CESÀRO SUMMABILITY OF FOURIER SERIES

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(Received September 28, 1953)

Let $f(x)$ be an integrable function with period 2π and its Fourier series be

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx). \tag{1}$$

We have proved the following results [2], [3]:

(i) If $f(x) \in L$ and $0 < \alpha < 1$, then the series

$$\sum_{n=1}^{\infty} n^{-\alpha} (a_n \cos nx + b_n \sin nx) \tag{2}$$

is summable $(C, -\alpha)$ except a set of $(1-\alpha)$-capacity zero.

(ii) If $f(x) \in L^2$ and $0 < \alpha < 1$, then the series

$$\sum_{n=1}^{\infty} n^{-\alpha/2} (a_n \cos nx + b_n \sin nx) \tag{3}$$

is summable $(C, -\alpha/2)$ except a set of $(1-\alpha)$-capacity zero.

(iii) If $f(x) \in L^p$, $1 \leqq p$ and $0 < \alpha < 1$, then the series

$$\sum_{n=1}^{\infty} n^{-\alpha/p} (a_n \cos nx + b_n \sin nx) \tag{4}$$

is summable $(C, -\alpha/p)$ except a set of β-capacity zero for any $\beta > 1-\alpha$.

In this paper these results are completed as follows:

THEOREM. *(a) If $f(x) \in L^p$, $1 \leqq p \leqq 2$, and $0 < \alpha < 1$, then the series (4) is summable $(C, -\alpha/p)$ except a set of $(1-\alpha)$-capacity zero.*

(b) If $f(x) \in L^p$, $p > 2$, and $0 < \alpha < 1$, then the series (4) is summable $(C, -\alpha/p)$ except a set of β-capacity zero for any $\beta > 1-\alpha$.

(c) In the case (b), β cannot be replaced by $1 - \alpha$.

Proof. The statement (b) is identical with (iii) above, and since N. du Plessis [1] has given an example of a function of L^p, $p > 2$, such that the series (4) assoiated with it does not even converge on the points of a set of $(1-\alpha)$-capacity positive, we have only to prove the statement (a).

For the proof we need the following lemma.

LEMMA. *Let $\mu(x)$ be an increasing function on $(0, 2\pi)$ and let*

$$V_{1-\alpha}(\mu) = \sup_{0 \leqq x \leqq 2\pi} \int_0^{2\pi} |x - t|^{\alpha-1} d\mu(t).$$

Then we have for $2 \leqq q \leqq \infty$

$$\left\{ \int_0^{2\pi} \left[\int_0^{2\pi} |x - t|^{\alpha/p - 1} d\mu(t) \right]^q dx \right\}^{1/q} \leqq A_\alpha V_{1-\alpha}^{1/p}(\mu), \quad (1/p + 1/q = 1), \tag{6}$$

where A_α is a constant depending only on α.

This is due to du Plessis [1].

To prove (a) it is sufficient to show that if E is a set of $(1-\alpha)$-capacity positive and $\mu(x)$ is a distribution concentrated on E then we have the inequality

$$(7) \qquad \int_0^{2\pi} \sup_n |N_n^{(\alpha/p)}(x;f)|\,d\mu(x) \leqq A_{\alpha,p}(\mu) \left\{ \int_0^{2\pi} |f(x)|^p dx \right\}^{1/p},$$

where $N_n^{(\alpha/p)}(x;f)$ is the Cesàro means of order $-\alpha/p$ of the series (4) and $A_{\alpha,p}(\mu)$ is a constant depending only on α, p and the distribution μ. From this maximal inequality the conclusion in (a) can be obtained quite analogously as in [2], [3].

Now, we already know [3, (16), p. 34]

$$|N_n^{(\alpha/p)}(x;f)| \leqq A_{\alpha,p} \int_0^{2\pi} |x-t|^{\alpha/p-1} |f(t)|\,dt,$$

where $A_{\alpha,p}$ is dependent only on α and p. In virtue of the lemma, we have

$$\begin{aligned}\int_0^{2\pi} \sup_n |N_n^{(\alpha/p)}(x;f)|\,d\mu(x) &\leqq A_{\alpha,p} \int_0^{2\pi} |f(t)| \left\{ \int_0^{2\pi} |x-t|^{\alpha/p-1} d\mu(x) \right\} dt \\ &\leqq A_{\alpha,p} \left\{ \int_0^{2\pi} |f(t)|^p dt \right\}^{1/p} \left\{ \int_0^{2\pi} \left[\int_0^{2\pi} |x-t|^{\alpha/p-1} d\mu(x) \right]^2 dt \right\}^{1/q} \\ &\leqq A_{\alpha,p} V_{1-\alpha}^{1/p}(\mu) \left\{ \int_0^{2\pi} |f(x)|^p dx \right\}^{1/p},\end{aligned}$$

where $1/p+1/q=1$.

Taking $A_{\alpha,p} V_{1-\alpha}^{1/p}(\mu)$ for $A_{\alpha,p}(\mu)$, we obtain the desired inequality (6) and the proof of (a) is completed.

REFERENCES

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