A THEOREM ON THE MAJORATION OF HARMONIC MEASURE AND ITS APPLICATIONS

MASATSUGU TSUJI

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1. Let D be a domain on the z-plane, which contains $z=0$ and $z=\infty$ belongs to its boundary Γ, which consists of at most a countable number of analytic curves. Let θ_r be the part of $|z|=r$ which is contained in D and $r\theta(r)$ be its length.

We put $\bar{\theta}(r)=\theta(r)$, if $|z|=r$ meets Γ and $\bar{\theta}(r)=\infty$, if $|z|=r$ does not meet Γ and is contained entirely in D.

Let $w(z)$ be one-valued and regular in D and on Γ, such that $|w(z)|>1$ in D and $|w(z)|=1$ on Γ and

$$M(r)=\operatorname*{Max}_{\theta_r}|w(z)|.$$

Then modifying Carleman's method[1], K. Arima[2] proved that

$$\log\log M(r)>\pi\int_0^{\alpha r}\frac{dr}{r\bar{\theta}(r)}-\text{const.}\qquad(0<\alpha<1),$$

where α is any postive number less than 1.

This is an extension of Ahlfors' theorem[3], who assumed that D is simply connected and is bounded by a single curve.

In this paper, by modifying Arima's method, we will prove a theorem on the majoration of harmonic measure and by which we will prove an extension of Arima's theorem.

2. Let $D=D_R$ be a domain on the z-plane, which lies in $|z|<R$, such that a part θ_R of $|z|=R$ belongs to its boundary. We denote the boundary of D, which lies in $|z|<R$ by $\Gamma=\Gamma_R$, so that $\Gamma_R+\theta_R$ is the whole boundary of D.

$z=0$ may or may not belong to D and let ρ_0 be the shortest distance from $z=0$ to D, such that $\rho_0=\inf\limits_{z\in D}|z|$.

Now $|z|=r(\rho_0<r<R)$ separates D into at most a countable number

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2) K. ARIMA, On maximum modulus of an integral function. To appear in the Jour. Math. Soc. Japan.

3) L. AHLFORS, Über die asymptotischen Werte der meromorphen Funktionen endlicher Ordnung. Acta.Acad.Aboensis.Mat.et Phys.6(1932).

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of connected domains. We consider only such connected ones $\{D_r^{(i)}\}$, which contains $z=0$, if $z=0$ belongs to D or have boundary points on $|z| = \rho_0$, if $z=0$ does not belong to D and put

$$D_r = \sum_i D_r^{(i)}. \tag{1}$$

Let θ_r be the part of $|z| = r$, which belongs to the boundary of D_r and Γ_r be the part of Γ, which belongs to the boundary of D_r, so that $\Gamma_r + \theta_r$ is the whole boundary of D_r. We denote the length of θ_r by $r\theta(r)$.

If Γ consists of a finite number of analytic curves, then we see easily that there exists a finite number of values $\rho_0 < r_1 < r_2 < \cdots < r_n < R$, such that $\theta(r)$ is continuous in $r_i < r < r_{i+1}$ and is discontinuous at r_i, such that

$$\theta(r_i - 0) = \theta(r_i) < \theta(r_i + 0), \quad D_{r_i} \subset D_{r_i+0} \quad . \tag{2}$$

Let $u(z) = u_R(z)$ be the generalized sequence solution of the Dirichlet problem for D, with the boundary value $u(z) = 1$ on θ_R and $u(z) = 0$ on Γ. Then $u(z)$ is harmonic in D, such that $0 < u(z) < 1$ in D and takes the given boundary value except a set of logarithmic capacity zero. $u(z)$ is the harmonic measure of θ_R with respect to D. We put

$$m(r) = \frac{1}{2\pi}\int_{\theta_r} [u(re^{i\theta})]^2 d\theta, \quad (\rho_0 \leqq r \leqq R), \tag{3}$$

$$S(r) = \frac{1}{\pi}\iint_{D_r}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) dxdy, \quad (z = x + iy). \tag{4}$$

Then $0 \leqq m(r) \leqq 1$. We will prove

THEOREM 1. *$m(r)$ is an increasing function of r, such that*

$$m(r) - m(\rho) \geqq \int_\rho^r \frac{S(r)}{r} dr, \qquad (\rho_0 \leqq \rho < r \leqq R).$$

If Γ consists of a finite number of analytic curves, then $m(r)$ is a convex function of $\log r$ *in (r_i, r_{i+1}), such that*

$$m'(r) = S(r)/r, \qquad (r \neq r_i).$$

PROOF. First we suppose that Γ consists of a finite number of analytic curves. Then for $r \neq r_i$,

$$\frac{\partial m(r)}{\partial \log r} = \frac{1}{\pi}\int_{\theta_r} u \frac{\partial u}{\partial \log r} d\theta, \qquad (\rho_0 < r < R), \tag{5}$$

$$\frac{\partial^2 m(r)}{\partial \log r^2} = \frac{1}{\pi}\int_{\theta_r}\left(\left(\frac{\partial u}{\partial \log r}\right)^2 + u\frac{\partial^2 u}{\partial \log r^2}\right)d\theta \tag{6}$$

$$= \frac{1}{\pi}\int_{\theta_r}\left(\left(\frac{\partial u}{\partial \log r}\right)^2 - u\frac{\partial^2 u}{\partial \theta^2}\right)d\theta = \frac{1}{\pi}\int_{\theta_r}\left(\left(\frac{\partial u}{\partial \log r}\right)^2 + \left(\frac{\partial u}{\partial \theta}\right)^2\right)d\theta > 0,$$

so that $m(r)$ is a convex function of $\log r$ in (r_i, r_{i+1}).

Let $\Gamma_r + \theta_r$ be the whole boundary of D_r and ν be its outer normal and

ds be its line element, then since $u(z)=0$ on Γ_r, if we apply Green's formula for D_r, we have

$$r\,m'(r)=\frac{\partial m(r)}{\partial \log r}=\frac{1}{\pi}\int_{\theta_r} u\frac{\partial u}{\partial r}\,r d\theta=\frac{1}{\pi}\int_{\Gamma_r+\theta_r} u\frac{\partial u}{\partial \nu}\,ds$$
$$=\frac{1}{\pi}\iint_{D_r}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy=S(r),$$

or

(7) $$m'(r)=S(r)/r, \qquad (r\neq r_i),$$

so that

(8) $$m(r)-m(\rho)\geqq\int_\rho^r \frac{S(r)}{r}dr, \quad (\rho<r).$$

In the general case, we approximate D by a sequence of domains

(9) $$D^{(1)}\subset D^{(2)}\subset\cdots\subset D^{(n)}\rightarrow D,$$

where $D^{(n)}$ is bounded by a finite number of analytic curves and we define $u_n(z)$, $\theta_r^{(n)}$, $\theta_n(r)$, $m_n(r)$ for $D^{(n)}$, such that

(10) $$m_n(r)=\frac{1}{2\pi}\int_{\theta_r^{(n)}}[u_n(re^{i\theta})]^2 d\theta, \quad m(r)=\frac{1}{2\pi}\int_{\theta_r}[u(re^{i\theta})]^2 d\theta.$$

Then since

(11) $$u_1(z)<u_2(z)<\cdots<u_n(z)\rightarrow u(z)$$

uniformly in the wider sense in D and

(12) $$\theta_r^{(1)}\subset\theta_r^{(2)}\subset\cdots\subset\theta_r^{(n)}\rightarrow\theta_r,$$

we have by Lebesgue's theorem,

$$m_n(r)\rightarrow m(r), \quad (n\rightarrow\infty).$$

By (8),

(13) $$m_n(r)-m_n(\rho)\geqq\int_\rho^r\frac{S_n(r)}{r}dr, \quad (\rho<r).$$

where

$$S_n(r)=\frac{1}{\pi}\iint_{D_r^{(n)}}\left(\left(\frac{\partial u_n}{\partial x}\right)^2+\left(\frac{\partial u_n}{\partial y}\right)^2\right)dxdy.$$

Since $D_r^{(p)}\subset D_r^{(n)}$ for $p<n$,

(14) $$S_n(r)\geqq\frac{1}{\pi}\iint_{D_r^{(p)}}\left(\left(\frac{\partial u_n}{\partial x}\right)^2+\left(\frac{\partial u_n}{\partial y}\right)^2\right)dxdy, \quad (p<n)$$

and by (11), $\frac{\partial u_n}{\partial x}\rightarrow\frac{\partial u}{\partial x}$, $\frac{\partial u_n}{\partial y}\rightarrow\frac{\partial u}{\partial y}$ uniformly in $D_r^{(p)}$, so that from (14)

$$\lim_{n\to\infty} S_n(r)\geqq\frac{1}{\pi}\iint_{D_r^{(p)}}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)dxdy,$$

hence for $p\rightarrow\infty$,

$$(15) \qquad \lim_{n\to\infty} S_n(r) \geqq \frac{1}{\pi}\iint_{D_r}\left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right)drdy = S(r).$$

From (13), (15) and Fatou's lemma, we have

$$(16) \qquad m(r)-m(\rho) \geqq \int_\rho^r \frac{S(r)}{r}dr, \quad (\rho < r).$$

Hence Theorem 1 is proved.

REMARK. From (16),

$$(17) \qquad 1 \geqq m(R)-m(\rho_0) \geqq \int_{\rho_0}^R \frac{S(r)}{r}dr,$$

or

$$\int_{\rho_0}^R \frac{S(r)}{r}dr \leqq 1.$$

Hence $S(r) < \infty$ for $\rho_0 \leqq r < R$.

3. Ler $D = D_R$ be a domain in $|z| < R$ and we define $u(z) = u_R(z)$, and θ_r as in § 2.

We define $\overline{\theta}(r)$ $(\rho_0 < r < R)$ as follows. If a cirle $|z| = r$ meets Γ, then θ_r consists of at most a countable number of arcs $\{\theta_r^{(i)}\}$ of lengths $r\,\theta^{(i)}(r)$ on $|z| = $ r, then we put $\theta(r) = \operatorname*{Sup}_i \theta^{(i)}(r)$ and if $|z| = r$ does not meet Γ and is contained entirely in D, then we put $\overline{\theta}(r) = \infty$. Then we will prove

THEOREM. 2. (Main theorem). *For any* $0 < \alpha < 1$, $k > 1$,

$$u_R(z) \leqq C.\ \exp\left(-\pi\int_{k|z|}^{\alpha R}\frac{dr}{r\theta(r)}\right), \quad (\rho_0 \leqq k|z| < \alpha R),$$

where

$$C = C(\alpha, k) = \frac{k+1}{k-1}\sqrt{\frac{2e}{1-\alpha}}.$$

If $z = 0$ *belongs to* D,

$$u_R(0) \leqq \sqrt{\frac{2e}{1-\alpha}}\exp\left(-\pi\int_0^{\alpha R}\frac{dr}{r\theta(r)}\right), \qquad (0 < \alpha < 1).$$

PROOF. First we suppose that Γ consists of a finite number of analytic curves. Then from (5),

$$\left(\frac{\partial m(r)}{\partial \log r}\right)^2 \leqq \frac{1}{\pi^2}\int_{\theta_r} u^2 d\theta\int_{\theta_r}\left(\frac{\partial u}{\partial \log r}\right)^2 d\theta = \frac{2m(r)}{\pi}\int_{\theta_r}\left(\frac{\partial u}{\partial \log r}\right)^2 d\theta,$$

hence

$$(18) \qquad \frac{1}{\pi}\int_{\theta_r}\left(\frac{\partial u}{\partial \log r}\right)^2 d\theta \geqq \frac{1}{2m(r)}\left(\frac{\partial m(r)}{\partial \log r}\right)^2.$$

Suppose that $|z| = r$ meets Γ, then θ_r consists of a finite number of arcs

$\theta_r = \sum_i \theta_r^{(i)}$ of length $r\theta^{(i)}(r)$. Since $u(z) = 0$ at the both ends of $\theta_r^{(i)}$, by the well known inequality,

$$\int_{\theta_r^{(i)}} \left(\frac{\partial u}{\partial \theta}\right)^2 d\theta \geqq \frac{\pi^2}{\theta^{(i)}(r)^2} \int_{\theta_r^{(i)}} u^2 d\theta \geqq \frac{\pi^2}{\bar{\theta}(r)^2} \int_{\theta_r^{(i)}} u^2 d\theta,$$

hences summing up for i,

$$\int_{\theta_r} \left(\frac{\partial u}{\partial \theta}\right)^2 d\theta \geqq \frac{\pi^2}{\bar{\theta}(r)^2} \int_{\theta_r} u^2 d\theta = \frac{2\pi^3}{\bar{\theta}(r)^2} m(r),$$

or

$$\frac{1}{\pi} \int_{\theta_r} \left(\frac{\partial u}{\partial \theta}\right)^2 d\theta \geqq \frac{2\pi^2}{\bar{\theta}(r)^2} m(r). \tag{19}$$

From (6), (18), (19), we have

$$\frac{\partial^2 m(r)}{\partial \log r^2} \geqq \frac{1}{2m(r)} \left(\frac{\partial m(r)}{\partial \log r}\right)^2 + \frac{2\pi^2}{\bar{\theta}(r)^2} m(r). \tag{20}$$

If $|z| = r$ does not meet Γ, then similarly

$$\frac{\partial^2 m(r)}{\partial \log r^2} \geqq \frac{1}{2m(r)} \left(\frac{\partial m(r)}{\partial \log r}\right)^2, \tag{21}$$

so that in any case, we have

$$\frac{\partial^2 m(r)}{\partial \log r^2} \geqq \frac{1}{2m(r)} \left(\frac{\partial m(r)}{\partial \log r}\right)^2 + \frac{1}{2}\left(\frac{2\pi}{\bar{\theta}(r)}\right)^2 m(r). \tag{22}$$

If we put

$$t = \log r, \quad \mu(t) = m(r), \quad Q(t) = 2\pi/\bar{\theta}(r), \tag{23}$$

then (22) becomes

$$\mu''(t) \geqq \frac{1}{2\mu(t)} \mu'(t)^2 + \frac{1}{2} Q(t)^2 \mu(t), \tag{24}$$

or

$$\lambda'(t)^2 + 2\lambda''(t) \geqq Q(t)^2, \qquad (\lambda(t) = \log \mu(t)),$$

so that

$$(\lambda'(t) + \lambda''(t)/\lambda'(t))^2 \geqq \lambda'(t)^2 + 2\lambda''(t) \geqq Q(t)^2.$$

Since by Theorem 1,

$$\lambda'(t) + \lambda''(t)/\lambda'(t) = \mu''(t)/\mu'(t) > 0 \text{ in } (t_i,\ t_{i+1}),\ (t_i = \log r_i),$$

we have

$$\mu''(t)/\mu'(t) \geqq Q(t) \text{ in } (t_i,\ t_{i+1}).$$

Since $\mu'(t_i - 0) < \mu'(t_i + 0)$ from $m'(r) = S(r)/r$ $(r \neq r_i)$, we have

$$\log \mu'(t) - \log \mu'(\tau) \geqq \int_\tau^t \frac{\mu''(t)}{\mu'(t)} dt \geqq \int_\tau^t Q(t)\, dt, \quad (\tau < t),$$

or

$$\mu'(t) \geqq \mu'(\tau) \exp\left(\int_\tau^t Q(t)\, dt\right), \qquad (\tau < t).$$

If we put

(25) $$T = \log R,\ t = \log r,\ \tau = \log \rho,$$
we have for any $0 < \alpha < 1$, $(\rho \leqq \alpha R)$,

$$1 \geqq \mu(T) - \mu(\tau) \geqq \int_\tau^T \mu'(t)\,dt \geqq \mu'(\tau)\int_\tau^T dt \exp\left(\int_\tau^t Q(t)\,dt\right)$$

$$= \mu'(\tau)\int_\rho^R \frac{dr}{r}\exp\left(2\pi\int_\rho^r \frac{dr}{r\bar\theta(r)}\right) \geqq \mu'(\tau)\int_{\alpha R}^R \frac{dr}{r}\exp\left(2\pi\int_\rho^r \frac{dr}{r\bar\theta(r)}\right)$$

$$\geqq \mu'(\tau)\log\frac{1}{\alpha}\exp\left(2\pi\int_\rho^{\alpha R}\frac{dr}{r\theta(r)}\right)$$

$$\geqq \mu'(\tau)(1-\alpha)\exp\left(2\pi\int_\rho^{\alpha R}\frac{dr}{r\theta(r)}\right).$$

Hence if we write t instead of τ, we have

(26) $$\mu'(t) \leqq \frac{1}{1-\alpha}\exp\left(-2\pi\int_r^{\alpha R}\frac{dr}{r\bar\theta(r)}\right), \qquad (0 < \alpha < 1).$$

From (24),

$$\mu''(t) \geqq \frac{1}{2}Q(t)^2\mu(t)$$

and since $(\bar\theta(r))^2 \leqq 2\pi\bar\theta(r)$, we have for $\tau < t$,

(27) $$\mu'(t) \geqq \mu'(t) - \mu'(\tau) \geqq \int_\tau^t \mu''(t)dt \geqq \frac{1}{2}\int_\tau^t Q(t)^2\mu(t)dt$$

$$\geqq \frac{\mu(\tau)}{2}\int_\tau^t Q(t)^2dt = 2\pi^2 m(\rho)\int_\rho^r \frac{dr}{r\bar\theta(r)^2} \geqq \pi\, m(\rho)\int_\rho^r \frac{dr}{r\bar\theta(r)}.$$

Hence from (26),

$$\frac{1}{1-\alpha}\exp\left(-2\pi\int_r^{\alpha R}\frac{dr}{r\bar\theta(r)}\right) \geqq \pi m(\rho)\int_\rho^r \frac{dr}{r\bar\theta(r)},$$

$$\frac{1}{1-\alpha}\exp\left(2\pi\int_\rho^r \frac{dr}{r\bar\theta(x)} - 2\pi\int_\rho^{\alpha R}\frac{dr}{r\theta(r)}\right) \geqq \frac{m(\rho)}{2}2\pi\int_\rho^r \frac{dr}{r\bar\theta(r)}.$$

If $2\pi\int_\rho^{\alpha R}\frac{dr}{r\theta(r)} > 1$, then we choose r $(\rho < r < \alpha R)$, so that $2\pi\int_\rho^r \frac{dr}{r\bar\theta(r)} = 1$, then

$$\frac{e}{1-\alpha}\exp\left(-2\pi\int_\rho^{\alpha R}\frac{dr}{r\theta(r)}\right) \geqq \frac{1}{2}m(\rho),$$

or

(28) $$m(\rho) \leqq \frac{2e}{1-\alpha}\exp\left(-2\pi\int_\rho^{\alpha R}\frac{dr}{r\bar\theta(r)}\right).$$

If $2\pi\int_\rho^{\alpha R}\frac{dr}{r\bar\theta(r)} \leqq 1$, then (28) holds, since $m(\rho) \leqq 1$, $2/(1-\alpha) > 1$.

Hence (28) holds in general. If $z=0$ belongs to D, then

(29) $$u(0)=\sqrt{m(0)}\leqq\sqrt{\frac{2e}{1-\alpha}}\exp\left(-\pi\int_0^{\alpha R}\frac{dr}{r\bar{\theta}(r)}\right),\quad (0<\alpha<1).$$

To obtain the majorant of $u(z)$, we add the inside of $|z|=\rho$ to D and D_0 be the thus enlarged domain, then $|z|<\rho$ belong to D_0. We define $u_0(z)$, $\bar{\theta}_0(r)$ for D_0, then $u(z)\leqq u_0(z)$ and $\bar{\theta}_0(r)=\infty$, $(0\leqq r<\rho)$, $\bar{\theta}_0(r)=\bar{\theta}(r)$ $(\rho\leqq r\leqq R)$, so that by (29)

$$u_0(0)\leqq\sqrt{\frac{2e}{1-\alpha}}\exp\left(-\pi\int_\rho^{\alpha R}\frac{dr}{r\bar{\theta}(r)}\right).$$

Since $u_0(z)>0$ in $|z|<\rho$, we have for $|z|=\lambda\rho$, $(0<\lambda<1)$,

$$u_0(z)\leqq\frac{1+\lambda}{1-\lambda}u_0(0)\leqq\frac{1+\lambda}{1-\lambda}\sqrt{\frac{2e}{1-\alpha}}\exp\left(-\pi\int_\rho^{\alpha R}\frac{dr}{r\bar{\theta}(r)}\right),$$

so that

$$u(z)\leqq u_0(z)\leqq\frac{1+\lambda}{1-\lambda}\sqrt{\frac{2e}{1-\alpha}}\exp\left(-\pi\int_\rho^{\alpha R}\frac{dr}{r\bar{\theta}(r)}\right).$$

Hence if we put $k=1/\lambda>1$, we have $\rho=k|z|$, so that

(30) $$u(z)\leqq C\exp\left(-\pi\int_{k|z|}^{\alpha R}\frac{dr}{r\bar{\theta}(r)}\right),\quad (0<\alpha<1, k>1),$$

where

$$C=C(\alpha,k)=\frac{k+1}{k-1}\sqrt{\frac{2e}{1-\alpha}}\,.$$

Hence the theorem is proved, when Γ consists of a finite number of analytic curves. In the general case, we approximate D by a sequence of domains $D^{(1)}\subset D^{(2)}\subset\cdots\subset D^{(n)}\to D$, where $D^{(n)}$ is bounded by a finite unmber of analytic curves. Let $u_n(z)$, $\bar{\theta}_n(r)$, be defined for $D^{(n)}$, then

$$u_1(z)<u_2(z)<\cdots<u_n(z)\to u(z)$$

uniformly in the wider sense in D and it is easily seen that

$$\bar{\theta}_1(r)\leqq\bar{\theta}_2(r)\leqq\cdots\leqq\bar{\theta}_n(r)\to\bar{\theta}(r).$$

Since by (30),

$$u_n(z)\leqq C.\ \exp\left(-\pi\int_{k|z|}^{\alpha R}\frac{dx}{r\bar{\theta}_n(r)}\right),$$

we have by Lebesgue's theorem,

$$u(z)\leqq C.\ \exp\left(-\pi\int_{k|z|}^{\alpha R}\frac{dr}{r\bar{\theta}(r)}\right),\quad (0<\alpha<1,\ k>1).$$

Hence the theorem is proved in the general case.

4. As an application of Theorem 2, we will prove the following extension of Arima's theorem mentioned in § 1.

THEOREM 3. *Let D be an infinite domain on the z-plane and w(z) be one-valued and regular in D and on its boundary* Γ, *such that*

$$|w(z)| \leqq \lambda \text{ on } \Gamma, \text{ and } M(r) = \operatorname*{Max}_{\theta_r} |w(z)|.$$

If there exists a point z_0 $(|z_0| = r_0)$ *in* D, *such that* $|w(z_0)| > \lambda$, *then*

$$\log\log\frac{M(r)}{\lambda} \geqq \pi\int_{r_0}^{\alpha r}\frac{dr}{r\bar{\theta}(r)} - \text{const.}, \quad (0<\alpha<1).$$

PROOF. Let D_r be defined as before and $u_r(z)$ be defined for D_r, then by Theorem 2,

$$u_r(z) \leqq C\exp\left(-\pi\int_{k|z|}^{\alpha r}\frac{dr}{r\bar{\theta}(r)}\right), \quad (0<\alpha<1,\ k>1).$$

Since $\log^+(|w(z)|/\lambda)$ is subharmonic and vanishes on Γ, we have

$$\log^+(|w(z)|/\lambda) \leqq \log(M(r)/\lambda)\, u_r(z) \text{ in } D_r.$$

Hence

$$0 < \log^+\frac{|w(z_0)|}{\lambda} \leqq \log\frac{M(r)}{\lambda}u_r(z_0) \leqq C.\ \log\frac{M(r)}{\lambda}\exp\left(-\pi\int_{kr_0}^{\alpha r}\frac{dr}{r\bar{\theta}(r)}\right),$$

so that

$$\log\log\frac{M(r)}{\lambda} \geqq \pi\int_{r_0}^{\alpha r}\frac{dr}{r\bar{\theta}(r)} - \text{const.}, \quad (0<\alpha<1), \ q.e.d.$$

From Theorem 3, we have the following extension of the classical theorem of Lindelöf-Phragmén :

THEOREM 4. *Let D be an infinite domain on the z-plane and w(z) be one-valued and regular in D and on its boundary* Γ, *such that*

$$|w(z)| \leqq \lambda \text{ on } \Gamma \text{ and } M(r) = \operatorname*{Max}_{\theta_r} |w(z)|.$$

If

$$\varlimsup_{r\to\infty}\left(\pi\int_{r_0}^{\alpha r}\frac{dr}{r\bar{\theta}(r)} - \log\log\frac{M(r)}{\lambda}\right) = \infty, \quad (0<\alpha<1),$$

then $|w(z)| \leqq \lambda$ *in D.*

5. Let D be a domain, which lies in $|z| < R$ and $z=0$ belongs to its boundary Γ. As well known, $z=0$ is a regular point for the Dirichlet problem, if and only if there exists a barrier $w_\rho(z)$ for any neighbourhood U_ρ of $z=0$, where a barrier is, by definition, a positive superharmonic function in *D*, such that $\lim_{z\to0} w_\rho(z) = 0$ uniformly in *D* and $w_\rho(z) \geqq a_\rho > 0$ for $|z| \geqq \rho$, where a_ρ depends on U_ρ. Let $u_\rho(z)$ be defined for D_ρ as Theorem 2 and let

$$m_\rho(r) = \frac{1}{2\pi}\int_{\theta_r}[u_\rho(re^{i\theta})]^2 d\theta, \quad (r<\rho).$$

Then by Theorem 1, $m_\rho(r)$ is an increasing function of r, so that

$$\lim_{r\to 0} m_\rho(r) = A_\rho \geqq 0$$

exists. We will prove

LEMMA. *The necessary and sufficient condition, that $z=0$ is a regular point is that*

$$A_\rho = 0$$

for any $\rho > 0$.

PROOF. Suppose that $z=0$ is a regular point, then $\lim_{z\to 0} u_\rho(z) = 0$, so that $A_\rho = 0$ for any $\rho > 0$.

Next suppose that $A_\rho = 0$ for any $\rho > 0$. Then

$$m_\rho(r) < \varepsilon^2 \text{ for } r \leqq r(\varepsilon) < \rho.$$

Let $U_\rho(z)$ be a harmonic function in $|z| < r$, such that $U_\rho(z) = u_\rho(z)$ on θ_r and $U_\rho(z) = 0$ on the complementary arc of θ_r on $|z| = r$. Then

(31) $$u_\rho(z) \leqq U_\rho(z) \text{ in } D_r.$$

Since $U_\rho(z) > 0$ in $|z| < r$, we have for $|z| \leqq kr\,(0 < k < 1)$,

$$U_\rho(z) \leqq \frac{1+k}{1-k} U_\rho(0) \leqq \frac{1+k}{1-k}\left(\frac{1}{2\pi}\int_0^{2\pi} U_\rho(re^{i\theta})^2 d\theta\right)^{1/2}$$

$$= \frac{1+k}{1-k}\sqrt{m_\rho(r)} \leqq \frac{1+k}{1-k}\varepsilon,$$

so that from (31),

$$u_\rho(z) \leqq \frac{1+k}{1-k}\varepsilon, \quad (|z| \leqq kr).$$

Hence $\lim_{z\to 0} u_\rho(z) = 0$.

We define $w_\rho(z)$ as follows.

$$w_\rho(z) = u_\rho(z) \text{ in } D_\rho, \quad w_\rho(z) = 1 \text{ in } D - D_\rho.$$

Then $w_\rho(z) \geqq 0$ is superharmonic in D and $w_\rho(z) = 1$ on θ_ρ and $\lim_{z\to 0} w_\rho(z) = 0$, so that $w_\rho(z)$ is a barrier, hence $z = 0$ is a regular point.

We will prove

THEOREM 5. *If* $\displaystyle\int_0^R \frac{dr}{r\theta(r)} = \infty$, *then $z = 0$ is a regular point.*

Let E be the set of r, such that $|z| = r$ meets the boundary Γ of D, then Beurling[4] proved that if $\displaystyle\int_E d\log r = \infty$, then $z = 0$ is a regular point. This is a special case of our theorem.

PROOF. Let $u_\rho(z)$ be defined for D_ρ as Theorem 2, then

$$u_\rho(z) \leqq C.\exp\left(-\pi\int_{k|z|}^{\alpha\rho} \frac{dr}{r\theta(r)}\right), \quad (0 < \alpha < 1,\ k > 1).$$

4) A. BEURLING, Thèse. Upsala (1933).

Since $\int_0^R \frac{dr}{r\bar{\theta}(r)} = \infty$, we have $\lim_{z\to 0} u_\rho(z) = 0$, so that $A_\rho = 0$ for any $\rho > 0$, hence by the lemma, $z = 0$ is a regular point, q. e. d.

We will prove a more general theorem:

THOREM. 6. *If* $\int_0^R \frac{dr}{r\bar{\theta}(r)^2} = \infty$, *or if* $\int_0^R \frac{dr}{r\bar{\theta}(r)^2} < \infty$ *and* $\lim_{r\to 0} \log \frac{1}{r} \int_0^r \frac{dr}{r\bar{\theta}(r)^2} > 0$, *then* $z = 0$ *is a regular point.*

PROOF. (i) First we suppose that

$$\int_0^R \frac{dr}{r\bar{\theta}(r)^2} = \infty. \tag{32}$$

We approximate D by a sequence of domains $D^{(1)} \subset D^{(2)} \subset \cdots \subset D^{(n)} \to D$, where $D^{(n)}$ is bounded by a finite number of analytic curves and let $u_n(z)$, $m_n(r) = \mu_n(t)$ $(t = \log r)$, $\theta_n(r)$, $Q_n(t)$ be defined for $D^{(n)}$. Then by (27),

$$\mu_n'(t) \geqq \frac{1}{2}\int_\tau^t Q_n^2(t)\mu_n(t)dt \geqq \frac{\mu_n(\tau)}{2}\int_\tau^t Q_n(t)^2 dt, \quad (\tau < t), \tag{33}$$

so that

$$1 \geqq \mu_n(T) - \mu_n(\tau) \geqq \int_\tau^T \mu_n'(t)dt \geqq \frac{\mu_n(\tau)}{2}\int_\tau^T dt\int_\tau^t Q_n(t)^2 dt, \quad (T = \log R).$$

Since $Q_n(t) \to Q(t)$ $(n \to \infty)$ by decreasing, we have by Lebesgue's theorem,

$$1 \geqq \frac{\mu(\tau)}{2}\int_\tau^T dt\int_\tau^t Q(t)^2 dt = 2\pi^2 m(\rho)\int_\rho^R \frac{dr}{r}\int_\rho^r \frac{dr}{r\bar{\theta}(r)^2}$$

$$\geqq 2\pi^2 m(\rho)\int_{R/2}^R \frac{dr}{r}\int_\rho^r \frac{dr}{r\bar{\theta}(r)^2} \geqq 2\pi^2 m(\rho)\log 2\int_\rho^{R/2} \frac{dr}{r\bar{\theta}(r)^2} \quad (t = \log r, \tau = \log\rho).$$

Hence by (32), we have $\lim_{r\to 0} m(r) = 0$. Similarly we have $\lim_{r\to 0} m_\rho(r) = A_\rho = 0$ for any $\rho > 0$, so that by the lemma, $z = 0$ is a regular point.

(ii) Next suppose that

$$\int_0^R \frac{dr}{r\bar{\theta}(r)^2} < \infty,$$

then making $\tau \to -\infty$ in (33),

$$\mu_n'(t) \geqq \frac{1}{2}\int_{-\infty}^t Q_n(t)^2\mu_n(t)dt,$$

so that

$$1 \geqq \mu_n(T) - \mu_n(\tau) \geqq \int_\tau^T \mu_n'(t)dt \geqq \frac{1}{2}\int_\tau^T dt\int_{-\infty}^t Q_n(t)^2\mu_n(t)\,dt,$$

hence for $\tau \to -\infty$,

$$(35) \qquad 1 \geqq \frac{1}{2}\int_{-\infty}^{T} dt \int_{-\infty}^{t} Q_n(t)^2 \mu_n(t) dt = 2\pi^2 \int_0^R \frac{dr}{r} \int^{r} \frac{m_n(r)}{r\theta_n(r)^2} dr.$$

Now $m_n(r) \to m(r)$ $(n \to \infty)$ and $m_n(r)$ is uniformly bounded $(0 \leqq m_n(r) \leqq 1)$ and $1/(r\bar{\theta}_n(r)^2)$ decreases with n, so that we can easily prove that we may make $n \to \infty$ under the integral sign of the right hand side of (35), so that

$$1 \geqq 2\pi^2 \int_0^R \frac{dr}{r} \int_0^{r} \frac{m(r)}{r\bar{\theta}(r)^2} dr.$$

Similarly we have for any $\rho < R$,

$$(36) \qquad 1 \geqq 2\pi^2 \int_0^{\rho} \frac{dr}{r} \int_0^{r} \frac{m_\rho(r)}{r\bar{\theta}(r)^2} dr.$$

Suppose that $z = 0$ is an irregular point, then by the lemma, for some $\rho > 0$, $\lim_{r \to 0} m_\rho(r) = A_\rho > 0$. Since $m_\rho(r)$ is an increasing function of r,

$$m_\rho(r) \geqq A_\rho > 0 \text{ for } 0 < r \leqq \rho.$$

Hence from (36),

$$\infty > \frac{1}{2\pi^2 A_\rho} \geqq \int_0^{\rho} \frac{dr}{r} \int_0^{r} \frac{dr}{r\bar{\theta}(r)^2},$$

so that

$$\frac{1}{2} \log \frac{1}{r} \int_0^{r} \frac{dr}{r\bar{\theta}(r)^2} \leqq \int_r^{\sqrt{r}} \frac{dt}{t} \int_t^{t} \frac{dt}{t\theta(t)^2} < \varepsilon, \quad (r \leqq r_0(\varepsilon)),$$

or

$$\lim_{r \to 0} \log \frac{1}{r} \int_0^{r} \frac{dr}{r\bar{\theta}(r)^2} = 0.$$

Hence if

$$\overline{\lim_{r \to 0}} \log \frac{1}{r} \int_0^{r} \frac{dr}{r\bar{\theta}(r)^2} > 0,$$

then $z = 0$ is a regular point.

MATHEMATICAL INSTITUTE, TOKYO UNIVERSITY.