NOTES ON FOURIER ANALYSIS (XLIV): ON THE SUMMATION OF FOURIER SERIES

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(Received August 2, 1950)

This paper consists of three parts. In § 1, we investigate the relation between the logarithmic order of a certain Cesàro mean of function and that of Cesàro sum of its Fourier series (Theorem 1. 2). Our theorem is best possible in a sense (Theorem 1. 3) and incidentally it is proved that there is an integrable function which is (C, α) continuous and whose Fourier series is not (C, α) summable. In § 2 we prove a theorem generalizing F. T. Wang's [5] (Theorem 2. 1). We give further a relation between Riesz continuity of function and Cesàro summability of its Fourier series (Theorem 2. 3). In § 3 we prove a theorem concerning absolute Riesz summability with its converse (Theorem 3. 1).

1. We suppose that $\varphi(t)$ is an even integrable function, with period 2π. We denote the α-th integral of $\varphi(t)$ by

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\varphi(u)\,du \qquad (\alpha > 0), \tag{1.1}$$

and the α-th mean of $\varphi(t)$ by

$$\varphi_\alpha(t) = \Gamma(\alpha+1)\, t^{-\alpha}\Phi_\alpha(t). \tag{1.2}$$

Let us write the Fourier series of $\varphi(t)$ as

$$\mathfrak{S}[\varphi] = \sum_{n=0}^{\infty} a_n \cos nt$$

and denote its (C,α)-mean by $C_\alpha(\omega)$.

By the result of Bosanquet [2, pp. 26–27], we have

$$C_\alpha(\omega) = \omega\int_0^\pi \varphi_\alpha(t) J_\alpha^\alpha(\omega t)\,dt + o(1), \qquad \text{as } \omega\to\infty, \tag{1.3}$$

where $J_\alpha^\alpha(u)$ satisfies the relation

$$|J_\alpha^\alpha(u)| \leqq K \tag{1.4}$$

for $u \geqq 0$, and

$$J_\alpha^\alpha(u) = (\sin(u-\pi\alpha))/u + O(1/u^2) \tag{1.5}$$

for large u.

We have then the following theorem:

THEOREM 1. 1. *If*

$$\int_0^t |\varphi_\alpha(t)|\,dt = o(t(\log 1/t)^r), \qquad (\alpha > 0,\ -1 < r < \infty), \tag{1.6}$$

as $t \to 0$, *then we have*

$$(1.7) \qquad C_\alpha(\omega) = o((\log \omega)^r) + o(1) + \int_{\pi\alpha/\omega}^{\pi} \varphi_\alpha(t) \frac{\sin(\omega t - \pi\alpha)}{t} dt,$$

as $\omega \to \infty$.

PROOF. We start from (1.3).

$$C_\alpha(\omega) = \omega \int_0^\pi \varphi_\alpha(t) J_\alpha^\alpha(\omega t)\, dt + o(1)$$

$$= \omega \int_0^{\pi\alpha/\omega} + \omega \int_{\pi\alpha/\omega}^{\pi} + o(1) = J_1(\omega) + J_2(\omega) + o(1), \text{ say.}$$

Then from (1.4) and (1.6), we have

$$|J_1(\omega)| \leqq K\omega \int_0^{\pi\alpha/\omega} |\varphi_\alpha(t)| dt = K\omega o\Big(\frac{1}{\omega} (\log \omega)^r\Big) = o\Big((\log \omega)^r\Big).$$

On the other hand, from (1.5),

$$J_2(\omega) = \int_{\pi\alpha/\omega}^{\pi} \varphi_\alpha(t) \frac{\sin(\omega t - \pi\alpha)}{t} dt + O\Big(\frac{1}{\omega} \int_{\pi\alpha/\omega}^{\pi} |\varphi_\alpha(t)| \frac{dt}{t^2}\Big)$$

$$= \int_{\pi\alpha/\omega}^{\pi} \varphi_\alpha(t) \frac{\sin(\omega t - \pi\alpha)}{t} dt + O(I(\omega)),$$

where

$$I(\omega) = \frac{1}{\omega} \int_{\pi\alpha/\omega}^{\pi} |\varphi_\alpha(t)| \frac{dt}{t^2}$$

$$= \frac{1}{\omega} [\Phi(t)/t^2]_{\pi\alpha/\omega}^{\pi} + \frac{2}{\omega} \int_{\pi\alpha/\omega}^{\pi} \Phi(t) \frac{dt}{t^2}$$

$$= o((\log \omega)^r) + o(1) + \begin{cases} o((\log \omega)^r), & \text{for } r \geqq 0 \\ o(1), & \text{for } r < 0, \end{cases}$$

where
$$\Phi(t) = \int_0^t |\varphi_r(t)| dt.$$

Summing up above estimations we get the theorem.

THEOREM 1.2. *If*

$$(1.6) \qquad \int_0^t |\varphi_\alpha(t)| dt = o(t(\log 1/t)^r), \qquad (\alpha > 0,\ -1 < r < \infty),$$

as $t \to 0$, *then we have*

$$(1.7) \qquad C_\alpha(\omega) = o((\log \omega)^{1+r}),$$

as $\omega \to \infty$.

PROOF. In order to prove the theorem, it is sufficient to show that the last integral in (1.7) is $o((\log\omega)^{1+r})$. Now

$$\left|\int_{\pi\alpha/\omega}^{\pi}\varphi_\alpha(t)\frac{\sin(\omega t-\pi\alpha)}{t}dt\right| \leqq \int_{\pi\alpha/\omega}^{\pi}|\varphi_\alpha(t)|\frac{dt}{t}$$

$$=[\Phi(t)/t]_{\pi\alpha/\omega}^{\pi}-\int_{\pi\alpha/\omega}^{\pi}\frac{\Phi(t)}{t^2}dt$$

$$=o((\log\omega)^r)+O(1)+o\left(\int_{\pi\alpha/\omega}^{\pi}(\log 1/t)^r/t\,dt\right)$$

$$=o((\log\omega)^r)+O(1)+o((\log\omega)^{1+r})=o((\log\omega)^{1+r}).$$

Thus the theorem is proved.

THEOREM 1.3. *There is an even periodic, integrable function $\varphi(t)$ such that*

$$\varphi_\alpha(t)=o((\log 1/t)^r), \qquad (\alpha>0,\ -1<r<\infty) \tag{1.8}$$

and that

$$C_\alpha(\omega)\neq o((\log\omega)^{r'}) \tag{1.9}$$

for any $0<r'<1+r$.

PROOF. Let us put $1+r=s>0$ and $r'=sa$, and then $0<a<1$. Take a number $d>0$ such as $a+d<1$ and then take a positive integer b such as

$$1/b+1<1/(a+d). \tag{1.10}$$

For $i=1,2,\cdots,$ let us put

$$l_i=3^{i^b};\ r_0=1,$$
$$r_i=(l_i/\alpha+1)r_{i-1};\ \alpha_i=\pi\alpha/r_i.$$

Then from (1.10), we have (see the following Addendum 1)

$$r_i/r_{i-1}>e(\log r_i)^{a+d}, \tag{1.11}$$

for large i. We define an even periodic function $\sigma(t)$ such that

$$\sigma(t)=c_it^\alpha(\log 1/t)^r\sin(r_it-\pi\alpha) \tag{1.12}$$

in the interval $[\alpha_i,\alpha_{i-1}]$ and $\sigma(t)=0$ elsewhere in $(0,\pi)$. Then we have $\sigma(\alpha_i)=\sigma(\alpha_{i-1})=0$. If (c_i) converges as $i\to\infty$, then $\sigma^{([\alpha]+1)}(t)$ is $([\alpha]+1)$-time integrable in $[0,\pi]$ (see the Addendum 2). By the convolution theorem, we have

$$\varphi(t)=\Gamma(1-\alpha)^{-1}\int_0^t(t-u)^{-\{\alpha\}}\sigma^{([\alpha]+1)}(u)\,du.$$

and $\varphi(t)\in L(0,2\pi)$, where $\{\alpha\}$ denotes the fractional part of α. From a theorem of fractional integral we have

$$\Phi_\alpha(t)=\Gamma(\alpha)^{-1}\int_0^t(t-u)^{\alpha-1}\varphi(u)\,du$$

$$= \int_0^t (t-u)^{[\alpha]} \sigma^{([\alpha]+1)}(u)du = \sigma(t).$$

If we take (c_i) such as $c_i \to 0$, then

(1.13) $$\Phi_\alpha(t)/t^\alpha = o((\log 1/t)^r), \qquad \text{as } t \to 0.$$

If we take (n_i) with sufficiently large gaps, then

(1.14) $$\int_{\alpha_{n_{i-1}}}^{\pi} \varphi_\alpha(t) \frac{\sin(r_{n_i}t - \pi\alpha)}{t} dt = O(1)$$

and

(1.15) $$\int_{\alpha_{n_i}}^{\alpha_{n_{i-1}}} \varphi_\alpha(t) \frac{\sin(r_{n_i}t - \pi\alpha)}{t} dt = c_{n_i} \int_{\alpha_{n_i}}^{\alpha_{n_{i-1}}} \frac{|\sin(r_{n_i}t - \pi\alpha)|^2}{t(\log 1/t)^{-r}} dt$$
$$= c_{n_i}(\log r_{n_i}/r_{n_{i-1}})^{1+r} = c_{n_i}(\log r_{n_i}/r_{n_{i-1}})^s.$$

If we take

$$c_{n_i} = (\log r_{n_i})^{-ds/2},$$

then, by the relation

$$C_\alpha(\omega) = o(\log \omega)^r + \int_{\pi\alpha/\omega}^{\pi} \varphi_\alpha(t) \frac{\sin(\omega t - \pi\alpha)}{t} dt + o(1),$$

we have (from (1.11), (1.14), (1.15))

$$\frac{C_\alpha r_{n_i}}{(\log r_{n_i})^{r'}} > (\log r_{n_i})^{-ds/2}(\log r_{n_i})^{(a+d)s}(\log r_{n_i})^{-sa}$$
$$= (\log r_{n_i})^{ds/2}.$$

Thus we get the theorem.

ADDENDUM 1. Proof of (1.11). Since

$$r_i = (l_1/\alpha + 1)(l_2/\alpha + 1) \cdots (l_i/\alpha + 1),$$

where $l_i = 3^{i^b}$, we have

$$\log r_i = O(i^{b+1}),$$
$$(\log r_i)^{a+d} = O(i^{(a+d)(b+1)}).$$

On the other hand

$$\log(r_i/r_{i-1}) = \log(l_i/\alpha + 1) = O(i^b).$$

Since, from (1.10), $1 + 1/b < 1/(a+d)$, we have

$$(\log r_i)^{a+d} < \log(r_i/r_{i-1}),$$

which is the required.

ADDENDUM 2. Proof of the integrability of $\sigma^{([\alpha]+1)}(t)$. We give the proof for $0 < \alpha < 1$. The proof of the general case is quitely analogous. We have

$$\sigma(t) = c_i t^\alpha (\log 1/t)^r \sin(r_i t - \pi\alpha)$$

and
$$\sigma'(t) = c_i\{t^{\alpha-1}(\log 1/t)^r - t^{\alpha-1}(\log 1/t)^{r-1}\}\sin(r_i t - \pi\alpha)$$
$$+ r_i c_i t^\alpha (\log 1/t)^r \cos(r_i t - \pi\alpha)$$
$$= P(t) + Q(t), \quad \text{say.}$$

Then since we can take ε such as $\alpha > \varepsilon > 0$ and

$$|P(t)| \leqq |c_i| t^{\alpha-\varepsilon-1},$$

$P(t)$ is integrable in $(0, \pi)$. On the other hand

$$|Q(t)| \leqq c_i r_i t^{\alpha} (\log 1/t)^r \leqq c_i r_i t^{\alpha-\varepsilon} \leqq c_i r_i t^{\alpha'},$$

where $\alpha' = \alpha - \varepsilon > 0$. We have consequently

$$\int_{\alpha_i}^{\alpha_{i-1}} |Q(t)| dt = \Big[c_i r_i t^{\alpha'+1} \Big]_{\pi\alpha/r_i}^{\pi\alpha/r_{i-1}}$$

$$= O\Big(c_i 3^{i^b} \Big/ \prod_{\nu=1}^{i-1} 3^{\alpha' \nu^b} \Big) = O(c_i 3^{i^b} / 3^{\alpha' i^{b+1}}) = O(c_i / 3^{\alpha' i^{b+1/2}})$$

for large i. Hence, if $|c_i| \leqq M$, then

$$\int_0^{\pi} |Q(t)| dt \leqq \sum_{i=0}^{\infty} \int_{\alpha_i}^{\alpha_{i-1}} |Q(t)| dt < \infty.$$

REMARK. If we put $r = 0$ in Theorem 1.3, we have the following corollary. There is an integrable function such that $\varphi_\alpha(t) = o(1)$, but $C_\alpha(\omega) \neq o((\log \omega)^{r'})$, for any $0 \leqq r' < 1$. The case $r' = 0$ is given by Professor S. Izumi [4]. Our example is somewhat easier.

2. We begin by stating a lemma due to A. Zygmund [6].

LEMMA 2.1. *If* $\sum a_n = o(R, n, \beta)$ *for some* β *and* $C_\alpha^{(n)}(\omega) = o[\psi^\alpha(\omega)]$ *where* $\psi(\omega) \uparrow \infty$ *as* $\omega \to \infty$ *and* $C_\alpha^{(n)}(\omega)$ *denotes the* (R, n, α)*-mean of the series* $\sum a_n$, *then*

$$\sum a_n = o(R, \mu, \alpha),$$

where

$$\mu(x) = \exp\Big(\int^x \frac{dt}{t\psi(t)} \Big).$$

THEOREM 2.1. *If*

$$\int_0^t |\varphi_\alpha(u)| du = o(1), \qquad \alpha > 1,$$

then the Fourier series $\mathfrak{S}[\varphi]$ *is* (R, μ, α)*-summable at* $t = 0$, *where*

$$\mu(x) = \exp\{(\log x)^{1-1/\alpha}\}.$$

PROOF. By the assumption of the theorem $\mathfrak{S}[\varphi]$ is $(C, \alpha+\varepsilon)$-summable and then the conditions of lemma 2.1 are satisfied from Theorem 1.2. Hence the theorem follows immediately from Lemma 2.1.

This Theorem is finer than Wang's [5].

THEOREM 2.2. *If*

$$\int_0^t |\varphi_\alpha(u)| du = o(t(\log 1/t)^{\alpha-1}), \qquad 1 \geqq \alpha > 0,$$

then the Fourier series $\mathfrak{S}[\varphi]$ *is* $(R, \log, \alpha)$*-summable.*

This is an immediate corollary of Theorem 1.2 and Lemma 2.1.

REMARK. It seems to be probable that if Σa_n is $(R, \log, \alpha)$-summable and $C_{\alpha+\varepsilon}^{(n)}(\omega) = o(1)$, then $C_{\alpha}^{(n)}(\omega) = o((\log \omega)^{\alpha})\ (\alpha > 0)$. This is easy for integral α, but the author did not succeed for fractional α. If this is valid, the result of Theorem 2.2 is best posible in view of Theorem 1.3.

The function $\varphi(t)$ is said to be $(R, \log, \alpha)$-summable to s as $t \to 0$, if

$$\lim_{t\to 0} \frac{1}{\Gamma(\alpha)(\log 1/t)} \int_t^{\pi} \left(\log \frac{u}{t}\right)^{\alpha-1} \frac{\varphi(u)}{u}\, du = s$$

We have then the following theorem.

THEOREM 2.3. *If $\mathfrak{S}[\varphi]$ is (C, α)-summable $(\alpha > 0)$ at $t = 0$, then $\varphi(t)$ is $(R, \log, \alpha + 1)$-summable as $t \to 0$.*

This is easy from the following two lemmas.

LEMMA 2.2. (Hyslop [3]). *If $C_{\alpha}(\omega) = o(\alpha > 0)$ as $\omega \to \infty$, then $\varphi_{\alpha+1}(t) = o(\log 1/t)$ as $t \to 0$.*

LEMMA 2.3. *If $\varphi(t)$ is Cesàro summable for some order and $\varphi_{\alpha}(t) = o((\log 1/t)^{\alpha})$, $\alpha > 0$, then $\varphi(t)$ is $(R, \log, \alpha)$-summable.*

Proof of the latter is analogous to Zygmund's theorem [7], where $\omega \to \infty$.

3. We shall finally treat the absolute summability. For the sake of simplicity of estimation, we consider only the summability of the integral order. In the case of fractional α, the problem is open.

THEOREM 3.1. *If the function $\varphi(t)$ is $|C, \alpha|$-summable (that is, $\varphi_{\alpha}(t)$ is bounded variation in $(0, \pi)$) for an integral $\alpha \geqq 2$, then its Fourier series $\mathfrak{S}[\varphi]$ is $|R, \log, \alpha|$-summable. Conversely if the Fourier sesies $\mathfrak{S}[\varphi]$ is $|C, \alpha|$-summable for an integral $\alpha \geqq 1$, then the function $\varphi(t)$ is $|R, \log, \alpha + 1|$-summable.*

The former half of the theorem follows from the following two lemmas.

LEMMA 3.1. (Bosanquet [1]) *If the function $\varphi(t)$ is $|C, \alpha|$-summable, then*

$$\int^{N} \left| d\left(\frac{S_n^{\alpha}(x)}{x^{\alpha}}\right)\right| = O(\log N),$$

where $S_n^{\alpha}(x)$ denotes the x-th Cesàro sum of α order of the Fourier series $\mathfrak{S}[\varphi]$.

LEMMA 3.2. *If Σa_n is $|C, \beta|$-summable for some β and $\int^{N} \left| d\left(\frac{S_n^{\alpha}(x)}{x^{\alpha}}\right)\right| = O(\log^{\alpha} N)$ for some integral $\alpha \geqq 2$, then the series Σa_n is $|R, \log, \alpha|$-summable, where $S_n^{\alpha}(x)$ denotes the (R, n, α)-sum of the series Σa_n.*

PROOF. Put $\lambda = \log n$, and

$$S_\lambda^{(0)}(x) = \sum_{\log n < x} a_n,$$

$$\alpha^{-1} S_\lambda^{(\alpha)}(x) = \int_1^x (x-t)^{\alpha-1} S_\lambda^{(0)}(t)dt = \int_1^x (x-t)^{\alpha-1} S_n^{(0)}(e^t)dt$$

and

$$\alpha^{-1} S_\lambda^{(\alpha)}(\log x) = \int_1^x \frac{(\log x - \log t)^{\alpha-1}}{t} S_n^{(0)}(t)\, dt \tag{3.1}$$

$$= \frac{1}{\beta!} \int_1^x \frac{(\log x - \log t)^{\alpha-1}}{t} \frac{d^\beta}{dt^\beta} S_n^{(\beta)}(t)dt$$

$$= \frac{1}{\beta!} \int_1^x \psi(t)\, B^{(\beta)}(t)dt,$$

where $\psi(t) = (\log x - \log t)^{\alpha-1}/t$, $B(t) = S_n^{(\beta)}(t)$, and β is an integer. Integrating by parts, we have

$$\int_1^x \psi(t) B^{(\beta)}(t)\, dt = [\psi(t)B^{(\beta-1)}(t) - \psi'(t)B^{(\beta-2)}(t) + \cdots \tag{3.2}$$

$$\cdots + (-1)^{\beta-1}\psi^{(\beta-1)}(t)B(t)]_1^x + (-1)^\beta \int_1^x \psi^{(\beta)}(t)B(t)dt,$$

where

$$B(1) = B'(1) = \cdots = 0,$$

and

$$\psi^{(k)}(x) = 0, \quad \text{for } k = 0, 1, 2, \cdots, \alpha - 2.$$

Put

$$V_k(t) \equiv (-1)^k \psi^{(k)}(t) B^{(\beta-1-k)}(t)$$
$$= A_k \psi^{(k)}(t) S_n^{(k+1)}(t),$$

for $k = \alpha - 1, \cdots, \beta - 1$, where A_k are constants. Since

$$\psi^{(k)}(t) = \frac{1}{t^{c+1}} \sum_s H_{k,s} (\log x - \log t)^{\alpha-1-s}, \qquad (0 \leqq s \leqq \alpha - 1)$$

where $H_{k,s}$ are constants, we have

$$V_s(t) = A_k S_n^{(k+1)}(t) t^{-k-1} \sum_s H_{k,s} (\log x - \log t)^{\alpha-1-s}, (k = \alpha - 1, \cdot\cdot, \beta - 1) \tag{3.3}$$

$$= \sum_{\alpha-1-s>0} + \sum_{\alpha-1-s=0} = \overline{V}_k(t) + \overline{\overline{V}}_k(t), \qquad \text{say.}$$

Then

$$[\overline{V}_k(t)]_1^x = 0.$$

Since

$$\int^N \left| d\left(\frac{S_n^{(\alpha)}(x)}{x^\alpha}\right)\right| = O(\log N) \tag{3.4}$$

and

$$\int^N \left| d\left(\frac{S_n^{(r)}(x)}{x^r}\right)\right| = O(\log N),$$

fog $r > \alpha$, we have

$$\int^N \left| d\left(\frac{\overline{\overline{V}}_k(x)}{\log^\alpha x}\right)\right| \leqq \int^N \left| \frac{d\overline{\overline{V}}_k(x)}{\log^\alpha x}\right| + \int^N \frac{|\overline{\overline{V}}_k(x)|}{x\log^{\alpha+1}x}\,dx \tag{3.5}$$

$$= I_1 + I_2, \qquad \text{say.}$$

Form (3.4), we have

$$I_1 = \left[\int^x |dV_k(x)|/\log^\alpha x\right]^N - \int^N \left\{\int^x |d\overline{\overline{V}}_k(x)|/x\log^{\alpha+1}x\right\}dx \tag{3.6}$$

$$= O\,[\log x/\log^\alpha x]^N - O\left(\int^N \log x/x\log^{\alpha+1}x\,dx\right)$$

$$= O(\log^{1-\alpha}N) + O\left(\int^N \frac{dx}{x\log^\alpha x}\right) < \infty,$$

since $\alpha > 1$.

$$I_2 = \int^N O(\log x)\frac{dx}{x\log^{\alpha+1}x} = \int \frac{dx}{x\log^\alpha x} < \infty.$$

Lastly

$$\frac{1}{(\log x)^\alpha}\int_1^x \psi^{(\beta)}(t)B(t)dt = \frac{1}{(\log x)^\alpha}\int_1^x \frac{t^\beta\log^{\alpha-1}x}{t^{\beta+1}}\frac{S_n^{(\beta)}(t)}{t^\beta}\,dt$$

$$= \frac{1}{(\log x)^\alpha}\int_1^x \frac{\log^{\alpha-1}x}{t}\,\Gamma(t)\,dt,$$

where

$$\Gamma(t) = S_n^{(\beta)}(t)/t^\beta.$$

Since from the hypothesis,

$$\int |d\Gamma(t)| = O(1),$$

and $\Gamma(N) = O(1)$ as $N \to \infty$.

The last term equals to

$$\frac{1}{\log x}\int_1^x \Gamma(t)/t\,dt \equiv A(x), \qquad \text{say.}$$

Since

$$A'(x) = (\log x)^{-2}\left[\Gamma(x)\log x/x - \frac{1}{x}\int_1^x \Gamma(t)/t\,dt\right]$$

$$= x^{-1}(\log x)^{-2}\left[\Gamma(x)\log x - \int_1^x \Gamma(t)/t\,dt\right],$$

we have

$$\int |A'(x)|\,dx \leq \int^N \left|\Gamma(x)\log x - \int_1^x \Gamma(t)/t\,dt\right| d(\log x)^{-1}$$

$$= \left[\,|\Gamma(x)\log x - \int_1^x \Gamma(t)/t dt|\,(\log x)^{-1}\right]^N$$

$$- \int^N (\log x)^{-1}|\Gamma'(x)\log x|\,dx$$

$$= |\Gamma(N)| + \operatorname*{Max}_{0<x<N}|\Gamma(x)| + \int^N |\Gamma'(x)|\,dx$$

$$= O(1).$$

Summing up the above estimations we get the lemma.

The latter half of Theorem 3.1 is proved analogously.

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