-2}\sum_{\eta=1,2,i(1),\ldots,i(k-2)}\begin{pmatrix}1, i(1), \ldots, i(u-1), & 2, & i(u), & \ldots\\ \eta, i(1), \ldots, i(u-1), & i(\lambda)i(\mu), & j(u)i(u), & \ldots\end{pmatrix.$$

$$\begin{matrix}\ldots, & i(k-2)\\ \ldots, & j(k-2)i(k-2)\end{pmatrix}(\Delta_{i(\mu)}\varepsilon_{i(\lambda)\eta}+\Delta_{i(\lambda)}\varepsilon_{\eta i(\mu)}).$$

Moreover, by the same method as we get the equality (91), we have

$$(\mathrm{d}) = (-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\sum_{\lambda,\mu}^{1,\ldots,k-2}\sum_{\eta=1,2,i(1),\ldots,i(k-2)}\begin{pmatrix}1, i(1), \ldots, i(u-1), & 2, & i(u), & \ldots & \ldots, & i(k-2)\\ \eta, i(1), \ldots, i(u-1), & i(\mu)i(\lambda), & j(u)i(u), & \ldots & \ldots, & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\mu)}\varepsilon_{\eta i(\lambda)}.$$

Then

$$(98)\quad \begin{aligned}(\mathrm{c}) + (\mathrm{d}) = (-1)^{u-1}&\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\\ &\cdots\Delta_{i(k-2)}^{a(k)}\cdot\Bigg[\sum_{\lambda=1}^{k-2}\sum_{\eta=1,2,i(1),\ldots,i(k-2)}\begin{pmatrix}1, i(1), \ldots\\ \eta, i(1), \ldots\end{pmatrix.\\ &\begin{matrix}\ldots, i(u-1), & 2, & i(u), & \ldots & i(k-2)\\ \ldots, i(u-1), & j(\lambda)i(\lambda), & j(u)i(u), & \ldots & j(k-2)i(k-2)\end{pmatrix}\varepsilon_{\eta i(\lambda)}\\ &+\sum_{\lambda=1}^{k-2}\sum_{\eta=1,2,i(1),\ldots,i(k-2)}\sum_{\xi=1,2}\begin{pmatrix}1, i(1), \ldots, i(u-1), & 2, & i(u), & \ldots\\ \eta, i(1), \ldots, i(u-1), & \xi i(\lambda), & j(u)i(u), & \ldots\end{pmatrix.\\ &\begin{matrix}\ldots, & i(k-2)\\ \ldots, & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{\xi\eta}\Bigg].\end{aligned}$$

In this first term $[(\mathrm{c})+(\mathrm{d})]_1$, the summation $\sum_{\lambda=1}^{k-2}$ is really $\sum_{\lambda=1}^{u-1}$, and therefore it remains only in the case of $u\geqq 2$, and then, there exists always the element Δ.

We can prove that

$$(99)\quad\begin{aligned}&[(\mathrm{c}) + (\mathrm{d})]_1 - S\\ &= -H_{u-2}\\ &-(-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-2}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-2)}^{a(k)}\\ &\cdot\sum_{\lambda,\mu}^{1,\ldots,k-2}\begin{pmatrix}1, & i(1), \ldots, i(u-1), & 2, & i(u), & \ldots\\ i(\mu), & i(1), \ldots, i(u-1), & j(\lambda)i(\lambda), & j(u)i(u), & \ldots\end{pmatrix.\\ &\begin{matrix}\ldots, & i(k-2)\\ \ldots, & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\mu)}\delta_{i(\lambda)}.\end{aligned}$$

For, by virtue of (21),

$$(100)\quad\begin{aligned}[(\mathrm{c}) + (\mathrm{d})]_1 = (-1)^{u-1}&\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-2}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\\ &\cdots\Delta_{i(k-2)}^{a(k)}\cdot\Bigg[\sum_{\lambda=1}^{k-2}\sum_{\eta=1,2,i(i),\ldots,i(k-2)}\begin{pmatrix}1, i(1), \ldots\\ \eta, i(1), \ldots\end{pmatrix.\\ &\begin{matrix}\ldots, i(u-1), & 2, & i(u), & \ldots, & i(k-2)\\ \ldots, i(u-1), & j(\lambda)i(\lambda), & j(u)i(u), & \ldots, & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\lambda)}\delta_\eta\end{aligned}$$

$$-\sum_{\lambda=1}^{k-2}\sum_{\eta=1,2}\begin{pmatrix}1, i(1), \dots, i(u-1), & 2, & i(u), & \dots, & i(k-2) \\ \eta, i(1),\dots,i(u-1), & j(\lambda)i(\lambda), & j(u)i(u), & \dots, & j(k-2)i(k-2)\end{pmatrix}\Delta_\eta\delta_{i(\lambda)}$$

$$-\sum_{\lambda=1}^{k-2}\sum_{\mu=1}^{k-2}\begin{pmatrix}1, & i(1),\dots i(u-1), & 2, & i(u), & \dots, & i(k-2)\\ i(\mu), & i(1),\dots i(u-1), & j(\lambda)i(\lambda), & j(u)i(u), & \dots, & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\mu)}\delta_{i(\lambda)}\Big],$$

and, by the same reason as we have got (85) from (84), the first and the second term is equal to

$$(-1)^{u-2}\sum_{\substack{2<i(1)<\dots<i(u-2)\\2<i(u-1)<\dots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\dots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\Big[\sum_{\eta=1,2,i(1),\dots,i(k-2)}\sum_{\lambda=1}^{k-2}\begin{pmatrix}1, & 2, i(1),\dots, i(u-2), & i(u-1), & \dots & i(k-2)\\ i(\lambda), & \eta, i(1),\dots, i(u-2), & j(u-1)i(u-1), & \dots & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\lambda)}\delta_\eta$$

$$+\sum_{\xi=i(1),\dots,i(k-2)}\sum_{\eta=1,2}\begin{pmatrix}1, 2, i(1),\dots,i(u-2), & i(u-1), & \dots, & i(k-2)\\ \eta,\xi,i(1),\dots,i(u-2), & j(u-1)i(u-1), & \dots, & j(k-2)i(k-2)\end{pmatrix}\Delta_\eta\delta_\xi\Big],$$

and, adding $-S$ to this, we get $-H_{u-2}$, and then, our equality (99).

The second term of (98) is equal to

$$(-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u-1)\\2<i(u)<\dots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\dots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\Big[\sum_{\lambda=1}^{k-2}\begin{pmatrix}1, i(1),\dots,i(u-1), & 2, & i(u) & \dots, & i(k-2)\\ 1, i(1),\dots,i(u-1), & i(\lambda)2, & j(u)i(u), & \dots, & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{12}$$

$$+\sum_{\lambda=1}^{k-2}\begin{pmatrix}1, i(1),\dots,i(u-1), & 2, & i(u) & \dots, & i(k-2)\\ 2, i(1),\dots,i(u-1), & 1i(\lambda), & j(u)i(u), & \dots, & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\lambda)}\cdot\varepsilon_{12}$$

$$+\sum_{\lambda=1}^{k-2}\sum_{\mu=1}^{k-2}\sum_{\nu=1,2}\begin{pmatrix}1, & i(1),\dots,i(u-1), & 2, & i(u), & \dots, & i(k-2)\\ i(\mu), & i(1),\dots,i(u-1), & \nu i(\lambda), & j(u)i(u), & \dots, & j(k-2)i(k-2)\end{pmatrix}\Delta_{i(\lambda)}\varepsilon_{\nu i(\mu)}\Big],$$

and the first and the second term are P and Q respectively.

Therefore, we have our equality (96), if we prove that R is equal to the sum of last term of (99) and the last term of above; that is

$$R=(-1)^{u-1}\sum_{\substack{2<i(1)<\dots<i(u-1)\\2<i(u)<\dots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\dots i(k-2)}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\dots\Delta_{i(k-2)}^{a(k)}$$

(101) $$\cdot\Big[-\Delta^{u-2}\sum_{\lambda,\mu=1}^{k-2}\begin{pmatrix}1, & i(1),\dots,i(u-1), & 2, & i(u), & \dots\\ i(\mu), & i(1),\dots,i(u-1), & j(\lambda)i(\lambda), & j(u)i(u), & \dots\end{pmatrix}$$

$$\left.\begin{matrix} \ldots, & i(k-2) \\ \ldots, & j(k-2)i(k-2) \end{matrix}\right)\Delta_{i(\mu)}\delta_{i(\lambda)}\Bigg]$$

$$+\Delta^{u-1}\sum_{\lambda,\mu=1}^{k-1}\sum_{\nu=1,2}\begin{pmatrix} 1, & i(1), \ldots, i(u-1), & 2, & i(u), & \ldots \\ i(\mu), & i(1), \ldots, i(u-1), & \nu i(\lambda), & j(u)i(u), & \ldots \end{pmatrix}$$

$$\left.\begin{matrix} \ldots & i(k-2) \\ \ldots & j(k-2)i(k-2) \end{matrix}\right)\Delta_{i(\lambda)}\varepsilon_{\nu i(\mu)}\Bigg].$$

We shall prove this now. Using the methed by which we may easily deduce (84) from (85), the right side of (101) is equal to

$$(-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u) \\ 2<i(u+1)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\ldots\Delta_{i(k-2)}^{a(k)}$$

$$\cdot\Bigg[-\Delta^{u-2}\sum_{\lambda,\mu}^{1,\ldots,k-2}\begin{pmatrix} i(1), & \ldots, & i(u), & 1, & 2, & i(u+1), & \ldots \\ i(1), & \ldots, & i(u), & j(\lambda)i(\lambda), & j(\mu)i(\mu), & j(u+1)i(u+1), & \ldots \end{pmatrix}$$

$$\left.\begin{matrix} \ldots, & i(k-2) \\ \ldots, & j(k-2)i(k-2) \end{matrix}\right)\Delta_{i(\mu)}\delta_{i(\lambda)}$$

$$+\Delta^{u-1}\sum_{\lambda,\mu}^{1,\ldots,k-2}\sum_{\nu=1,2}\begin{pmatrix} i(1), \ldots, i(u), & 1, & 2, & i(u+1), & \ldots \\ i(1), \ldots, i(u), & \nu i(\lambda), & j(u)i(u), & j(u+1)i(u+1), & \ldots \end{pmatrix}$$

$$\left.\begin{matrix} \ldots, & i(k-2) \\ \ldots, & j(k-2)i(k-2) \end{matrix}\right)\Delta_{i(\lambda)}\varepsilon_{\nu i(\mu)}\Bigg].$$

To avoid the complication, we write

$$(-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u) \\ 2<i(u+1)<\ldots<i(k-2)}}\sum_{a,j}\theta_{12i(1)\ldots i(k-2)}\Delta^{u-1}\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\ldots\Delta_{i(k-2)}^{a(k)}$$

simply by $\sum$, and the determinant

$$\begin{pmatrix} i(1), \ldots, i(u), 1, & 2, & i(u+1), & \ldots, & i(k-2) \\ i(1), \ldots, i(u), pq, & rs, & j(u+1)i(u+1), & \ldots, & j(k-2)i(k-2) \end{pmatrix}$$

simply by $(pq,\ rs)$. Then, calling the property (21) in mind, our expression is equal to

$$\sum\Bigg[\sum_{\lambda<\mu}^{1,\ldots,k-2}(j_1(\lambda)i(\lambda), j_2(\mu)i(\mu))\varepsilon_{i(\mu)i(\lambda)}+\sum_{\lambda,\mu}^{1,\ldots,k-2}\sum_{\nu=1,2}(\nu i(\lambda), j_3(\mu)i(\mu))\Delta_{i(\lambda)}\varepsilon_{\nu i(\mu)}\Bigg].$$

We divide this expression into the following

(i) $j_i=1,2 \quad (i=1,2,3)$;

(ii) $j_1=1,2, \qquad j_2=i(1),\ldots,i(k-2)$;

$\quad j_2=1,2, \qquad j_1=i(1),\ldots,i(k-2)$;

$\quad j_3=i(1),\ldots,i(k-2)$;

(iii) $j_1=i(1),\ldots,i(k-2),\ j_2=i(1),\ldots,i(k-2)$;

and we shall prove that

(102) $\qquad$ (i) $=R$,

(103) $\qquad$ (ii) $=0$,

(104) $\qquad$ (iii) $=0$,

and then, we have our equality (101) as was required.

To calculate the expression (i), we divide it into three parts

(1°) if $j_i=1 (i=1,2,3)$ and $\nu=1$, then, we have

$$\sum\Bigg[\sum_{\lambda<\mu}^{1,\ldots,k-2}(1i(\lambda), 1i(\mu))\Delta_1^2\varepsilon_{i(\mu)i(\lambda)}+\sum_{\lambda,\mu}^{1,\ldots,k-2}(1i(\lambda), 1i(\mu))\Delta_1\Delta_{i(\lambda)}\varepsilon_{1i(\mu)}\Bigg]$$

$$= \sum \sum_{\lambda<\mu}^{1,\ldots,k-2} (1i(\lambda), 1i(\mu))\Delta_1(\Delta_1\varepsilon_{i(\mu)i(\lambda)} + \Delta_{i(\lambda)}\varepsilon_{1i(\mu)} + \Delta_{i(\mu)}\varepsilon_{i(\lambda)1})$$
$$= 0$$

by virtue of the property (20).

(2°) if $j_i=2$, $(i=1, 2, 3)$ and $\nu=2$, then, these terms vanish by the same reason. Therefore, there remains only the following terms.

(3°) $j_1=1, j_2=2$; $j_1=2, j_2=1$; $j_3=1, \nu=2$; $j_3=2, \nu=1$; and then

$$\begin{aligned}(\text{i}) = \sum\Big[&\sum_{\lambda<\mu}^{1,\ldots,k-2} (1i(\lambda), 2i(\mu))\Delta_1\Delta_2\varepsilon_{i(\mu)i(\lambda)} \\ &+ \sum_{\lambda<\mu}^{1,\ldots,k-2} (2i(\lambda), 1i(\mu))\Delta_1\Delta_2\varepsilon_{i(\mu)i(\lambda)} \\ &+ \sum_{\lambda,\mu}^{1,\ldots,k-2} (1i(\lambda), 2i(\mu))\Delta_2\Delta_{i(\lambda)}\varepsilon_{1i(\mu)} \\ &+ \sum_{\lambda,\mu}^{1,\ldots,k-2} (2i(\lambda), 1i(\mu))\Delta_1\Delta_{i(\lambda)}\varepsilon_{2i(\mu)}\Big],\end{aligned}$$

and the sum of the first and the second term is $\sum_{\lambda,\mu}^{1,\ldots,k-2} (1i(\lambda), 2i(\mu)) \cdot\Delta_1\Delta_2\varepsilon_{i(\mu)i(\lambda)}$ and, from (20), we get

$$\begin{aligned}(\text{i}) &= \sum \sum_{\lambda,\mu}^{1,\ldots,k-2} (1i(\lambda), 2i(\mu)\Delta_{i(\lambda)}\Delta_{i(\mu)}\varepsilon_{12} \\ &= R.\end{aligned}$$

We have thus the equality (102).

Next

$$\begin{aligned}(\text{ii}) = \sum\Big[&\sum_{\lambda<\mu}^{1,\ldots,k-2} \sum_{\nu=1,2} \sum_{\xi=1}^{k-2} (\nu i(\lambda), i(\xi)i(\mu))\Delta_\nu\Delta_{i(\xi)}\varepsilon_{i(\mu)i(\lambda)} \\ &+ \sum_{\lambda<\mu}^{1,\ldots,k-2} \sum_{\nu=1,2} \sum_{\xi=1}^{k-2} (i(\xi)i(\lambda), \nu i(\mu))\Delta_\nu\Delta_{i(\xi)}\varepsilon_{i(\mu)i(\lambda)} \\ &+ \sum_{\lambda,\mu}^{1,\ldots,k-2} \sum_{\nu=1,2} \sum_{\xi=1}^{k-2} (\nu i(\lambda), i(\xi)i(\mu))\Delta_{i(\lambda)}\Delta_{i(\xi)}\varepsilon_{\nu i(\mu)}\Big].\end{aligned}$$

Exchanging letters λ, μ in the second term, we get by (20)

$$\begin{aligned}(\text{ii}) &= \sum \sum_{\nu=1,2} \sum_{\lambda,\mu,\xi}^{1,\ldots,k-2} (\nu i(\lambda), i(\xi)i(\mu))\Delta_{i(\xi)}(\Delta_\nu\varepsilon_{i(\mu)i(\lambda)} + \Delta_{i(\lambda)}\varepsilon_{\nu i(\mu)}) \\ &= \sum \sum_{\nu=1,2} \sum_{\lambda,\mu,\xi}^{1,\ldots,k-2} (\nu i(\lambda), i(\xi)i(\mu))\Delta_{i(\xi)}\Delta_{i(\mu)}\varepsilon_{\nu i(\lambda)} \\ &= \sum \sum_{\nu=1,2} \sum_{\lambda=1}^{k-2} \sum_{\mu<\xi}^{1,\ldots,k-2} \Big[(\nu i(\lambda), i(\xi)i(\mu))\Delta_{i(\xi)}\Delta_{i(\mu)}\varepsilon_{\nu i(\lambda)} \\ &\quad + (\nu i(\lambda), i(\mu)i(\xi))\Delta_{i(\xi)}\Delta_{i(\mu)}\varepsilon_{\nu i(\lambda)}\Big] \\ &= 0.\end{aligned}$$

Finally,

$$(\text{iii}) = \sum \sum_{\lambda<\mu}^{1,\ldots,k-2} \sum_{\xi,\eta} (i(\xi)i(\lambda), i(\eta)i(\mu))\Delta_{i(\xi)}\Delta_{i(\eta)}\varepsilon_{i(\mu)i(\lambda)}.$$

Putting the letter ξ into $\lambda<\mu$, we get

$$(\text{iii}) = \sum^{1,\ldots,k-2}\sum_{\lambda<\mu<\xi}\sum_{\eta}\Big[(i(\xi)i(\lambda),\, i(\eta)i(\mu))\Delta_{i(\xi)}\Delta_{i(\eta)}\varepsilon_{i(\mu)i(}$$
$$+ (i(\mu)i(\lambda),\, i(\eta)i(\xi))\Delta_{i(\mu)}\Delta_{i(\eta)}\varepsilon_{i(\xi)i(\lambda)}$$
$$+ (i(\lambda)i(\mu),\, i(\eta)i(\xi))\Delta_{i(\lambda)}\Delta_{i(\eta)}\varepsilon_{i(\mu)i(\xi)}\Big]$$
$$+ \sum\sum_{\lambda<\mu}\sum_{\eta}(i(\mu)i(\lambda),\, i(\eta)i(\mu))\Delta_{i(\mu)}\Delta_{i(\eta)}\varepsilon_{i(\mu)i(\lambda)},$$

and, putting the letter η into the inequality $\lambda<\mu<\xi$ in the first term, we can easily see that there remains only the terms $\eta=\lambda$ or $\eta=\mu$ or $\eta=\xi$. Therefore

$$(\text{iii}) = \sum\sum_{\lambda<\mu<\xi}\Big[(i(\xi)i(\lambda),\, i(\lambda)i(\mu))\Delta_{i(\lambda)}\Delta_{i(\xi)}\varepsilon_{i(\mu)i(\lambda)}$$
$$+ (i(\mu)i(\lambda),\, i(\lambda)i(\xi))\Delta_{i(\lambda)}\Delta_{i(\xi)}\varepsilon_{i(\mu)i(\lambda)}$$
$$+ (i(\mu)i(\lambda),\, i(\mu)i(\xi))\Delta_{i(\mu)}\Delta_{i(\xi)}\varepsilon_{i(\mu)i(\lambda)}$$
$$+ (i(\xi)i(\lambda),\, i(\xi)i(\mu))\Delta^2_{i(\xi)}\varepsilon_{i(\mu)i(\lambda)}\Big]$$
$$+ \sum\sum_{\lambda<\mu}\sum_{\eta}(i(\mu)i(\lambda),\, i(\eta)i(\mu))\Delta_{i(\mu)}\Delta_{i(\eta)}\varepsilon_{i(\mu)i(\lambda)},$$

and, putting η into $\lambda<\mu$ in the second term, and then, adding to the first term, we can easily show that $(\text{iii})=0$ from (20).

17. The proof of the equality (95). To prove the equality (95), we divide the summation with respect to t of the expression M_{u-1} into two parts,

(a) $\qquad t = 1, 2, i(1), \ldots, i(k-2),$

(b) $\qquad t \neq 1, 2, i(1), \ldots, i(k-2).$

Then the part (a) is really the expression H_{u-1}.

In case (b), we change the letter $t, i(u), \ldots, i(k-2)$ as $t\to i(u)\to i(u+1)\to\ldots\to i(k-3)\to i(k-2)\to i(k-1)$, then, by (23)

$$(\text{b}) = (-1)^{u-1}\sum_{\substack{2<i(1)<\ldots<i(u-1)\\ 2<i(u+1)<\ldots<i(k-1)}}\ \sum_{2<i(u)}\ \sum_{a,j}\theta_{12i(1)\ldots i(k-1)}\Delta^u\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-1)}^{a(k+1)}$$
$$\cdot\begin{pmatrix}1, i(1), \ldots, i(u-1), & 2, & i(u+1), & \ldots, & i(k-1)\\ j, i(1), \ldots, i(u-1), & i(u), & j(u+1)i(u+1), & \ldots, & j(k-1)i(k-1)\end{pmatrix}\alpha_{i(u)}.$$

Puttting the letter $i(u)$ into the inequality $2<i(1)<\ldots<i(u-1)$, and, arranging the rows and columns in order, we get really $-G_{2,u}$. Therefore, we have the equality (95), as was required.

As was described before, we have the equality (73), since we had proved the equality (94) and (95).

18. From the equality (68), (72), and in general (73), we can prove by induction that

$$(\Xi_1 + \Xi_2 + \ldots + \Xi_k)\varepsilon_{12}$$
$$= \sum_{u=0}^{k-1}\ \sum_{\substack{2<i(1)<\ldots<i(u)\\ <i(u+1)<\ldots<i(k-}}\ \sum_{a,j}(-1)^{u+1}\theta_{12i(1)\ldots i(k-1)}\Delta^u\Delta_1^{a(1)}\Delta_2^{a(2)}\Delta_{i(1)}^{a(3)}\cdots\Delta_{i(k-1)}^{a(k+1)}$$

$$\cdot\begin{pmatrix} 1, 2, i(1), \ldots i(u), & i(u+1), & \ldots, & i(k-1) \\ j, \alpha, i(1), \ldots i(u), & j(u+1)i(u+1), & \ldots, & j(k-1)i(k-1) \end{pmatrix}$$

$=$ *the first term of the equality (73).*

On the other hand, it is easy to see from our proof of the equality (73) that $\Xi_n \cdot \varepsilon_{12}$ is equal to the second term of the right side of the equality (73) only, where n is the number of the generators ξ_i of the group $\mathfrak{H}/\mathfrak{G}'$. Therefore, we can conclude that

$$(\Xi_1 + \Xi_2 + \ldots + \Xi_n)\varepsilon_{12} = 0,$$

as was expected.

REFERENCES

(1) Prof. Tannaka indicated me that it is not necessary to reduce our theorem to the case of p-groups. For, the generators of $\mathfrak{G}'$ may be considered as $u(\xi_i)$ with $e_i=1$. (Schuman [1])

(2) This Reduction theorem is also due to Prof. Tannaka. My original proof started from Reduction theorem 3, and in § 3, I had to add the treatment with respect to $\Delta_k \delta_j$ besides ε_{rs}, although this can be seen easily.

(3) In this paper, I use the letters $i(1)$, $i(2)$,...without notice under the condition $i(r) \neq i(s)$ for $r \neq s$.

(4) By the same reason as in (5) below, the Δ-product $\Delta_{i(1)}^{a(1)} \ldots \Delta_{i(m+1)}^{a(m+1)}$ of this expression is equal to $\Delta_{i(1)}^{1+a(1)} \ldots \Delta_{i(m+1)}^{a(m+1)}$ as in the paper of Furtwängler.

(5) When $j(2)=i(\lambda)$, we consider $j(\lambda)$, and if $j(\lambda)=i(\mu)$, we consider $j(\mu)$, and so on. Then we get a permutation $(2, \lambda, \mu, \ldots, \nu, \ldots, \nu)$, and from our assumption that $j(k)=i(1)$ for all $k=2, \ldots, m+1$, this contains a cycle $(\nu, \xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_n, \nu)$. In this case, we take a determinant which corresponds to the inverse of this cycle, that is, $(2, \lambda, \mu, \ldots, \nu, \xi_n, \xi_{n-1}, \ldots, \xi_2, \xi_1, \nu)$. Then this determinant is either equal to the former, and is equal to 0; or is equal to the former with negative sign, and also the parts of Δ-product in the former and latter are equal.

(6) In the case of $n=2$, we get only the first and the second term.

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