ON THE METHODS OF SUMMATION OF INFINITE SERIES*)

By

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1. Let $\mathfrak{A} = (a_{ik})$ be a matrix of a method of summation which transforms a sequence $x = \{\xi_k\}$ into

$$\{A_i(x)\}, \qquad A_i(x) = \sum_{k=1}^{\infty} a_{ik}\xi_k, \qquad (i = 1, 2, 3, \cdots)$$

when the limit $\lim_{i\to\infty} A_i(x)$ exists, we call it the *generalized limit*. The method is said to be regular if every convergent sequence $x = \{\xi_k\}$ is transformed into a convergent sequence $\{A_i(x)\}$ with the same limit.

The necessary and sufficient condition for regularity of $\mathfrak{A}$ is given by O. Toeplitz and T. Kojima as follows.

THEOREM A. *The necessary and sufficient condition that the method $\mathfrak{A}$ should be regular is that the following three conditions are satisfied simultaneously:*

$$(1) \qquad \sum_{k=1}^{\infty} |a_{ik}| \leqq M \qquad \text{for all } i = 1, 2, \cdots, \text{ where } M \text{ is a constant,}$$

$$(2) \qquad \lim_{i\to\infty} a_{ik} = 0 \qquad \text{for all } k = 1, 2, \cdots,$$

$$(3) \qquad \lim_{i\to\infty} \sum_{k=1}^{\infty} a_{ik} = 1.$$

Moreover S. Banach established the following[1)]

THEOREM B. *Let $\mathfrak{A}$ be a regular method and $y_0 = \{\eta_i^0\}$ be a convergent sequence. If for any sequence $\{\alpha_i\}$ the condition*

$$(4) \qquad \sum_{i=1}^{\infty} |\alpha_i| < \infty, \quad \sum_{i=1}^{\infty} \alpha_i a_{ik} = 0 \qquad \text{for all } k = 1, 2, \cdots$$

implies $\sum_{i=1}^{\infty} \alpha_i \eta_i^0 = 0$, then, for any $\varepsilon > 0$, there exists a convergent sequence x

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1) S. Banach; Théorie des opérations linéaires, Warszawa, 1933, p. 91.

such that

$$|A_i(x) - \eta_i^0| < \varepsilon \qquad \textit{for all } i = 1, 2, \cdots.$$

In this paper we shall establish a converse of Theorem B and give its applications.

2. A converse of Theorem B. THEOREM 1. *Let* $\mathfrak{A}$ *be a regular method and* $y_0 = \{\eta_i^0\}$ *a convergent sequence. If for any* $\varepsilon > 0$, *there exists a convergent sequence* x *such that*

$$|A_i(x) - \eta_i^0| < \varepsilon \qquad \textit{for all } i = 1, 2, \cdots,$$

then for any sequence $\{\alpha_i\}$ *the comdition* (4) *implies* $\sum_{i=1}^{\infty} \alpha_i \eta_i^0 = 0$.

PROOF. Since $\sum_{i=1}^{\infty} |\alpha_i| < \infty$,

$$f(x) = C \lim_{i\to\infty} \xi_i + \sum_{i=1}^{\infty} \alpha_i \xi_i \tag{1}$$

where $x = \{\xi_i\} \varepsilon (c)$ and C is a constant, is a continuous linear functional defined in the space (c).[2]

For any monotone sequence of positive numbers $\{\varepsilon_j\}$ tending to zero, there exists a sequence $\{x_j\}$ of points of the space (c) such that

$$|A_i(x_j) - \eta_i^0| < \varepsilon_j \qquad \text{for all } i = 1, 2, \cdots. \tag{2}$$

On accout of the regularity of $\mathfrak{A}$, $\lim_{i\to\infty} A_i(x_j) = \lim_{n\to\infty} \xi_n^j$, where $x_j = \{\xi_n^j\}$. Therefore the sequence $\{A_i(x_j)\}$ is a point of the space (c). Let the sequence $\{A_i(x_j)\}$ be denoted by $A(x_j)$. From (2), we get $\lim_{j\to\infty} A(x_j) = y_0$. From (1), we have

$$f\{A(x_j)\} = C \lim_{i\to\infty} A_i(x_j) + \sum_{i=1}^{\infty} \alpha_i A_i(x_j).$$

Since $\lim_{i\to\infty} A_i(x_j) = \lim_{n\to\infty} \xi_n^j$, we get

$$f\{A(x_j)\} = C \lim_{n\to\infty} \xi_n^j + \sum_{i=1}^{\infty} \alpha_i \Big(\sum_{k=1}^{\infty} a_{ik} \xi_k^j \Big).$$

On the other hand, by Theorem A,

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} |\alpha_i| \cdot |a_{ik}| \cdot |\xi_k^j| \leqq M \cdot \|x_j\| \cdot \sum_{i=1}^{\infty} |\alpha_i|,$$

2) (c) is the space of all convergent sequences which is a Banach space with the norm $\|x\| = \text{l.u.b.}\, |\xi_i|$.

therefore

$$f\{A(x_j)\} = C \lim_{n\to\infty} \xi_n^j + \sum_{k=1}^{\infty} \xi_k^j \Big(\sum_{i=1}^{\infty} \alpha_i a_{ik}\Big).$$

In virtue of $\sum_{i=1}^{\infty} \alpha_i a_{ik} = 0$ for all $k = 1, 2, \cdots$, we ge

$$f\{A(x_j)\} = C \lim_{n\to\infty} \xi_n^j.$$

Since $f(x)$ is a continuous linear functional in (c), we get

$$\lim_{j\to\infty} f\{A(x_j)\} = f(y_0),$$

therefore $C \lim_{n\to\infty} \eta_n^0 + \sum_{i=1}^{\infty} \alpha_i \eta_i^0 = C \lim_{j\to\infty} (\lim_{n\to\infty} \xi_n^j)$.

From (2), we have

$$\lim_{i\to\infty} | A_i(x_j) - \eta_i^0 | \leqq \varepsilon_j,$$

therefore $|\lim_{n\to\infty} \xi_n^j - \lim_{n\to\infty} \eta_n^0 | \leqq \varepsilon_j$.

If we put $\lim_{n\to\infty} \eta_n^0 = \eta^0$, then

$$\lim_{j\to\infty} (\lim_{n\to\infty} \xi_n^j) = \eta^0.$$

consequently we get $\sum_{i=1}^{\infty} \alpha_i \eta_i^0 = 0$. Thus the theorem is completely proved.

3. **Application.** Let $S(\mathfrak{A})$ be the set of all sequences $\{A_i(x)\}$ where $\mathfrak{A}$ is a regular method of summation and x varies on the space (c). Then we have $S(\mathfrak{A}) \subset (c)$. From Theorem B and Theorem 1 we obtain the following

THEOREM 2. *In order that the set $S(\mathfrak{A})$ should be everywhere dense in the space (c), it is necessary and sufficient that (4) implies $\alpha_i = 0$ for all $i = 1, 2, \cdots$.*

PROOF. *Necessity.* Let $y_0 = \{\eta_i^0\}$ be the sequence such as $\eta_1^0 = 1$, $\eta_i^0 = 0$ $(i \neq 1)$. It follows from Theorem 1 that (4) implies $\sum_{i=1}^{\infty} \alpha_i \eta_i^0 = \alpha_1 = 0$. In the same manner we deduce $\alpha_i = 0$ $(i = 2, 3, \cdots,)$.

Sufficiency is evident by the lemma of S. Banach.[3] Thus the theorem is completely proved.

The method $\mathfrak{A}$ is said to be reversible if for any convergent sequence $\{\eta_i\}$ there exists only one sequence x (convergent or not) such that

$$A_i(x) = \eta_i \qquad \text{for } i = 1, 2, \cdots.$$

3) cf. S. Banach, loc. cit., p. 93.

Then we have the following

THEOREM 3. *Let the method* $\mathfrak{A}$ *be regular and reversible. If there is a non-zero sequence satisfying* (4), *then the cardinal number of the set of all linearly independent divergent sequences summable by* $\mathfrak{A}$ *is at least enumerable.*

PROOF. In virtue of Theorem 2, the set $S(\mathfrak{A})$ is not dense in the space (c). Therefore there exists a sphere K in the space (c) such that $K \cap S(\mathfrak{A}) = 0$. Let $x_0 = \{\xi_k^0\}$ and r_0 be the centre and the radius of the sphere K, respectively. Now let us introduce the convergent sequence $x_n = \{\xi_k^n\}$ such that $\xi_k^n = \xi_k^0$ for $n \neq k$ and $\xi_n^n = \xi_n^0 + r$ where $0 < |r| < r_0$. Then $\{x_n\}$ are evidently linearly independent and contained in the sphere K.

On the other hand, since the method $\mathfrak{A}$ is regular and reversible, all the sequence y_n such that $x_n = A(y_n)$ $(n = 1, 2, \cdots)$ are linearly independent. This proves the theorem.

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