A NOTE ON ABSOLUTE NEIGHBORHOOD RETRACTS

HIROSHI NOGUCHI

(Received, October 15, 1951)

1. The concept of an absolute neighborhood retract was defined at first by K. Borsuk [2][1] for compact metric spaces. Next C. Kuratowski [5] extended it for separable metric spaces[2] as follows: A space A is said to be an absolute neighborhood retract if A is a neighborhood retract of every space which contains it and in which A is closed. But S.-T. Hu [4] extended the definition of K. Borsuk as follows: A space A is said to be an absolute neighborhood retract if A is a neighborhood retract of every space which contains it. We shall call an ANR which was defined by C. Kuratowski as "an ANR in the weak sense (W-ANR)" and an ANR which was defined by S.-T. Hu as "an ANR in the strong sense (S-ANR)". In this note we shall show that, for locally compact spaces, W-ANR and S-ANR are equivalent (Theorem 2).

2. Let us denote by Q_ω a fundamental cube in Hilbert space.

THEOREM 1. *A separable metric space, A, is an absolute neighborhood retract in the strong sense if, and only if, $f(A)$ is a neighborhood ratract of Q_ω for every homeomorphism f of A in Q_ω.*

PROOF. Necessity: It is trivial.

Sufficiency: Let Z be a space which contains A. By Urysohn's theorem [1] Z is imbedded in Q_ω by a homeomorphism f of Z in Q_ω. Then, by our assumption, $f(A)$ is a neighborhood retract of Q_ω i. e., there exists a neighborhood V of $f(A)$ and a retraction θ of V onto $f(A)$. Let $U=f^{-1}(f(Z)\cap V)$. Since U is open in Z and contains A, U is a neighborhood of A in Z. Let $r=f^{-1}\theta f|U$[3]. Since f is a homeomorphism of Z onto $f(Z)$, r is a retraction of U onto A. Q.E.D.

3. When A is contained in Z, we denote by $\bar{A}$ the closure of A in Z and define $\Gamma_z[A]=\{x;x\in\bar{A},\ x\bar{\in} A\}$.

LEMMA 1. *If, for every homeomorphism f of an W-ANR A in Q_ω, $\Gamma Q_\omega[f(A)]$ is closed in Q_ω, then A is an S-ANR.*

PROOF. Since A is a W-ANR, by R.H. Fox [3] $f(A)\times[0]$ is a neighborhood retract of $f(A)\times[0]+Q_\omega\times(0, 1]$, i. e., there exists a neighborhood V of $f(A)\times[0]$ in $f(A)\times[0]+Q_\omega\times(0, 1]$ and a retraction θ of V onto

1) Numbers in brackets refer to the bibliography at the end of this paper.

2) In the remaining part of this paper, "space" means always a separable metric space.

3) $f|U$ means the partial mapping of f on U.

$f(A) \times [0]$. Let $\rho(x) = \min(1, \text{distance}(x, \overline{f(A)}))$ for every $x \in Q_\omega - \Gamma_{Q_\omega}[f(A)]$ and let $F(x) = (x, \rho(x))$, so that F is a mapping defined on $Q - \Gamma_{Q_\omega}[f(A)]$ with values in $f(A) \times [0] + Q_\omega \times (0, 1]$ which has the property $F(Q_\omega - \overline{f(A)}) \subset Q_\omega \times (0, 1]$. Let U be $F^{-1}\{F(Q_\omega - \Gamma_{Q_\omega}[f(A)]) \cap V\}$, then U is open in $Q_\omega - \Gamma_{Q_\omega}[f(A)]$ and open also in Q_ω, because $\Gamma[_{Q_\omega} f(A)]$ is closed in Q_ω. Therefore U is a neighborhood of $f(A)$. Let $r = F^{-1}\theta F|U$, so that r is a retraction of U onto $f(A)$, accordingly $f(A)$ is a neighborhood retract of Q_ω. Using Theorem 1, A is an S-ANR. Q. E. D.

LEMMA 2. *For every space Z which contains S-ANR A, $\Gamma_z[A]$ is closed in Z.*

PROOF. Let us suppose that there exists a space Z which contains A and in which $\Gamma_z[A]$ is not closed, then there exists a sequence $\{x_n\} \in \Gamma_z[A]$ which converges to a point $x \bar{\in} \Gamma_z[A]$. Since $\{x_n\} \in \Gamma_z[A]$ we have $x \in \overline{A}$, but since $x \bar{\in} A$, we can see $x \in A$.

Let D be $A + \sum_{n=1}^{\infty} x_n$, then D is a space which contains A. Each neighborhood V of A in D has a form $V = A + \sum_{n=1}^{\infty} x^n - (x_{n1} + \dots + x_{n_m})$, where m is finite. Let us suppose that, for any V, there exists a mapping r of V onto A. Then, by the continuity of r and Hausdorff's separation axiom, we can choose neighborhoods $U(x_n)$ and $U(r(x_n))$ of all $x_n \in \Gamma_z[A] \cap V$ such that, for each x_n, $U(x_n)$ and $U(r(x_n))$ are disjoint and both contains some point of A and finally $r(U(x_n)) \subset U(r(x_n))$. Hence r is not a retraction of V onto A, accordingly A is not a neighborhood retract of D. Q. E. D.

LEMMA 3. *Let B be a locally compact space which is contained in a compact space C, then $\Gamma_c[B]$ is closed.*

PROOF. If $\Gamma_c[B]$ is a finite set, our Lemma is trivial, hence suppose that $\Gamma_c[B]$ be an infinite set which is not closed. There exists a sequence $\{x_n\} \in \Gamma_c[B]$ which converges to some point x of B. Since B is locally compact, there exists a neighborhood U of x in B such that the closure $\overline{U \cap B_B}$ of $U \cap B$ in B is compact, and so closed in C and in $\overline{B}$. By our construction, $\overline{U \cap B^B}$ does not contain any point of the sequence $\{x_n\}$.

On the other hand, the sequence $\{x_n\}$ converges to x, hence there exists a neighborhood V_n of x_n in $\overline{B}$ such that $V_n \cap B$ is contained in U for sufficiently large n, x_n is a limit point of B, hence there exists a sequence $\{x_m^n\}$ of B which converges to x_n and, for sufficiently large N, x_m^n $(m > N)$ are contained in $V_n \subset B$, hence they are also contained in $\overline{U \cap B^B}$. But $\overline{U \cap B^B}$ is closed in $\overline{B}$. Therefore, x_n is contained in $\overline{U \cap B^B}$. This is a contradiction. Q. E. D.

4. THEOREM 2. *For locally compact spaces, an absolute neighborhood retract*

in the strong sense and an absolute neighborhood retract in the weak sense are equivalent.

PROOF. An S-ANR, A, is a W-ANR and we may assume that A is contained in a compact space Z (for example, by Urysohn's theorem [1]). It follows, by Lemma 2, that $A = \bar{A} - \Gamma_Z[A]$ is open in $\bar{A}$, hence A is locally compact.

Let A be a locally compact W-ANR and f be an arbitrary homeomorphism of A in Q_ω. Then Lemma 3 implies that $\Gamma_{Q\omega}[f(A)]$ is closed in Q_ω. Thus by Lemma 1, A is an S-ANR. Q.E.D.

COROLLARY. *A locally finite n-dimensional infinite polyhedron is an S-ANR.*

PROOF. In the small the polyhedron is a finite polyhedron then the polyhedron is locally compact and locally contractible. From the later the polyhedron is a W-ANR [6] and is an S-ANR by Theorem 2. Q.E.D.

BIBLIOGRAPHY

[1] P. ALEXANDROFF, H. HOPF. Topologie I, 1935.
[2] K. BORSUK, Über eine Klasse von lokal zusammenhängenden Räumen, Fund. Math., 19(1932), 220-242.
[3] R. H. FOX, A characterization of absolute neighborhood retracts, Bull. Amer. Math. Soc., 48 (1942), 271-275.
[4] S.-T. HU., On homotopy and deformation retracts, Proc. Combridge Phil. Soc., 43 (1947), 314-320.
[5] C. KURATOWSKI, Sur les espaces localement connexes et péaniens en dimension n, Fund. Math., 24 (1935), 269-287.
[6] S. D. LIAO, On locally connected sets and absolute neighborhood retracts, Portugaliae Mathematicae, 8 (1949), 137-142.

MATHEMATICAL INSTITUTE, TÔHOKU UNIVERSITY.