NOTES ON FOURIER ANALYSIS (XIX):

A REMARK ON RIEMANN SUMS[*]

By

SHIGEKI YANO

Let $f(x)$ be an integrable function of period 1 and $f_n(x)$ be its Riemann sum

(1) $$f_n(f,\ x) = f_n(x) = \frac{1}{n}\sum_{k=1}^{n} f(x+k/n).$$

Marcinkiewicz and Salem[1] have proved that, if $f(x)$ satisfies the condition

(2) $$\int_0^1 |f(x+t) - f(x)|^2\,dx = O(t^{\varepsilon})$$

for any positive ε, then $\{f_n(x)\}$ converges to $\int_0^1 f(x)dx$ almost everywhere, and that the exponent 2 in the left hand side of (2) cannot be improved.

Concerning this theorem we begin to prove the following theorem:

THEOREM 1. *If $f(x)$ belongs to* Lip $(1/p - 1/2 + \varepsilon, p)$ $(\varepsilon > 0, 1 \leqq p \leqq 2)$, *then* $\{f_n(x)\}$ *converges to* $\int_0^1 f(x)dx$ *almost everywhere*[2]. *If* $1 \leqq p < 2$ *and* $\varepsilon = 0$, *then the theorem does not hold in general.*

PROOF. Hardy and Littlewood[3] proved that, if $g(x)$ belongs to Lip(α, p), $0 < \alpha \leqq 1$, $p \geqq 1$, $\alpha p \leqq 1$, then $g(x)$ belongs to Lip$(\alpha - 1/p + 1/q,\ q)$ where $p < q < p/(1-\alpha p)$. If we put $\alpha = 1/p - 1/2 + \varepsilon$, then $\varepsilon < 1/2$ implies $\alpha p < 1$ and $p/(1-\alpha p) = 1/(1/2 - \varepsilon) > 2$. Hence we can take $q = 2$ in the Hardy-Littlewood theorem, so that $f(x)$ belongs to Lip$(\varepsilon, 2)$, and then $f_n(x)$ converges to $\int_0^1 f(x)dx$ almost everywhere by the Marcinkiewicz-Salem theorem above mentioned.

On the other hand $f(x) = 1/x^s$ $(1 > s \geqq 1/2)$ satisfies the condition

(3) $$\int_0^1 |f(x+t) - f(x)|^p\,dx = O(t^{1-ps})$$

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1) J. Marcinkiewicz and R. Salem, Sur les sommes riemanniennes, Compositio Mathematica, **7** (1940).

2) The case $p=2$ is the Marcinkiewicz-Salem theorem.

3) G. H. Hardy and J. E. Littlewood, A convergence criterion for Fourier series. Mathematische Zeitschrift, **28** (1928).

for $p<1/s$, and then $f(x)$ belongs to $\mathrm{Lip}(\alpha, p)$, $\alpha=1/p-s$, $1\leqq p<2$. It is well known that for this function, $\{f_n^*(x)\}$ does not converge to $\int_0^1 f(x)dx$ almost everywhere.

THEOREM 2. *If $f(x)$ belongs to* $\mathrm{Lip}(\alpha, p)$, $0<\alpha\leqq 1$, $p\geqq 1$, *and* $\sum_{k=1}^{\infty} 1/n_k^{\alpha}$ *converges, then* $\{f_{n_k}(x)\}$ *converges to* $\int_0^1 f(x)dx$ *almost everywhere.*

For an example of $\{n_k\}$, *we can take* $\{k^{\beta}\}$, $\alpha\beta>1$ (*which satisfies the condition* $n_{k+1}/n_k \to 1$ *as* $k\to\infty$).

For the proof we need a lemma.

LEMMA. *If $f(x)$ belongs to* $\mathrm{Lip}(\alpha, p)$, $0<\alpha<1$, *then there exists a trigonometrical polynomial $P_n(x)$ of order at most n such that*

$$\int_0^1 |f(x) - P_n(x)|\,dx = O(1/n^{\alpha}). \tag{4}$$

PROOF. Let $K_n(x)$ be the Fejér kernel, that is,

$$K_n(x) = \frac{\sin^2(n+1)\pi t}{2(n+1)\sin^2 \pi t},$$

and let

$$P_n(x) = 2\int_0^1 f(x+t)K_n(t)dt$$

which is of order n. It is well known that $P_n(x)$ has the required property.

We will now prove Theorem 2. Without any loss of generality we can suppose that $\int_0^1 f(x)dx = 0$. Then

$$f_n(f, x) = f_n(f - P_{n-1}, x),$$

$$\int_0^1 |f_n(f, x)|\,dx \leqq \int_0^1 |f - P_{n-1}|\,dx = O(1/n^{\alpha}).$$

By the convergence of the series $\sum_{k=1}^{\infty} 1/n_k^{\alpha}$, we see the convergence of the series

$$\int_0^1 \sum_{k=1}^{\infty} |f_{n_k}(x)|\,dx$$

which implies the almost everywhere convergence of $\{f_{n_k}(x)\}$.

THEOREM 3. *If $f(x)$ satisfies the condition*

$$\int_0^1 |f(x+t) - f(x)|\,dx = O(1/\log^s 1/t) \qquad (s>1) \tag{5}$$

and

$$\sum 1/\log^s n_k < \infty$$

then $\{f_{n_k}(x)\}$ *converges to* $\int_0^1 f(x)dx$ *almost everywhere*[4]. *In particular, we can take for* $\{n_k\}$ *such a sequence which satisfies* $n_{k+1}/n_k \geqq q > 1$.

Lemma. *If the function* $f(x)$ *satisfies the relation* (5), *there exsits a trigonometrical polynomial* $P_n(x)$ *of order at most n such that*

$$\int_0^1 |f(x) - P_n(x)| dx = O(1/\log^s n).$$

We can take for $P_n(x)$ the same one as in the preceding lemma. This is due to Marcinkiewicz and Salem[5].

The proof of Theorem 3 runs similarly as that of Theorem 2[6].

Mathematical Institute,
Tôhoku University.

4) Under the condition of Theorem 3 Marcinkiewicz and Salem proved that the arithmetic mean of Riemann sum converges to the integral almost everywhere. Theorem 3 of our paper can be proved by an argument analogous to their one.

5) loc. cit. 1).

6) After preparation of this paper, the author learned in Mathematical Review that R. Salem (SAERTRYK AF MATEMATISK TIDSSKRIFT, B. 1948 pp. 60—62) has dealt the same problem as Theorem 3. But the result of this paper is a little stronger than that of Salem's.