AN EXTENSION OF BLOCH'S THEOREM AND ITS APPLICATIONS TO NORMAL FAMILY

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In the former paper[1], I have proved the following extension of Bloch's theorem:

THEOREM 1. *Let $w = f(z)$ be meromorphic in $|z| < 1$ and*

$$\frac{|f'(0)|}{1 + |f(0)|^2} = 1.$$

Then the Riemann surface F generated by $w = f(z)$ on the w-sphere contains a schlicht spherical disc, whose radius is $\geqq \rho_0 > 0$, where ρ_0 is a constant independent of $f(z)$.

In this paper, I shall apply this theorem to normal family.

THEOREM 2.[2] *Let $D_1, \cdots, D_q$ $(q \geqq 3)$ be q disjoint simply connected domains on the w-sphere and $1 \leqq m_i \leqq \infty$ be positive integers or ∞, such that*

$$\sum_{i=1}^{q} (1 - 1/m_i) > 2.$$

Let $w = f(z)$ be mermorphic in $|z| < R$ and F be the Riemann surface generated by $w = f(z)$ on the w-sphere. If every simply connected island of F, which lies above D_i is of multiplicity $\geqq m_i$[3]*, then*

$$R \leqq \kappa \frac{1 + |f(0)|^2}{|f'(0)|}, \qquad \frac{|f(0)|}{1 + |f(0)|^2} \leqq \frac{\kappa}{R},$$

where κ is a constant, which depends on $D_1, \cdots, D_q$ only.

PROOF. It can be proved easily that if $\sum_{i=1}^{q}(1 - 1/m_i) > 2$, then

$$\sum_{i=1}^{q} (1 - 1/m_i) - 2 \geqq 1/42, \tag{1}$$

where the minimum value $1/42$ is attained, when $m_1 = 2$, $m_2 = 3$, $m_3 = 7$, $m_4 = \cdots = m_q = 1$.

First suppose that

$$\frac{|f'(0)|}{1 + |f(0)|^2} = 1. \tag{2}$$

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1) M. TSUJI, On an extension of Bloch's theorem. Proc. Imp. Acad., 18(1942).

2) J. DUFRESNOY, Sur les domaines couverts par les valeurs d'une fonction méromorphe ou algébroïde. (Thèse, 1935). Z. YÛJÔBÔ, An application of Ahlfors' theory of covering surfaces. Jour. Math. Soc. Japan., 4(1952).

3) $m_i = \infty$ means that there is no island of F above D_i.

We put

$$L(r) = \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2} r\,d\theta \qquad (0 \leqq r < R),$$

$$S(r) = \frac{1}{\pi}\int_0^r \int_0^{2\pi} \left(\frac{|f'(te^{i\theta})|}{1+|f(te^{i\theta})|^2}\right)^2 t\,dt\,d\theta \qquad (0 \leqq r < R),$$

then

(3) $$(L(r))^2 \leqq 2\pi^2 r \frac{dS(r)}{dr}.$$

By Ahlfors' theorem[4], there exists a constant $h > 0$, which depends on $D_1, \dots, D_q$ only, such that

$$\sum_{i=1}^{q}(1-1/m_i) \leqq 2 + h\frac{L(r)}{S(r)},$$

so that

$$\delta = \frac{1}{42} \leqq \sum_{i=1}^{q}(1-1/m_i) - 2 \leqq h\frac{L(r)}{S(r)},$$

hence by (3),

$$\frac{\delta^2}{h^2}(S(r))^2 \leqq (L(r))^2 \leqq 2\pi^2 r\frac{dS(r)}{dr} \qquad (0 \leqq r < R),$$

so that if $R > 1$,

$$\log R = \int_1^R \frac{dr}{r} \leqq \frac{2\pi^2 h^2}{\delta^2}\int_1^R \frac{dS(r)}{(S(r))^2} \leqq \frac{2\pi^2 h^2}{\delta^2}\frac{1}{S(1)}.$$

By Theorem 1, $S(1) \geqq h_0 > 0$, where h_0 is a constant. Hence

(4) $$R \leqq \exp(2\pi^2 h^2/\delta^2 h_0) = \kappa.$$

(4) holds evidently, if $R \leqq 1$.

In the general case

(5) $$\frac{|f'(0)|}{1+|f(0)|^2} = \alpha \neq 0,$$

we put $z = x/\alpha$, $F(x) = f(z)$, then $|F'(0)|/(1+|F(0)|^2) = 1$, so that by (4)

(6) $$R \leqq \frac{\kappa}{\alpha} = \kappa\frac{1+|f(0)|^2}{|f'(0)|},$$

which holds evidently, if $|f'(0)|/(1+|f(0)|^2) = 0$. Hence our theorem is proved.

Let z be any point in $|z| < R$, then

4) L. AHLFORS, Zur Theorie der Überlagerungsflächen. Acta Math., 65(1935).

$$\varphi(\zeta) = f\left(\frac{R^2(\zeta + z)}{R^2 + \bar{z}\zeta}\right)$$

satisfies the condition of Theorem 2, so that

$$\frac{|\varphi'(0)|}{1 + |\varphi(0)|^2} \leqq \frac{\kappa}{R}.$$

Since $\varphi(0) = f(z)$, $\varphi'(0) = f'(z)(R^2 - |z|^2)/R^2$, we have

$$\frac{|f'(z)|}{1 + |f(z)|^2} \leqq \frac{\kappa R}{R^2 - |z|^2}. \tag{7}$$

Hence

THEOREM 3.[5] *Let $f(z)$ be meromorphic in $|z| < R$ and satisfy the condition of Theorem 2, then the family $\{f(z)\}$ is normal.*

Almost all criteria on normal family can be deduced from Theorem 3.

THEOREM 4.[6] *Let $w = f(z)$ be meromorphic in $|z| < R$ and the Riemann surface F generated by $w = f(z)$ on the w-sphere is a covering surface of a closed Riemann surface Φ of genus $p \geqq 2$, then*

$$R \leqq \kappa \frac{1 + |f(0)|^2}{|f'(0)|}, \qquad \frac{|f'(0)|}{1 + |f(0)|^2} \leqq \frac{\kappa}{R},$$

where κ is a constant, which depends on Φ only.

PROOF. Let $L(r)$, $S(r)$ have the same meanings as the proof of Theorem 2 and n be the number of sheets of Φ. Then by Ahlfors' fundamental theorem on covering surfaces[7],

$$\rho^+(r) \geqq \frac{\rho_0}{n} S(r) - hL(r), \quad \rho^+(r) = \text{Max } (\rho(r),\ 0),$$

where $\rho_0 = 2(p-1)$ is the Euler's characteristic of Φ and $\rho(r)$ is that of the Riemann surface F_r generated by $w = f(z)$ $(0 \leqq |z| \leqq r)$ and $h > 0$ is a constant, which depends on Φ only. Since F_r is simply connected, $\rho^+(r) = 0$, hence

$$S(r) \leqq \frac{nh}{\rho_0} L(r).$$

From this we can prove the theorem similarly as Theorem 2.
Similarly as Theorem 3, we have

THEOREM 5.[8] *Let $F(x, y) = 0$ be an algebraic curve of genus $p \geqq 2$ and $x = x(z)$, $y = y(z)$ be uniformizing functions, which are meromorphic in a domain D, then the family $\{x(z)\}$, $\{y(z)\}$ are normal in D.*

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5) Dufresnoy, l. c. 2).
6) A. BLOCH, Les théorèmes de M. Valiron sur les fonctions entières et la théorie de uniformisation, C. R., 178(1924).
7) Ahlfors, l. c. 4).
8) Bloch, l. c. 6).