NOTES ON FOURIER ANALYSIS (XVI): ON THE STRONG LAW OF LARGE NUMBERS AND GAP SERIES

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Part I. On the strong law of large numbers.

1. Let $f(x)$ be an integrable function with period 1 such that $\int_0^1 f(x)dx = 0$ and $\int_0^1 f^2(x)dx = 1$. Recently P. Erdös [1] proved the following theorem:

THEOREM A. *If, for some $\varepsilon > 0$,*

$$(1.1) \qquad \left(\int_0^1 [f(x+t) - f(x)]^2 dx\right)^{1/2} = O\left(1\Big/\left(\log\log\frac{1}{t}\right)^{1+\varepsilon}\right)$$

as $t \to 0$, then for any sequence of positive integers (n_k), $n_k/n_{k-1} > q > 1$,

$$(1.2) \qquad \lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(n_k x) = 0,$$

almost everywhere.

Roughly speaking, the strong law of large numbers holds for $\{f(n_k x)\}$. We shall here prove that, if $1+\varepsilon$ is replaced by $2+\varepsilon$ in the condition (1.1), then the series

$$(1.3) \qquad \sum_{k=1}^{\infty} \frac{1}{k} f(n_k x)$$

converges almost everywhere. This is stronger than Theorem 1 in both hypothesis and conclusion. The method of proof is quitely different from that of P. Erdös, and is that used by Marcinkiewicz and the author [2]. By this method we give an alternative proof of the following theorem due to M. Kac, R. Salem and A. Zygmund [3]:

THEOREM B. *If, for $0 < \alpha < 1$,*

$$(1.4) \qquad \left(\int_0^1 [f(x+t) - f(x)]^2 dx\right)^{1/2} = O\left(1\Big/\left(\log\frac{1}{t}\right)^{\alpha}\right),$$

then the series

$$(1.5) \qquad \sum_{k=1}^{\infty} \frac{1}{k^{\beta}} f(n_k x)$$

converges almost everywhere for $\beta > 1 - \alpha/2$.

They proved the theorem for the case $\alpha \geqq 1$. By our method we get a little better theorem than theirs in our case. In the case $\alpha = 1$, the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}\,(\log k)^{\beta}} f(n_k x) \tag{1.6}$$

converges almost everywhere for $\beta > 1$, and in the case $\alpha > 1$, the series (1.6) converges almost eyerywhere for $\beta > 1/2$. We can see also that we can not take $\beta = 1/2$ in (1.6), but we can replace the factor $(\log k)^{\beta}$ in (1.6) by $\sqrt{\log k}\ (\log\log k)^{\beta}$ $(\beta > 1/2)$, and so on.

We prove a theorem concerning Riemann sum, due to R. Salem [6] and S. Yano [7] in a little modified form, by the above method. We further remark that our method enables us to prove theorems, replaced $f(n_k x)$ by $f_k(n_k x)$. In the following, (n_k) denote a sequence of integers with the Hadamard gap[1].

2. THEOREM 1. *If for some $\varepsilon > 0$*

$$\left(\int_0^1 [f(x+t) - f(x)]^2 dx\right)^{1/2} = O\left(1\Big/\left(\log\log\frac{1}{t}\right)^{2+\varepsilon}\right), \tag{2.1}$$

then the series

$$\sum_{k=1}^{\infty} \frac{1}{k} f(n_k x) \tag{2.2}$$

converges almost everywhere.

For the proof of Theorem 1 we need two lemmas.

LEMMA 1. *If $f(x)$ satisfies the condition (2.1) for some $\varepsilon > 0$, then*

$$\int_0^1 f(n_j x) f(n_k x) dx = O\,(1/(\log|j-k|)^{2+\varepsilon}) \tag{2.3}$$

for $j \neq k$.

Proof is similar as Lemma 1 in [2].

LEMMA 2. *If $f(x)$ satisfies the condition (2.1) for some $\varepsilon > 0$, then the series (2.2) converges to a function $\phi(x)$ in the L^2-mean.*

PROOF. Let $1 \leqq m \leqq n$. Then we have, by Lemma 1,

$$\int_0^1 \left(\sum_{k=m}^{n} \frac{1}{k} f(n_k x)\right)^2 dx = \sum_{j,k=m}^{n} \frac{1}{jk} \int_0^1 f(n_j x) f(n_k x)\, dx \tag{2.4}$$

$$= \sum_{k=m}^{n} \frac{1}{k^2} + O\left(\sum_{j,k=m}^{n} \frac{1}{jk\,(\log|j-k|)^{2+\varepsilon}}\right) = O\left(\frac{1}{(\log m)^{\varepsilon}}\right) = o(1),$$

1) Theorems in this paper were first proved for integral n_k. Mr. G. Sunouchi remarked me that these hold for real n_k with trivial modification of proof.

as $m, n \to \infty$. Thus the lemma is proved.

PROOF OF THEOREM 1. We can suppose that

$$f(x) \sim \sum_{\nu=1}^{\infty} a_\nu e^{2\pi i\nu x}.$$

Let us put

$$s_n(x) \equiv \sum_{\nu=1}^{n} a_\nu e^{2\pi i\nu x}.$$

Then we have easily

$$f(n_k x) \sim \sum_{\nu=1}^{\infty} a_\nu e^{2\pi i\nu n_k x}$$

and

$$\left(\int_0^1 [f(n_k x) - s_{\mu_k}(n_k x)]^2\, dx\right)^{1/2} = \left(\int_0^1 [f(x) - s_{\mu_k}(x)]^2 dx\right)^{1/2}$$
$$= O\,(1/(\log\log\mu_k)^{2+\epsilon}).$$

If we put $\mu_k \equiv 2^k$ and $\psi_k \equiv f(x) - s_{\mu_k}(x)$, then

$$\int_0^1 \left(\sum_{k=2}^{\infty} \frac{1}{k} \,|\psi_k(n_k x)|\right)^2 dx = \int_0^1 \left(\sum_{k=2}^{\infty} \frac{1}{\sqrt{k}\,\log k}\, \frac{\log k}{\sqrt{k}}\, |\psi_k(n_k x)|\right)^2 dx$$
$$\leqq \sum_{k=2}^{\infty} \frac{1}{k(\log k)^2} \sum_{k=2}^{\infty} \frac{(\log k)^2}{k} \int_0^1 [\psi_k(n_k x)]^2 dx$$
$$= O\left(\sum_{k=1}^{\infty} \frac{1}{k(\log k)^{2+2\epsilon}}\right) = O(1).$$

Thus the series

$$\sum_{k=1}^{\infty} \frac{1}{k}\, \psi_k(n_k x)$$

converges almost everywhere, and converges in the L^2-means. By Lemma 2, the series

$$\sum_{k=1}^{\infty} \frac{1}{k}\, s_{\mu_k}(n_k x)$$

converges in the L^2-mean. Using the Kolmogoroff -Kac theorem [4], we see that the sequence

$$\sum_{k=1}^{2^N} \frac{1}{k}\, s_{\mu_k}(n_k x)$$

and then the sequence

$$\sum_{k=1}^{2^N} \frac{1}{k} f(n_k x) \tag{2.5}$$

converges almost everywhere, from which it follows the almost everywhere convergence of (2.2), provided that

$$(2.6)\qquad \sum_{n=1}^{\infty}\int_0^1\Big(\max_{2^n<\nu\leqq 2^{n+1}}\Big|\sum_{k=2^n}^{\nu}\frac{1}{k}f(n_kx)\Big|\Big)^2dx<\infty.$$

For the proof of (2.6) we use the device of Menchoff [5]. Let

$$I_j\equiv\int_0^1\Big[\sum_{k=2^n+j2^\lambda}^{2^n+(j+1)2^\lambda}\frac{1}{k}f(n_kx)\Big]^2dx.$$

Then we have, by Lemma 1, putting $\alpha=2+\varepsilon$,

$$I_j\leqq\sum_{k=2^n+j2^\lambda}^{2^n+(j+1)2^\lambda}\frac{1}{k^2}+\sum_{k,l=2^n+j2^\lambda}^{2^n+(j+1)2^\lambda}\frac{1}{kl(\log|k-l|)^\alpha}$$
$$\equiv I_j^{(1)}+I_j^{(2)},$$

say. Now

$$I_j^{(1)}\leqq\frac{1}{2^n+j2^\lambda}-\frac{1}{2^n+(j+1)2^\lambda}\leqq\frac{2^\lambda}{(2^n+j2^\lambda)^2}$$

and

$$I_j^{(2)}=\sum_{l=2^n+j2^\lambda-1}^{2^n+(j+1)2^\lambda}\frac{1}{l}\sum_{k=2^n+j2^\lambda}^{l-1}\frac{1}{k(\log(l-k))^\alpha}$$

$$\leqq\sum_{l=2^n+j2^\lambda-1}^{2^n+(j+1)2^\lambda}\frac{1}{l(2^n+j2^\lambda)}\frac{l-(2^n+j2^\lambda)}{(\log(l-(2^n+j2^\lambda)))^\alpha}\leqq\frac{2^{2\lambda}}{\lambda^\alpha(2^n+j2^\lambda)^2}.$$

As easily may be seen, the n-th term of (2.6) is less than

$$n\sum_{\lambda=1}^{n}\sum_{j=0}^{2^{n-\lambda}}I_j\leqq An\sum_{\lambda=1}^{n}\sum_{j=0}^{2^{n-\lambda}}I_j^{(2)}$$

$$\leqq An\sum_{\lambda=1}^{n}\sum_{j=0}^{2^{n-\lambda}}\frac{2^{2\lambda}}{\lambda^\alpha(2^n+j2^\lambda)^2}$$

$$\leqq A\sum_{\lambda=1}^{[n/2]}\frac{1}{\lambda^\alpha}\frac{n}{2^{n/2}}+An\sum_{\lambda=[n/2]+1}^{n}\frac{2^\lambda}{\lambda^\alpha 2^n}$$

$$\leqq A\frac{n}{2^{n/2}}+\frac{A}{n^{\alpha-1}}\leqq\frac{A}{n^{\alpha-1}}.$$

Thus we get (2.6) as $\alpha-1>1$, and then the theorem is proved.

Similarly we can prove that, if

$$\Big(\int_0^1[f(x+t)-f(x)]^2dx\Big)^{1/2}=O\Big(1\Big/\Big(\log\log\log\frac{1}{t}\Big)^{2+\varepsilon}\Big),$$

then the series

$$\sum_{k=2}^{\infty}\frac{f(n_kx)}{k\log k}$$

converges almost everywhere.

3. THEOREM 2. *If for* $1>\alpha>0$,

$$\left(\int_0^1 [f(x+t)-f(x)]^2dx\right)^{1/2} = O\left(1\Big/\left(\log\frac{1}{t}\right)^\alpha\right), \tag{3.1}$$

then the series

$$\sum_{k=1}^\infty \frac{1}{k^\beta} f(n_kx) \tag{3.2}$$

converges almost everywhere for $\beta>1-\alpha/2$.

We will begin by two lemmas analogous to Lemma 1 and 2.

LEMMA 3. *If* $f(x)$ *satisfies the condition* (3.1) *for* $1>\alpha>0$, *then*

$$\int_0^1 f(n_jx)f(n_kx)dx = O(1/|j-k|^\alpha) \tag{3.3}$$

for $j\neq k$.

Proof is similar as Lemma 1.

LEMMA 4. *If* $f(x)$ *satisfies the condition* (3.1) *for* $1>\alpha>0$, *then the series* (3.2) *converges in the* L^2*-mean for* $\beta>1-\alpha/2$.

PROOF. Let $1\leqq m\leqq n$ and we can suppose that $\beta<1$. By Lemma 3 we have

$$\int_0^1\left(\sum_{k=m}^n \frac{1}{k^\beta} f(n_kx)\right)^2dx = \sum_{j,k=m}^n \frac{1}{j^\beta k^\beta}\int_0^1 f(n_jx)f(n_kx)\,dx$$

$$= \sum_{k=m}^n \frac{1}{k^{2\beta}} + O\Big(\sum_{\substack{j,k=m\\ j\neq k}}^n \frac{1}{j^\beta k^\beta |j-k|^\alpha}\Big)$$

$$= O\left(\frac{1}{m^{2\beta-1}}\right) + O\Big(\sum_{j=m+1}^n \frac{1}{j^\beta}\sum_{k=m}^{j-1}\frac{1}{k^\beta(j-k)^\alpha}\Big) \tag{3.4}$$

$$= O\left(\frac{1}{m^{2\beta-1}}\right) + O\Big(\sum_{j=m+1}^n \frac{1}{j^\beta}\Big[\sum_{k=m}^{[j/2]} + \sum_{k=[j/2]+1}^{j-1}\frac{1}{k^\beta(j-k)^\alpha}\Big]\Big)$$

$$= O\left(\frac{1}{m^{2\beta-1}}\right) + O\Big(\sum_{j=m+1}^n \frac{1}{j^{\alpha+2\beta-1}}\Big) = o(1)$$

as $m,\ n\to\infty$. Thus the lemma is proved.

PROOF OF THEOREM 2. We follow the line of the proof of Theorem 1 and use its notations. We have

$$\left(\int_0^1 [\psi_k(n_kx)]^2\,dx\right)^{1/2} = \left(\int_0^1 [\psi_k(x)]^2dx\right)^{1/2} = O\left(1/(\log\mu_k)^\alpha\right),$$

$$\int_0^1\Big(\sum_{k=1}^{\infty}\Big|\frac{1}{k^{\beta}}\psi_k(n_kx)\Big|\Big)^2dx=\int_0^1\Big(\sum_{k=1}^{\infty}\frac{1}{k^{\frac{\beta}{2}+\frac{\alpha}{4}}}\frac{1}{k^{\frac{\beta}{2}-\frac{\alpha}{4}}}|\psi_k(n_kx)|\Big)^2dx$$
$$\leq\sum_{k=1}^{\infty}\frac{1}{k^{\beta+\frac{\alpha}{2}}}\sum_{k=1}^{\infty}\frac{1}{k^{\beta-\frac{\alpha}{2}}(\log\mu_k)^{2\alpha}}$$

which is bounded if $\mu_k = 2^k$. Thus it is sufficient to prove that, corresponding to (2.6),

$$\sum_{n=1}^{\infty}\int_0^1\Big(\max_{2^n<\nu\leq 2^{n+1}}\Big|\sum_{k=2^n}^{\nu}\frac{1}{k^{\beta}}f(n_kx)\Big|\Big)^2dx<\infty. \tag{3.5}$$

Now, by (3.4), we have

$$I_j\leq\int_0^1\Big[\sum_{k=2^n+j2^{\lambda}}^{2^n+(j+1)2^{\lambda}}\frac{1}{k^{\beta}}f(n_kx)\Big]^2dx\leq\frac{2}{(2^n+j2^{\lambda})^{2\beta}}.$$

Thus the n-th term of (3.5) is less than

$$n\sum_{\lambda=1}^{n}\sum_{j=0}^{2^{n-\lambda}}I_j\leq n\sum_{\lambda=1}^{n}\sum_{j=0}^{2^{n-\lambda}}\frac{2^{\lambda}}{(2^n+j2^{\lambda})^{2\beta}}\leq\frac{n^2}{2^{(2\beta-1)n}}.$$

Thus the theorem is proved.

4. THEOREM 3. *If*

$$\Big(\int_0^1[f(x+t)-f(x)]^2dx\Big)^{1/2}=O\Big(1\Big/\log\frac{1}{t}\Big), \tag{4.1}$$

then the series

$$\sum_{k=1}^{\infty}\frac{f(n_kx)}{\sqrt{k}\,(\log k)^{\beta}} \tag{4.2}$$

converges almost everywhere for $\beta>1$.

Proof is similar as Theorem 2. This is the case $\alpha=1$ in Theorem 2.

THEOREM 4. *If for any $\varepsilon>0$,*

$$\Big(\int_0^1[f(x+t)-f(x)]^2dx\Big)^{1/2}=O\Big(1/\Big(\log\frac{1}{t}\Big)^{1+\varepsilon}\Big), \tag{4.3}$$

then the series

$$\sum_{k=1}^{\infty}\frac{f(n_kx)}{\sqrt{k}\,(\log k)^{\beta}} \tag{4.4}$$

converges almost everywhere for $\beta>1/2$.

Proof is also similar as Theorem 2.

5. We will now prove theorems concerning Riemann sums.

THEOREM 5. *If $\alpha>0$,*

$$(5.1)\qquad \left(\int_0^1 [f(x+t)-f(x)]^2\,dx\right)^{1/2} = O\left(1\Big/\left(\log\frac{1}{t}\right)^{\alpha}\right)$$

and $\sum (\log n_k)^{-2\alpha} < \infty$, *then the Riemann sum*

$$(5.2)\qquad F_{n_k}(x) \equiv \frac{1}{n_k}\sum_{\mu=1}^{n_k} f\left(x+\frac{\mu}{n_k}\right)$$

converges to $\int_0^1 f(x)\,dx$ *almost everywhere.*

LEMMA 5. *Under the hypothesis of Theorem 5,*

$$(5.3)\qquad \int_0^1 F_{n_j}(x)F_{n_k}(x)\,dx = O\left(1/(\log n_j)^{\alpha}(\log n_k)^{\alpha}\right)$$

PROOF. We have, supposing $\int_0^1 f(t)dt = 0$,

$$\int_0^1 F_{n_j}(x)F_{n_k}(x)dx = \int_0^1\left[\frac{1}{n_j}\sum_{\mu=1}^{n_j} f\left(x+\frac{\mu}{n_j}\right)\right]\left[\frac{1}{n_k}\sum_{\nu=1}^{n_k} f\left(x+\frac{\nu}{n_k}\right)\right]dx$$

$$= \int_0^1\left[\frac{1}{n_j}\sum_{\mu=1}^{n_j} f\left(x+\frac{\mu}{n_j}\right) - \int_0^1 f(t)dt\right]\left[\frac{1}{n_k}\sum_{\nu=1}^{n_k} f\left(x+\frac{\nu}{n_k}\right) - \int_0^1 f(t)dt\right]dx$$

$$= \int_0^1\left[\sum_{\mu=1}^{n_j}\int_{\mu/n_j}^{(\mu+1)/n_j}\left[f\left(x+\frac{\mu}{n_j}\right)-f(x+t)\right]dt\right]\left[\sum_{\nu=1}^{n_k}\int_{\nu/n_k}^{(\nu+1)/n_k}\left[f\left(x+\frac{\nu}{n_k}\right)-f(x+t)\right]dt\right]dx.$$

If we put

$$I_j = \int_0^1\left[\sum_{\mu=1}^{n_j}\int_{\mu/n_j}^{(\mu+1)/n_j}\left[f\left(x+\frac{\mu}{n_j}\right)-f(x+t)\right]dt\right]^2 dx,$$

then

$$I_j = \sum_{\lambda,\mu=1}^{n_j}\int_0^1 dx\int_{\mu/n_j}^{(\mu+1)/n_j}\left[f\left(x+\frac{\mu}{n_j}\right)-f(x+t)\right]dt\int_{\lambda/n_j}^{(\lambda+1)/n_j}\left[f\left(x+\frac{\lambda}{n_j}\right)-f(x+u)\right]du$$

$$= \sum_{\lambda,\mu=1}^{n_j}\int_{\mu/n_j}^{(\mu+1)/n_j}dt\int_{\lambda/n_j}^{(\lambda+1)/n_j}du\int_0^1\left[f\left(x+\frac{\mu}{n_j}\right)-f(x+t)\right]\left[f\left(x+\frac{\lambda}{n_j}\right)-f(x+u)\right]dx.$$

The inner integral is less than in absoute value

$$\left\{\int_0^1\left[f\left(x+\frac{\mu}{n_j}\right)-f(x+t)\right]^2dx\int_0^1\left[f\left(x+\frac{\lambda}{n_j}\right)-f(x+t)\right]^2dx\right\}^{1/2}$$

which is $O(1/(\log n_j)^{2\alpha})$. Thus we have

$$\int_0^1 F(n_jx)F(n_kx)dx = O(\sqrt{I_jI_k}) = O\left(1/(\log n_j)^{\alpha}(\log n_k)^{\alpha}\right).$$

PROOF OF THEOREM 5. It is sufficient to follow the lines of proof of Theorem 1. By (5.3) ($j=k$) and the convergence of $\sum_{j=1}^{\infty} 1/(\log n_j)^{2\alpha}$, the series

$$\sum_{j=1}^{\infty}\int_0^1 F_{n_j}(x)^2 dx$$

converges almost everywhere and then the series $\sum_{j=1}^{\infty}\left(F^{n_j}(x)\right)^2$ converges almost everywhere. Hence $F_{n_j}(x)$ tends to zero almost everywhere as $j\to\infty$.

Similarly we can prove that if, $\alpha>1$,

$$\left(\int_0^1 [f(x+t)-f(x)]^2 dx\right)^{1/2} = O\left(1\Big/\left(\log\log\frac{1}{t}\right)\right)$$

and (n_k) has the Hadamard gap, then

$$\sum_{k=1}^{\infty}\frac{1}{k}F_{n_k}(x)$$

converges almost everywhere. Especially

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}F_{n_k}(x) = \int_0^1 f(x)dx,$$

almost everywhere.

6. We remark that we can replace $f(n_k x)$ by $f_k(n_k x)$ in the theorems above proved. For example, Theorem 2 and 5 can be generalized in the following form:

THEOREM 6. *If* $\int_0^1 f_k(x)dx=0$, $\int_0^1 f_k^2(x)dx=1$ *and for some* $\varepsilon>0$,

$$\left(\int_0^1\left[[f_k(x+t)-f_k(x)\right]^2 dx\right)^{1/2} \leqq A\Big/\left(\log\log\frac{1}{t}\right)^{2+\varepsilon},$$

A being a constant independent of k, then the series

$$\sum_{k=1}^{\infty}\frac{1}{k}f_k(n_k x)$$

converges almost everywhere, and then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=1}^{N}f_k(n_k x)=0$$

almost everywhere.

THEOREM 7. *If* $\alpha>1$,

$$\left(\int_0^1\left[f_k(x+t)-f_k(x)\right]^2 dx\right)^{1/2} \leqq A\Big/\left(\log\frac{1}{t}\right)^{\alpha}.$$

A being a constant independent of k, and $\sum \frac{1}{(\log n_k)^{2\alpha}}$ *converges, then, the Riemann sum*

$$F_{n_k}(x) \equiv \frac{1}{n_k} \sum_{\mu=1}^{n_k} f_k\left(x + \frac{\mu}{n_k}\right)$$

converges to

$$\lim_{k\to\infty} \int_0^1 f_k(t)dt$$

almost everywhere, provided that the last limit exists.

Part II. Gap Theorems.

1. Let $f(t)$ be a function considered in the beginning of Part I. M. Kac, R. Salem and A. Zygmund [3] proved that

THEOREM A. *If the sequence* (n_k) *has the Hadamard gap and*

$$\left(\int_0^1 [f(x+t) - f(x)]^2 dx\right)^{1/2} = O\left(1\Big/\left(\log \frac{1}{t}\right)^{\alpha}\right) \tag{1.1}$$

for $\alpha > 1$, *then the convergence of the series*

$$\sum_{n=1}^{\infty} c_n^2 (\log n)^2. \tag{1.2}$$

implies the almost everywhere convergence of the series

$$\sum_{k=1}^{\infty} c_k f(n_k t). \tag{1.3}$$

They proved the theorem for real sequence (n_k). We restrict ourselves that (n_k) is a sequence of integers.

In this case we have proved [2] that

THEOREM B. *If the sequence* (n_k) *of integers has the Hadamard gap and if* $f(t)$ *satisfies the condition* (1.1) *for* $\alpha > 1$, *then the convergence of the series*

$$\sum_{n=1}^{\infty} c_n^2 \tag{1.4}$$

implies the almost everywhere convergence of

$$\sum_{k=1}^{m_i} c_k f(n_k t), \tag{1.5}$$

where (m_i) *is a sequence with the Hadamard gap.*

We shall now show that the condition of (m_i) may be much improved by the method used in our paper [2] and that, if the condition (1.1) is replaced by the ordinary integrated Lipschitz condition:

$$\left(\int_0^1 \left[f(x+t)-f(x)\right]^2 dx\right)^{1/2} = O\,(t^{\alpha}) \tag{1.6}$$

for $0 < \alpha < 1$, then the convergence of the series

$$\sum_{n=1}^{\infty} c_n^2(\log\log n)^2 \tag{1.7}$$

implies the almost everywhere convergence of the series (1.3).

Finally we prove, generalizing the Kac theorem [8], that if the sequence (n_k) has larger gap than the Hadamard's such that

$$\sum_{k=1}^{\infty} (n_k/n_{k+1})^{2\alpha} < \infty, \tag{1.8}$$

and $f(t)$ satisfies the condition (1.6) for $0 < \alpha < 1$, then the convergence of (1.4) implies the almost everywhere convergence of (1.3).

2. THEOREM 1. *If the sequence (n_k) of imtegers has the Hadamad gap such that*

$$n_{k+1}/n_k > \lambda > 1 \quad (k = 1, 2, \cdots\cdots) \tag{2.1}$$

and

$$\left(\int_0^1 \left[f(x+t)-f(x)\right]^2 dx\right)^{1/2} = O\left(1\Big/\left(\log\frac{1}{t}\right)^{\alpha}\right) \tag{2.2}$$

for $\alpha > 1$, then the convergence of the series

$$\sum_{n=1}^{\infty} c_n^2 \tag{2.3}$$

implies the almost everywhere convergence of

$$\sum_{k=1}^{m_i} c_k f(n_k x), \tag{2.4}$$

where (m_i) is a sequence such that

$$m_{k+1} - m_k > m_k^{1/2\alpha}(\log m_k)^{1/\alpha} \quad (k = 1, 2, \cdots\cdots). \tag{2.5}$$

The condition (2.5) is satisfied when

$$m_{k+1} - m_k > \sqrt{m_k} \quad (k = 1, 2, \cdots\cdots),$$

more specially when $m_k = k^2$.
For the proof of theorem, we need some lemmas.

LEMMA 1. *If $f(x)$ satisfies the condition (2.2), then*

$$\left|\int^1 f(n_i x)\, f(n_j x) dx\right| \leqq A/|i-j|^{\alpha} \quad (i \neq j). \tag{2.6}$$

Proof is similar as Lemma 1 in [2].

LEMMA 2. *If the conditions of the theorem is satisfied, then the series*

$$\sum_{k=1}^{\infty} c_k f(n_k x) \tag{2.7}$$

converges in the L^2-mean.

PROOF. We have

$$\int_0^1 \Big(\sum_{k=N}^{M} c_k f(n_k x)\Big)^2 dx = \sum_{i,j=N}^{M} c_i c_j \int_0^1 f(n_i x) f(n_j x) dx$$

$$= \sum_{i=N}^{M} c_i^2 \int_0^1 (f(n_i x))^2 dx + \sum_{\substack{i,j=N \\ i \neq j}}^{M} c_i c_j \int f(n_i x) f(n_j x) dx$$

$$\equiv I + J,$$

say. Evidently

$$I = \sum_{i=N}^{M} c_i^2 \to 0 \qquad (M,\ N \to \infty).$$

By (2.6)

$$|J| \leqq \sum_{\substack{i,j=N \\ i \neq j}}^{M} \frac{|c_i c_j|}{|i-j|^\alpha} \leqq A \sum_{i=N}^{M} c_i^2 \to 0 \qquad (M, N \to \infty).$$

Thus the lemma is proved.

PROOF OF THEOREM 1. Let the Fourier series of $f(x)$ be

$$f(x) \sim \sum_{\nu=-\infty}^{\infty} a_\nu e^{2\pi i \nu x}. \tag{2.8}$$

We can suppose that $a_\nu = 0$ for $\nu \leqq 0$. Let the n-th partial sum of (2.8) be

$$s_n(x) \equiv \sum_{\nu=1}^{n} a_\nu e^{2\pi i \nu x}$$

Since we have

$$f(n_k x) \sim \sum_{\nu=1}^{\infty} a_\nu e^{2\pi i \nu n_k x},$$

we have, by the condition (2.2),

$$\Big(\int_0^1 \big[f(n_k x) - s_{\mu_k}(n_k)\big]^2 dx\Big)^{1/2} = \Big(\int_0^1 \big[f(x) - s_{\mu_k}(x)\big]^2 dx\Big)^{1/2}$$

$$= A/(\log \mu_k)^\alpha.$$

If we put

$$\psi_k(x) = f(x) - s_{\mu_k}(x),$$

then

$$\int_0^1 (\psi_k(n_k x))^2 dx \leqq A/(\log \mu_k)^{2\alpha}.$$

Let us take

$$\mu_k = \exp\,(k^{1/2\alpha} (\log k)^{1/\alpha}),$$

then $(\log \mu_k)^{2\alpha} = k (\log k)^2$, and then

$$\sum 1/(\log \mu_k)^{2\alpha} = \sum 1/k(\log k)^2 < \infty.$$

Hence the series

$$\sum_{k=1}^{\infty} c_k \psi_k(n_k x) \tag{2.9}$$

converges almost everywhere and converges in the L^2-mean. The series (2.7) is the sum of (2.9) and the series

$$\sum c_k s_{\mu_k}(n_k x). \tag{2.10}$$

The m_k -th term in (2.10) is the trigonometrical polynomial with the first term $c_{m_k} a_1 \exp(2\pi i n_{m_k} x)$ and with the last term $c_{m_k} a_{\mu_{m_k}} \exp(2\pi i \mu_{m_k} n_{m_k} x)$.

Let us now devide the series (2.10) into two parts

$$\sum c_k' s_{\mu_k}(n_k x), \qquad \sum c_k'' s_{\mu_k}(n_k x), \tag{2.11}$$

where $c_k' = c_k$ for $m_{2\nu-1} < k \leq m_{2\nu}$ $(\nu = 1, 2, \cdots\cdots)$ and $c_k' = 0$ otherwise, and $c_k'' = c_k - c_k'$ $(k = 1, 2, \cdots\cdots)$. By Lemma 2, the series (2.7) defines a function in L^2-class, and the series (2.9) does also. Hence the series (2.10) becomes the Fourier series of a function of the L^2-class. The same holds for the both of the series (2.11).

If, for a $p > 1$,

$$n_{m_k+1} > p n_{m_k} \mu_{m_k} \qquad (k = 1, 2, \cdots\cdots), \tag{2.12}$$

then m_k -th partial sum of the series (2.11) converges almost everywhere, by Kolmogoroff theorem. The same holds for the series (2.10) and then for the series (2.7).

Now, the condition (2.12)is satisfied when

$$\lambda^{m_{k+1}-m_k} > p\mu_{m_k} = p \exp(m_k^{1/2\alpha}(\log m_k)^{1/\alpha}),$$

and then when

$$m_{k+1} - m_k > m_k^{1/2\alpha}(\log m_k)^{1/\alpha},$$

for we can take $\lambda \geqq 3$. This is the condition (2.5). Thus the theorem is proved.

3. THEOREM 2. *If the sequence (n_k) of integers has Hadamard gap and*

$$\left(\int_0^1 \left[f(x+t) - f(x)\right]^2 dx\right)^{1/2} = O(t^\alpha) \tag{3.1}$$

for $0 < \alpha \leqq 1$, then the convergence of the series

$$\sum_{n=1}^{\infty} c_n^2 (\log \log n)^2 \tag{3.2}$$

implies the almost everywhere convergence of the series

$$\sum_{k=1}^{\infty} c_k f(n_k x). \tag{3.3}$$

LEMMA 3. *If the conditions of Theorem 2 are satisfied, then*

$$\int_0^1 \left(\max_{1 \leqq j \leqq n} \left| \sum_{k=1}^{j} c_k f(n_k x) \right| \right)^2 dx \leqq A (\log n)^2 \sum_{k=1}^{n} c_k^2. \tag{3.4}$$

This is an analogue of the Menchoff's lemma [9]. Proof runs on the similar lines. The difference lies in the point that orthogonality relation is replaced by

$$\left|\int_0^1 f(n_i t) f(n_j t)dt\right| \leqq A\Big/\lambda^{\alpha|i-j|},$$

which is proved in [2], and that the Bessell's inequality is replaced by

$$\int_0^1 \Big(\sum_{k=N}^{M} c_k f(n_k t)\Big)^2 dt \leqq A \sum_{k=N}^{M} c_k^2,$$

which is contained in the proof of Lemma 2.

PROOF OF THEOREM 2. Using the notations in the proof of Theorem 1, we obtain

$$\int_0^1 (\psi_k(n_k x))^2 dx \leqq A/\mu_k^{2\alpha}.$$

If we take

$$\mu_k = k^{1/2\alpha}(\log k)^{\beta/2\alpha}$$

for $\beta > 1$, then the series (2.9) converges almost everywhere and converges in the L^2-mean.

The condition (2.12) is satisfied when

(3.5) $$\lambda^{m_{k+1}-m_k} > p m_k^{1/2\alpha}(\log k)^{\beta/2\alpha}$$

for a $p > 1$. Let

(3.6) $$m_k = k(\log k)^\gamma \qquad (k = 1, 2, \cdots),$$

then

$$m_{k+1} - m_k \geqq A(\log k)^\gamma \qquad (k = 1, 2, \cdots)$$

Thus, if $\gamma > 1$ in (3.6), then the condition (3.5) is satisfied. Hence, for the sequence (m_k) in (3.6), the sequence

$$t_{m_k} = \sum_{j=1}^{m_k} c_j f(n_j t)$$

converges almost everywhere.

Let n be any integer and k be an integer such that $m_k < n \leqq m_{k+1}$. Then we have

(3.7) $$\int_0^1 \Big(\max_{m_k < n \leqq m_{k+1}} \Big|\sum_{j=m_k}^{n} c_j f(n_j t)\Big|\Big)^2 dt \leqq A\,(\log(m_{k+1} - m_k))^2 \sum_{j=m_k}^{m_{k+1}} c_j^2$$

by Lemma 3. Since

$$m_{k+1} - m_k \leqq A\,(\log k)^\gamma \qquad (k = 1, 2, \cdots),$$

the right-hand side of (3.7) is majorated by

$$\leqq A\,(\log\log k)^2 \sum_{j=m_k}^{m_{k+1}} c_j^2$$

$$\leqq A \sum_{j=m_k}^{m_{k+1}} c_j^2 (\log\log j)^2.$$

Hence

$$\sum_{k=1}^{\infty}\int_0^1\Big(\max_{m_k<n\leqq m_{k+1}}\Big|\sum_{j=m_k}^{n}c_jf(n_jt)\Big|\Big)^2dt<\infty.$$

Thus

$$t_n=t_{m_k}+\sum_{j=m_k}^{n}c_jf(n_jt)$$

converges almost everywhere as $n\to\infty$, which is the required.

4. THEOREM 3. *If*

$$\Big(\int_0^1\big[f(x+t)-f(x)\big]^2dx\Big)^{1/2}=O(t^{\alpha}) \tag{4.1}$$

for $0<\alpha\leqq 1$ *and* (n_k) *is a sequence of integers such that*

$$\sum_{k=1}^{\infty}(n_k/n_{k+1})^{2\alpha}<\infty. \tag{4.2}$$

Then the convergence of the series

$$\sum_{k=1}^{\infty}c_k^2. \tag{4.3}$$

implies the almost everywhere convergence of the series

$$\sum_{k=1}^{\infty}c_kf(n_kt). \tag{4.4}$$

This theorem is proved by M. Kac [8] for the case $n=2^{m_k}$, (m_k) being an increasing sequence of integers.

PROOF. Using the above notation, we take

$$\mu_k=[n_{k+1}/2n_k],$$

then $\sum 1/\mu_k^{2\alpha}<\infty$. Since $s_{\mu_k}(n_kt)$ is a trigonometrical polynomial ending by $c_ka_{\mu_k}e^{2\pi in_k\mu_kt}$ and $s_{\mu_{k+1}}(n_{k+1}t)$ is that beginning by $c_{k+1}a_1e^{2\pi in_{k+1}t}$. Thus, following the same idea as the proof of Theorem 1 we can show that

$$\sum_{k=1}^{\infty}c_ks_{\mu_k}(n_kt)$$

converges almost everywhere.

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