ON THE FUNCTIONS $Ci(x, y)$ AND $Si(x, y)$*)

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Mr. Gennosuke Hara has introduced the following functions[1]:

$$Ei(-ix, y) = Ci(x, y) - iSi(x, y),$$
$$Ci(x, y) = \int_{\infty}^{x} \frac{U_0(2t, 2y)}{t}\, dt,$$
$$Si(x, y) = \int_{0}^{x} \frac{U_1(2t, 2y)}{t}\, dt,$$

where $U_0(2t, 2y)$ and $U_1(2t, 2y)$ are Lommel's functions of order zero and one respectively.

In this note we shall establish some relations concerning these functions, especially Hara's function $\gamma(y)$.[2]

FORMULA (1). *If* $\Re\nu > -1$, *then*

$$J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \left\{1 - z\int_0^{\pi/2} J_1(z\sin\theta)\cos^{2\nu+1}\theta\, d\theta\right\}$$
$$= \frac{(z/2)^\nu}{\Gamma(\nu+1)} \left\{1 - z\int_0^{1} J_1(zt)(1-t^2)^\nu\, dt\right\}.$$

PROOF. We shall prove the first part.

$$-\frac{(z/2)^\nu z}{\Gamma(\nu+1)} \int_0^{\pi/2} J_1(z\sin\theta)\cos^{2\nu+1}\theta\, d\theta =$$
$$= -\frac{(z/2)^\nu z}{\Gamma(\nu+1)} \int_0^{\pi/2} \cos^{2\nu+1}\theta \sum_{m=0}^{\infty} (-1)^m \left(\frac{z\sin\theta}{2}\right)^{2m+1} \Big/ (m!\,(m+1)!)\, d\theta$$
$$= -\frac{(z/2)^\nu z}{\Gamma(\nu+1)} \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+1}}{m!\,(m+1)!} \int_0^{\pi/2} \cos^{2\nu+1}\theta \sin^{2m+1}\theta\, d\theta$$
$$= \sum_{m=0}^{\infty} (-1)^{m+1} \frac{(z/2)^{2(m+1)+\nu}}{(m+1)!\,\Gamma(m+1+\nu+1)} = \sum_{m=0}^{\infty} (-1)^m \frac{(z/2)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)}$$
$$- \frac{(z/2)^\nu}{\Gamma(\nu+1)} = J_\nu(z) - \frac{(z/2)^\nu}{\Gamma(\nu+1)}.$$

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*) Received May 1, 1946.

1) 旅順工大紀要, 第9卷 第8號.

2) In the sequel, we adopt the notations in the book: Watson, Theory of Bessel functions, 1922.

The second part is easily decuced from the first part.

FORMULA (2).

$$U_0(2t, 2y) = \cos t - 2y \int_0^1 J_1(2y\,u) \cos\{t(1-u)\}\,du,$$

$$U_1(2t, 2y) = \sin t - 2y \int_0^1 J_1(2y\,u) \sin\{t(1-u)\}\,du.$$

PROOF. By the fomula (1) we have

$$U_0(2t, 2y) = \sum_{m=0}^{\infty} (-1)^m (t/y)^{2m} J^{2m}(2y)$$

$$= \sum_{m=0}^{\infty} (-1)^m (t/y)^{2m} \frac{y^{2m}}{(2m)!}\left[1 - 2y \int_0^1 J_1(2y\,u)\,(1-u^2)^{2m}\right] du$$

$$= \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m}}{(2m)!} - 2y \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m}}{(2m)!} \int_0^1 J_1(2y\,u)\,(1-u^2)^{2m}\,du$$

$$= \cos t - 2y \int_0^1 J_1(2y\,u) \sum_{m=0}^{\infty} (-1)^m \frac{t^{2m}(1-u^2)^{2m}}{(2m)!}\,du$$

$$= \cos t - 2y \int_0^1 J_1(2y\,u) \cos\{t(1-u^2)\}\,du.$$

Similarly we can deduce the second formula.

FORMULA (3).

$$Ci(x,y) = Ci(x) - 2y \int_0^1 J_1(2y\,u)\,Ci\{x(1-u^2)\}\,du,$$

$$Si(x,y) = Si(x) - 2y \int_0^1 J_1(2y\,u)\,Si\{x(1-u^2)\}\,du.$$

PROOF. From the formula (2), we have

$$Ci(x,y) = \int_\infty^x \frac{U_0(2t, 2y)}{t}\,dt = \int_\infty^x \frac{\cos t}{t}\,dt - 2y \int_\infty^x dt \int_0^1 J_1(2y\,u) \frac{\cos\{t(1-u^2)\}}{t}\,du$$

$$= Ci(x) - 2y \int_0^1 J_1(2y\,u)\,Ci\{x(1-u)\}\,du.$$

The second part may be deduced similarly.

FORMULA (3.1). $Ci(x, 0) = Ci(x)$, $Si(x, \ ) = Si(x)$.

FORMULA (3.2). $Ci(\infty; y) = 0$, $Si(\infty, y) = {}^{\pi}/_{2} J_0(2y)$.

FORMULA (3.3). $Ei(ix,y) = Ei(ix) - 2y \int_0^1 J_1(2y\,u)\,Ei\{ix(1-u)\}\,du.$

FORMULA (4). $Ci(x,y) = \gamma(y) + Ci^*(x;y)$,

where

$$\gamma(y) = \gamma \cdot J_0(2y) - 2y\int_0^1 J_1(2yu)\log(1-u^2)\,du$$
$$= \pi/2\, Y_0(2y) - J_0(2y)\log y,$$
$$Ci^*(x,y) = J_0(2y)\log x + \sum_{m=1}^{\infty}(-1)^m \frac{1}{2m}\left(\frac{x}{y}\right)^{2m} J_{2m}(2y).$$

The function $\gamma(y)$ is a function due to Hara which corresponds to the Euler constant γ in the case of one variable.

PROOF. Applying the identity

$$Ci(x) = \gamma + \log x + \sum_{m=1}^{\infty}(-1)^m \frac{x^{2m}}{(2m)!\,(2m)}$$

to the expression $Ci(x,y)$ in Formula (3), we have

$$Ci(x,y) = Ci(x) - 2y\int_0^1 J_1(2yu)\Big[\gamma + \log x + \log(1-u^2)$$
$$+ \sum_{m=1}^{\infty}(-1)^m \frac{x^{2m}(1-u^2)^{2m}}{(2m)!\,(2m)}\Big]du$$
$$= Ci(x) + (\gamma + \log x)(J_0(2y) - 1) - 2y\int_0^1 J_1(2yu)\log(1-u)\,du$$
$$+ 2y\sum_{m=1}^{\infty}(-1)^{m+1}\frac{x^{2m}}{(2m)!\,(2m)}\int_0^1 J_1(2yu)(1-u^2)^m\,du$$
$$= \gamma + \log x + \sum_{m=1}^{\infty}(-1)^m \frac{x^{2m}}{(2m)!\,(2m)} + (\gamma + \log x)(J_0(2y) - 1)$$
$$- 2y\int_0^1 J_1(2yu)\log(1-u^2)\,du$$
$$+ \sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{(2m)!\,(2m)}\left[\frac{(2m)!}{y^{2m}} J_{2m}(2y) - 1\right]$$
$$= (\gamma + \log x)J_0(2y) - 2y\int_0^1 J_1(2yu)\log(1-u^2)\,du$$
$$+ \sum_{m=1}^{\infty}(-1)^m \frac{1}{2m}\left(\frac{x}{y}\right)^{2m} J_{2m}(2y).$$

It remains to prove the second expression of $\gamma(y)$. Now we have

$$\gamma(y) = \gamma \cdot J_0(2y) - 2y\int_0^1 (2yu)\log(1-u^2)\,du$$
$$= \gamma \cdot J_0(2y) + 2y\sum_{m=1}^{\infty}\frac{1}{m}\int_0^1 J_1(2yu)\,u^{2m}\,du$$

$$= \gamma \cdot J_0(2y) + 2y \sum_{m=1}^{\infty} \frac{1}{m} \sum_{n=0}^{\infty} (-1)^n \frac{y^{2n+1}}{n!(n+1)!} \int_0^1 u^{2m+2n+1} du$$

$$= \gamma \cdot J_0(2y) + \sum_{n=0}^{\infty} \frac{(-1)^n y^{2(n+1)}}{n!(n+\ \ )!} \sum_{m=1}^{\infty} \frac{1}{m(m+n+1)}$$

$$= \gamma \cdot J_0(2y) + \sum_{n=0}^{\infty} \frac{(-1)^n y^{2(n+1)}}{n!(n+1)!} \frac{1}{n+1} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(n!)^2} \left[\gamma - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\right]^{3)}$$

$$= - \sum_{n=0}^{\infty} \frac{(-1)^n y^{2n}}{(n!)^2} \psi(n+1) = \frac{1}{2} \mathbf{Y}_0(2y) - J_0(2y) \log y$$

$$= \frac{\pi}{2} Y_0(2y) - J_0(2y) \log y.$$

Formula (4.1).

$$z \int_0^1 J_1(zt) \log(1-t) \, dt = \left(\gamma + \log \frac{z}{2}\right) J_0(z) - \frac{\pi}{2} Y_0(z)$$
$$= J_0(z) \log z - Y^{(0)}(z)$$
$$= 2 \sum_{n=1}^{\infty} (-1)^n \frac{J_{2n}(z)}{n}.$$

The proof is immediate.[4]

Formula (4.2). $Ci(y\,;y) = \frac{\gamma + \log y}{4} J_0(2y) + \frac{3}{8} \pi Y_0(2y).$

Proof. From (4.1) we have

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{2n} J_{2m}(2y) = \frac{\gamma + \log y}{4} J_0(2y) - \frac{\pi}{8} Y_0(2y).$$

Hence

$$Ci^*(y, y) = J_0(2y) \log y + \frac{\gamma + \log y}{4} J_0(2y) - \frac{\pi}{8} Y_0(2y),$$

from which we have the required, using (4).

Formula (5).

$$Ci(x, y) = \frac{\pi}{2} Y_0(2y) + \sum_{n=1}^{\infty} (-1)^n \frac{1}{2n} \left(\frac{y}{x}\right)^{2n} J_{2n}(2y) -$$

3) Watson. loc. cit., p.60 (2).

4) Cf. ibid., p.71 (8), p.67 (1).

$$+\int_0^{\log\frac{x}{y}} \cos(2y\cosh t)\,dt,$$

$$Si(x,y)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\left(\frac{y}{x}\right)^{2n+1}J_{2n+1}(2y)+\int_0^{\log\frac{x}{y}}\sin(2y\cosh t)\,dt.$$

PROOF. Integrating the identies[5]

$$J_0(2y)+2\sum_{n=1}^{\infty}(-1)^n J_{2n}(2y)\cosh(2nt)=\cos(2y\cosh t),$$

$$2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(2y)\cosh(\overline{2n+1}\,t)=\sin(2y\cosh t)$$

from zero to $\log(x/y)$, we have easily

$$\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n}\left(\frac{x}{y}\right)^{2n}J_{2n}(2y)=J_0(2y)\log(y/x)$$

$$+\sum_{n=1}^{\infty}(-1)^n\frac{1}{2n}\left(\frac{y}{x}\right)^{2n}J_{2n}(2y)+\int_0^{\log\frac{x}{y}}\cos(2y\cosh t)\,dt.$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\left(\frac{x}{y}\right)^{n+1}J_{2n+1}(2y)=\sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}\left(\frac{x}{y}\right)^{2n+2}J_{2n+1}(2y)$$

$$+\int_0^{\log\frac{x}{y}}\sin(2y\cosh t)\,dt.$$

By the former and (4) we deduce the formula for $Ci(x,y)$, and the left-hand side of the latter is the function $Si(x,y)$.[6]

REMARK. Let $j_{0,n}$ be the positive zeros of $J_0(x)$ arranged in ascending order of magnitude, and let

$$Ci\left(0,\frac{1}{2}j_{0,n}\right)=\lim_{x\to 0}Ci\left(x,\frac{1}{2}j_{0,n}\right).$$

Then we have

FORMULA (6).

$$Ci\left(0,\frac{1}{2}j_{0,n}\right)=\frac{\pi}{2}Y_0(j_{0,n}).$$

For example[7]

5) Cf. N. W. Maclachlan, Bessel functions for Engineers, p.50.

6) Compare the formulas

$$\int_0^{\infty}\cos(2y\cosh t)\,dt=-\frac{\pi}{2}Y_0(2y),\quad \int_0^{\infty}\sin(2y\cosh t)\,dt=\frac{\pi}{2}J_0(2y),$$

(Watson, loc. cit., p. 180 (12) and (13)).

7) Calculated by the table of Watson, loc. cit.

$$Ci\left(0, \frac{1}{2} j_{0,1}\right) = Ci\,(0, 1.20241) = +\ 0.80099,$$

$$Ci\left(0, \frac{1}{2} j_{0,2}\right) = Ci\,(0, 2.76004) = -\ 0.53285,$$

$$Ci\left(0, \frac{1}{2} j_{0,3}\right) = Ci\,(0, 4.32686) = +\ 0.43569_5,$$

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