ON THE FUNCTION $t-[t]-\frac{1}{2}$

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Let a be a positive integer $\geqq 2$, and put

$$f(t)=t-[t]-\frac{1}{2},$$

where $[t]$ denotes the largest integer $\leqq t$. We shall consider the expression

$$\sum_{l=0}^{N} f(a^l t) \qquad (N=1,2\cdots).$$

If $a=2$, A. Khintchine [2] proved that

$$(1) \qquad \limsup_{N\to\infty}\Big(\sum_{l=0}^{N} f(2^l t)\Big)\Big/(N\log\log N)^{1/2}=1/\sqrt{2}$$

for almost all t. On the other hand J. F. Koksma [3] proved that for a positive integer $a\geqq 2$

$$\Big(\sum_{l=0}^{N} f(a^l t)\Big)\Big/(N^{3/2}(\varphi(N))^{1/3})=o(1)$$

and

$$\liminf_{N\to\infty}\Big(\sum_{l=0}^{N} f(a^l t)\Big)\Big/(N^{1/2}\psi(N))=0$$

for almost all t, where $\varphi(n)$ is any given positive non-decreasing function of integer $n\geqq n_0>0$ such that

$$\sum\frac{1}{n\varphi(n)}<\infty \quad\text{and}\quad \varphi(n+1)\leqq(1+K/n)\,\varphi(n) \qquad (n\geqq n_0),$$

K being a conveniently chosen positive constant, and $\psi(n)$ denotes any given positive function of the integer $n\geqq n_0$ such that $\psi(n)\to 0$ as $n\to\infty$.

The purpose of this note is to furnish more precise results than these estimations of Koksma, and a related theorem.

1. THEOREM 1. *If a is a positive integer $\geqq 2$, then for almost all t, we have*

$$(2) \qquad \limsup_{N\to\infty}\Big(\sum_{l=0}^{N} f(a^l t)\Big)\Big/(N\log\log N)^{1/2}=\Big(\frac{a+1}{6(a-1)}\Big)^{1/2},$$

$$(3) \qquad \liminf_{N\to\infty}\Big|\sum_{l=0}^{N} f(a^l t)\Big|\leqq\frac{1}{a-1}\Big(|f(t)|+\frac{1}{2}\Big).$$

PROOF. If we consider the decimal representation of a real number t, in the scale of a, then every digit can be regarded as a function of t. Hence we put

$$t=[t]+\frac{\varepsilon_1(t)}{a}+\frac{\varepsilon_2(t)}{a^2}+\cdots.$$

Every $\varepsilon_k(t)$ has its value region 0, 1, 2, $\cdots$, $a-1$; and clearly $\{\varepsilon_k(t)\}$ forms an indepenent system in the sense of M. Kac and H. Steinhaus. If we put

(4) $$\delta_k(t)=\varepsilon_k(t)-(a-1)/2 \qquad (k=1,2,\cdots),$$

then $\{\delta_k(t)\}$ is also an independent system and has the following properties:

(5) $$|\delta_k(t)|\leqq(a-1)/2 \qquad \text{for all } t \text{ and } k,$$

(6) $$\int_0^1 \delta_k(t)dt=0 \qquad \text{for all } k,$$

(7) $$\int_0^1 \delta_k^2(t)dt=(a^2-1)/12 \qquad \text{for all } k.$$

In fact, by (4), $|\delta_k(t)|\leqq(a-1)-(a-1)/2=(a-1)/2$;

$$\int_0^1 \delta_k(t)dt=\int_0^1 \varepsilon_k(t)dt-\frac{a-1}{2}=\frac{1}{a}\sum_{k=1}^{a-1}k-\frac{a-1}{2}=0;$$

and

$$\int_0^1 \delta_k^2(t)dt=\int_0^1\left(\varepsilon_k(t)-\frac{a-1}{2}\right)^2 dt$$

$$=\int_0^1 \varepsilon_k^2(t)dt-(a-1)\int_0^1 \varepsilon_k(t)dt+\frac{(a-1)^2}{4}$$

$$=\frac{1}{a}\sum_{k=1}^{a-1}k^2-\frac{a-1}{a}\sum_{k=1}^{a-1}k+\frac{(a-1)^2}{4}=\frac{a^2-1}{12}.$$

By the law of the iterated logarithm of Khintchine and Kolmogoroff [4], we see immediately from the above properties that

(8) $$\limsup_{N\to\infty}\left(\sum_{k=1}^{N}\delta_k(t)\right)\Big/(N\log\log N)^{1/2}=\left(\frac{a^2-1}{6}\right)^{1/2}$$

for almost all t.

Now we have

$$f(t)=t-[t]-\frac{1}{2}=\sum_{k=1}^{\infty}\frac{\varepsilon_k(t)}{a^k}-\frac{1}{2}$$

$$=\sum_{k=1}^{\infty}\frac{\delta_k(t)}{a^k}+\frac{a-1}{2}\sum_{k=1}^{\infty}\frac{1}{a^k}-\frac{1}{2}$$

$$=\sum_{k=1}^{\infty}\frac{\delta_k(t)}{a^k},$$

and obviously $\delta_{k+l}(t)=\delta_k(a^l t)$ for every positive integer k and l, we have then

$$
\begin{aligned}
\sum_{l=1}^{N} f(a^l t) &= \sum_{l=1}^{N}\sum_{k=1}^{\infty} \frac{\delta_k(a^l t)}{a_k} = \sum_{l=1}^{N}\sum_{k=1}^{\infty} \frac{\delta_{k+l}(t)}{a^k} \\
&= \sum_{l=1}^{N}\sum_{k=l}^{\infty} \frac{\delta_{k+1}(t)}{a^{k-l+1}} = \left(\sum_{k=1}^{N}\sum_{l=1}^{k} + \sum_{k=N+1}^{\infty}\sum_{l=1}^{N}\right) \frac{\delta_{k+1}(t)}{a^{k+1}}\, a^l \\
&= \frac{1}{a-1}\sum_{k=1}^{N} \delta_{k+1}(t) - \frac{a}{a-1}\sum_{k=1}^{\infty} \frac{\delta_{k+1}(t)}{a^{k+1}} + \frac{a^{N+1}}{a-1}\sum_{k=N+1}^{\infty} \frac{\delta_{k+1}(t)}{a^{k+1}} \\
&= \frac{1}{a-1}\sum_{k=1}^{N+1} \delta_k(t) - \frac{a}{a-1} f(t) + \frac{a^{N+1}}{a-1}\sum_{k=N+2}^{\infty} \frac{\delta_k(t)}{a^k}\,,
\end{aligned}
$$

or we may write

$$
\begin{aligned}
\sum_{l=0}^{N} f(a^l t) &= \frac{1}{a-1}\sum_{k=1}^{N+1} \delta_k(t) - \frac{1}{a-1} f(t) + \frac{a^{N+1}}{a-1}\sum_{k=N+2}^{\infty} \frac{\delta_k(t)}{a^k} \\
&= P_N(t) - \frac{1}{a-1} f(t) + Q_n(t)
\end{aligned}
\tag{9}
$$

say. Clearly we have

$$
\left| \frac{1}{a-1} f(t) \right| \leqq \frac{1}{2(a-1)} \qquad \text{for all } t,
$$

and by (5)

$$
|Q_n(t)| \leqq \frac{a^{N+1}}{a-1}\, \frac{a-1}{2} \sum_{k=N+2}^{\infty} \frac{1}{a^k} = \frac{1}{2(a-1)}
\tag{10}
$$

for all t. Hence from (9)

$$
\sum_{l=0}^{N} f(a^l t) = \frac{1}{a-1}\sum_{k=1}^{N+1} \delta_k(t) + O(1)\,.
\tag{11}
$$

Combining (8) and (11) we have the relation (2) for almost all t.

On the other hand as we see easily (or see e.g. [1])

$$
\liminf_{N\to\infty} \left| \sum_{k=1}^{N+1} \delta_k(t) \right| = 0
$$

for almost all t; and then we get by (9) and (10)

$$
\begin{aligned}
\liminf_{N\to\infty} \left| \sum_{l=0}^{N} f(a^l t) \right| &\leqq \frac{1}{a-1}\left\{ \liminf_{N\to\infty} \sum_{k=1}^{N+1} \delta_k(t) + |f(t)| + \frac{1}{2} \right\} \\
&= \frac{1}{a-1}\left(|f(t)| + \frac{1}{2} \right)
\end{aligned}
$$

for almost all t, and (3) is proved.

2. We shall add a category theorem.

THEOREM 2. *Let $\chi(N)$ be a function defined for every positive integer N such that $\chi(N) \to 0$ as $N \to \infty$. Then for every t, except perhaps for a set of the first category, we have*

$$
\limsup_{N\to\infty} \left| \left(\sum_{l=0}^{N} f(a^l t) \right) \Big/ (N\chi(N)) \right| = +\infty.
\tag{12}
$$

PROOF. In virtue of the relation (11) we may replace $f(a^l t)$ in (12) by $\delta_l(t)$, for we may suppose that $N\chi(N)\to\infty$ as $N\to\infty$. Put

$$\sigma_N(t)=\Big(\sum_{l=0}^{N}\delta_l(t)\Big)\Big/(N\chi(N)), \qquad (N=1,2,\cdots).$$

We may find easily a sequence of positive integers $\{N_i\}$ such that $N_1<N_2<\cdots\to\infty$ and $2i<N_i(1-\chi^{1/2}(N_i))$ $(i=1,2,\cdots)$, that is,

(13) $$N_i\chi(N_i)<(N_i-2i)\chi^{1/2}(N_i) \qquad (i=1,2,\cdots).$$

Denote by A the set of all $t\in(0,1)$ which are not of the form m/a^k (m,k being integers). For $p=1,2,\cdots$, let E_p be the set of all $t\in A$ for which $|\sigma_N(t)|\leqq p$ for all N; and let E be that of all $t\in A$ for which $\sigma_N(t)$ is bounded in N. Then clearly $E=\bigcup_p E_p$. If E is of the second category, so is the set E_{p_0} for some p_0. And E_{p_0} is closed in A in virtue of the continuity of $\delta_k(t)$, $t\in A$. Hence E_{p_0} contains an interval I of the space A. Let $t_0\in A$ be the point whose N_i-th digit in the decimal representation in the scale of a, is 0 $(i=1,2,\cdots)$ and other digits are all $a-1$. Then by (4) $\delta_k(t_0)=-(a-1)/2$ $(k=N_1,N_2,\cdots)$ and $\delta_k(t_0)=(a-1)/2$ $(k\neq N_i;\ i=1,2,\cdots)$; and we get by (13)

$$|\sigma_{N_i}(t_0)|=\Big|\sum_{l=0}^{N_i}\delta_l(t_0)\Big|\Big/(N_i\chi(N_i))$$
$$\geqq\frac{a-1}{2}(N_i-2i)/((N_i-2i)\chi^{1/2}(N_i)) \qquad \text{as } i\to\infty.$$

Since there is a point $t_1\in I$ such that the difference t_0-t_1 is a-adically rational, we see easily that

$$\lim_{i\to\infty}|\sigma_{N_i}(t_1)|=\lim_{i\to\infty}|\sigma_{N_i}(t_0)|=\infty,$$

which contradicts the fact $t_1\in I\subset E_{p_0}$. Hence the set E is of the first category, q. e. d.

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