ON PSEUDO-PARALLELISM IN EINSTEIN SPACES*)

BY

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In this paper we shall define a new parallelism in Einstein spaces making use of their Poincaré's and Klein's representations which were generalized by S. Sasaki [1], [2] and K. Yano [2][1]. We shall obtain the differential equations which give the parallelism and compute the parallel angle.

§1. **Preliminaries.** Consider an Einstein space E_n with a positive definite fundamental metric tensor g_{ij} $(i, j, k \cdots = 1, 2, \cdots, n)$, then the curvature tensor is given by the components

$$R^i_{jkl} = \frac{\partial \left\{ {}^{\,i}_{jk} \right\}}{\partial x^l} - \frac{\partial \left\{ {}^{\,i}_{il} \right\}}{\partial x^k} + \left\{ {}^{\,i}_{hl} \right\} \left\{ {}^{\,h}_{jk} \right\} - \left\{ {}^{\,i}_{hk} \right\} \left\{ {}^{\,h}_{il} \right\},$$

where $\left\{ {}^{\,i}_{jk} \right\}$ are Christoffel's symbols constructed from g_{ij}. Making use of the curvature tensor we put

$$R_{jk} = R^i_{jki}, \quad R = g^{ik} R_{ik}.$$

Now we construct the space with normal conformal connexion C_n [1], [3] corresponding to E_n, then the connexion of C_n is given by following equations:

$$\begin{aligned} dR_0 &= && dx^i R_i, \\ dR_j &= c g_{jk} dx^k R_0 + && \left\{ {}^{\,i}_{jk} \right\} dx^k R_i + g_{jk} dx^k R_\infty, \\ dR_\infty &= && c dx^i R_i, \end{aligned} \tag{1}$$

where R_A's $(A = 0, 1, \cdots, n, \infty)$ are Veblen's reperes corresponding to E_n and

$$c = -\frac{R}{2n\,(n-1)} \tag{2}$$

which is constant by assumption. Consider a hypersphere or a point $A \equiv R_\infty - cR_0$, then the group of holonomy of C_n fixes A. In the following we

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1) The brackets [ ] mean the order of papers to be referred which are given at the end of this paper.

assume that $c \neq 0$. Then the hypersphere A is the absolute in Sasaki's generalized Poincaré's representation and conformal circles in C_n which intersect A with a right angle become geodesics in E_n.

On the other hand if we construct from E_n the space with normal projective connexion P_n, then the group of holonomy of P_n fixes a hyperquadric B [2]. For the sake of convenience we perform a trivial conformal transformation $E_n \to E_n^*$ defined by

$$g_{ij}^* = k^2 g_{ij}, \qquad k = \sqrt{2\varepsilon c}, \tag{3}$$

where ε is $+1$ or -1 according to $c > o$ or $c < o$, then the connexion of P_n^* corresponding to E_n^* is given by

$$\begin{aligned} dA_0 &= \qquad\qquad dx^i A_i, \\ dA_j &= g_{ik}^* dx^k A_0 + \left\{ {i \atop jk} \right\}^* dx^k A_i, \end{aligned} \tag{4}$$

where A_λ's $(\lambda = 0, \ , \cdots, n)$ are semi-natural repères corresponding to E^*_n. Then the hyperquadric B^* invariant under the group of holonomy of P^* is given by the following equation:

$$\varepsilon g_{ij}^* X^i X^j + (X^0)^2 = 0,$$

where X^λ's are current coordinates in tangential projective spaces.

Now we consider a geodesic e in E_n and let $g(h)$ be the corresponding conformal circle (path) in $C_n(P_n^*)$. We develop $g(h)$ in a tangential Möbius' (projective) space at a point P_0 which lies on $g(h)$ and let P', P'' be points at which $g(h)$ and $A(B^*)$ intersect. Let $\overline{g(h)}$, $\overline{P}_0$, $\overline{P}'$, $\overline{P}''$ be the circle (path) and points defined in the same way for an another geodesic $\overline{e}$. Next we make following

Definition. Two geodesics e, $\overline{e}$ in E_n are said to be *$A(B)$-parallel*, if the image of $\overline{P}'$ coincides with P' when we develop the tangential Möbius' (projective) space at $\overline{P}_0$ on the one at P_0 along a suitable curve which joins $\overline{P}_0$ to P_0, provided that P' and $\overline{P}'$ lie in directions of increasing or decreasing arc length for both geodesics simultaneously.

§ 2. **A-parallelism.** At first we consider A-parallelism in C_n. We develop an arbitrary conformal circle in a tangential Möbius' space at P_0 and choose a projective parameter t suitably on the circle, then a variable point P on it is expressible in the following form [1].

$$P = \left(1 + \frac{t^2}{4} g_{jk}\, n^j\, n^k\right) \mathrm{R}_0 + \left(x'^i t + \frac{t^2}{2} n^i\right) \mathrm{R}_i + \frac{t^2}{2} \mathrm{R}_\infty, \tag{5}$$

where $x'^i = \left(\frac{dx^i}{ds}\right)_{P_0}$, $n^i = \left(\frac{\delta^2 x^i}{\delta s^2}\right)_{P_0}$ and δ means the covariant differentiation along the curve.

Now consider a geodesic e in E_n, then a variable point P on the corresponding conformal circle g in C_n is r presented as follows;

$$P = R_0 + t x'^i R_i + \frac{t^2}{2} R_\infty. \tag{6}$$

Since (6) meets with A as points corresponding to $t = \pm\sqrt{\frac{2}{c}}$ [1], the point of intersection P' is given by

$$P' = R_0 \pm \sqrt{\frac{2}{c}}\, x'^i R_i + \frac{1}{c} R_\infty. \tag{7}$$

In the same way, we have for another geodesic $\bar{e}$

$$\bar{P}' = \bar{R}_0 \pm \sqrt{\frac{2}{c}}\, \bar{x}'^i \bar{R}_i + \frac{1}{c} \bar{R}_\infty, \tag{8}$$

where the quantities carrying bar are considered at $\bar{P}_0$.

Now suppose that $\bar{P}_0(\bar{x}^i)$ lies indefinitely near to $P_0(x^i)$, then we can put $\bar{x}^j = x^j + \epsilon\lambda^j$, wher λ^i is a unit vector at P_0 and ϵ is an infinitesimal constant. Then from (1) we have

$$\begin{aligned} \bar{R}_0 &= R_0 && + \epsilon\lambda^i R_i, \\ \bar{R}_j &= R_j + \epsilon\, c\, g_{jk}\lambda^k R_0 && + \epsilon \left\{ {i \atop jk} \right\} \lambda_k R_i + \epsilon\, g_{jk}\lambda_k R_\infty, \\ \bar{R}_\infty &= R_\infty && + \epsilon\, c\, \lambda^i R_i. \end{aligned} \tag{9}$$

If e and $\bar{e}$ are A-parallel, then, by definition, the relation $\bar{P}' = \rho P'$ holds good, where ρ is some scalar function. Subs ituting (9) in (8) and putting the relation thus obtained into $\bar{P}' = \rho P'$, we obtain

$$\rho = 1 \pm \sqrt{2c}\;\epsilon\, g_{jk}\lambda^k \bar{x}'^j, \tag{10}$$

$$\pm \rho x'^i = \epsilon\sqrt{2c}\;\lambda^i \pm (\bar{x}'^i + \epsilon \left\{ {i \atop jk} \right\} \bar{x}'^j \lambda^k), \tag{11}$$

where the double sign $\pm$ must be used in he same order by the definition of A-parallelism. Eliminating ρ from (10) and (11) and neglecting terms of the higher order with respect to ϵ, we get

$$[x'^i{}_{;j} \pm \sqrt{2c}\; g_{jk}(x'^i x'^k - g^{ik})]\lambda^j = 0, \tag{12}$$

where semi-colom deno es the covarient derivative.

Equation (11) defines A-parallelism in consideration in the Einstein space E_n.

In the next place we shall compute the angle which may be called to be parallel angle. In order to do this we displace $\bar{x}'^i$ at $\bar{P}_0$ to P_0 in the sense of Levi-Civita's parallelism and subs itute the resulting vec or (neglecting terms of the higher order with respect to ϵ)

$$\bar{\bar{x}}'^i = \bar{x}'^i + \epsilon \left\{ {i \atop jk} \right\} \lambda^k \bar{x}'^j$$

in (11). Then we have

$$\epsilon \sqrt{2c}\, \lambda^i = \pm(\rho x'^i - \bar{\bar{x}}'^i). \tag{13}$$

Now suppose that $\lambda_i x'^i = 0$, then contracting λ_i with (13) we obtain

$$\mp \lambda_i \bar{\bar{x}}'^i = \epsilon \sqrt{2c}\,.$$

Let θ be the angle which $\bar{\bar{x}}'^i$ makes with λ^i, then we see that

$$\cos\theta = \mp \epsilon \sqrt{2c} \tag{14}$$

holds good, because $\bar{\bar{x}}'^i$ is a unit vector as $\bar{x}'^i$. (14) is the equation which gives the parallel angle of the A-parallel geodesics e and $\bar{e}$.

§ 3. **B-parallelism.** Next let us consider B-parallelism. We shall deal only with the case $c > 0$, since the case $c < 0$ can be dealt with similarly.

Consider a geodesic e in E_n, it is a path h in P^*_n. We develop h in the tangential projective space at a point P_0 on h, then a variable point P on h is expressible as follows [2]:

$$P = \cosh s^*. A_0, + \sinh s^*. A'_0,$$

where s^* means arc length along the curve in E^* and A_0, A_0' are those at point corresponding to P_0. h meets with B^* at points

$$P' = A_0 \pm A_0'. \tag{15}$$

In the same way, for $\bar{P}'$ corresponding to an another geodesic $\bar{e}$, we get

$$\bar{P}' = \bar{A}_0 \pm \bar{A}'_0. \tag{16}$$

Hereafter we agree that the quantities denoted by symbols with bar are those at $\bar{P}_0$.

Now as in § 2, we assume that $\overline{P}_0$ lies indefinitely near to P. Then the relation $\overline{x^i} = x^i + \epsilon\lambda^i$ holds good, where λ^i/k is a unit vector in E_n^*. From (4) we have

$$\overline{A}_0 = A_0 + \epsilon\lambda^i A_i,$$
$$\overline{A}'_0 = \overline{x}'^i \overline{A}_i = \overline{x}'^i (A_i + \epsilon\, g^*_{ik}\, \lambda^k A_0 + \epsilon \left\{ {j \atop ik} \right\}^* \lambda^k A_j). \tag{17}$$

If e and $\overline{e}$ are B-parallel, then substituting (17) into (16) and computing $\overline{P}' = \rho P'$, we have

$$\rho = 1 \pm \epsilon\, g^*_{jk}\, \overline{x}'^j \lambda^k,$$
$$\pm \rho\, x'^i = \epsilon\, \lambda^i \pm (\overline{x}'^i + \epsilon\, \overline{x}'^j \left\{ {i \atop jk} \right\}^* \lambda^k).$$

Since these equations are written in terms of quantities of E^*, we must translate them to those written in terms of quantities of E_n. For this it is ufficient to remark that vectors x'^i, $\overline{x}'^i$ are unit vetors with respect to g_{ij}. Thus, we obtain (10) and (11) again.

§ 4. **Pseudo-parallel vector fields.** In § 2, § 3 we observed that for two geodesics which lie indefinitely near to each other, A- and B-parallelism coincide with each other. Therefore we shall call that unit vectors v^i at (x^i) and $v^i + dv^i$ at $(x^i + dx^i)$ are $(+)$ (or $(-)$)-pseudo-parallel if their components satisfy the equations

$$[v^i{}_{;j} + (\text{or } -) \sqrt{2c}\; g_{jk}\,(v^i v^k - g^{jk})]\; dx^j = 0 \tag{18}$$

(Cf. (12)). A unit vector field v^i which satisfy (18) for arbitrary element (x^i, dx^i) will be called a $(+)$ (or $(-)$)-pseudo-parallel vector field. Then we can state the following

THEOREM. *If an Einstein space with non vanishing constants c admits n linearly independent pseudo-parallel vector fields, it is a space of constant curvature.*

PROOF. Let v^i be a pseudo-parallel vector fields, then from (18) we have

$$v_{i;k} \pm \sqrt{2c}\;(v_i v_k - g_{ik}) = 0,$$

where $\pm$ is taken either $+$ or $-$. Differentiating it covariantly, we get

$$v_{i;k;j} + 2c\,[-2\, v_i v_j v_k + g_{ji} v_k + g_{kj} v_i] = 0.$$

Interchanging k and j and subtracting the equation thus obtained from the first equation, we have

$$v_{i;k;j} - v_{i;j;k} + 2c\,(g_{ij}\,v_k - g_{ik}\,v_j) = 0.$$

Making use of Ricci's identities, we see that the relation

$$v_h\,Z^h{}_{ikj} = 0 \tag{19}$$

holds good, where

$$Z^h{}_{ikj} = R^h{}_{ikj} - 2c\,(g_{ij}\,\delta^h_k - g_{ik}\,\delta^h_j)$$

is the so-called concircular curvature tensor. From (19), we can easily see that our assertion is true.

§ 5. Up to the present we restricted ourselves only to Einstein spaces. But these results can be generalized to an arbitrary Riemannian space V_n as follows[2)]. Let c be an arbitrary constant, and we difine $\overrightarrow{C}_n$ and $\overrightarrow{P}^*_n$ (or $\overrightarrow{P}_n$) from V_n by (1) and (4) respectively. Then making use of $\overrightarrow{C}_n$ and $\overrightarrow{P}_n$ we obtain results analogous to those in sections 2, 3, 4. But generally $\overrightarrow{C}_n$ (or $\overrightarrow{P}_n$) is not a space with normal conformal (or projective) connexion.

References

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2) This fact was remarked by Prof. S. Sasaki.